Math 408 • Section 3963xD • Mathematical Statistics
Quiz # 6 Solution, April 16, 2013 Teaching Assistant: Greg Sokolov

This was a 20 minute quiz worth 10 points.

Let p be an unknown parameter, $0 < p < 1$, and suppose that X is a discrete random variable with p.m.f.

$$\pi_X(k|p) = P_p(X = k) = (1-p)^k\, p, \quad k = 0, 1, 2, \cdots .$$

Hint. $EX = (1-p)/p$ and $\mathrm{Var}X = (1-p)/p^2$.
You observe three values $6, 1, 2$.

1.) (2 points) Find the method of moments estimator of p.

Solution.

$$\frac{X_1 + \cdots + X_n}{n} = \frac{1 - \hat{p}_{\text{MoM}}}{\hat{p}_{\text{MoM}}} \quad \Rightarrow \quad \hat{p}_{\text{MoM}} = \frac{n}{n + X_1 + \cdots + X_n}.$$

In our case, $\hat{p}_{\text{MoM}} = 3/(3+6+1+2) = 1/4$.

2.) (4 points) Find the maximum-likelihood estimator of p.

Solution.

$$\mathcal{L}(p) = \prod_{i=1}^{n} (1-p)^{X_i} p = (1-p)^{X_1 + \cdots + X_n}\, p^n,$$
$$\log \mathcal{L}(p) = (X_1 + \cdots + X_n) \log(1-p) + n \log p,$$
$$\frac{\partial}{\partial p} \log \mathcal{L}(p) = -(X_1 + \cdots + X_n)\, \frac{1}{1-p} + \frac{n}{p}.$$

Therefore,

$$\hat{p}_{\text{MLE}} = \frac{n}{n + X_1 + \cdots + X_n} = \hat{p}_{\text{MoM}}.$$

In our case, $\hat{p}_{\text{MLE}} = 1/4$.

3.) (4 points) Find the Cramer-Rao bound for the variance of unbiased estimators of p.

Solution.

$$\pi(X|p) = (1-p)^X p,$$
$$\log \pi(X|p) = X \log(1-p) + \log p,$$
$$\frac{\partial}{\partial p} \log \pi(X|p) = -\frac{X}{1-p} + \frac{1}{p} = -\frac{1}{1-p}\left(X - \frac{1-p}{p}\right),$$
$$E\left(\frac{\partial}{\partial p} \log \pi(X|p)\right)^2 = \frac{1}{(1-p)^2}\, E\left(X - \frac{1-p}{p}\right)^2$$
$$= \frac{1}{(1-p)^2}\, \mathrm{Var}X$$
$$= \frac{1}{p^2(1-p)}.$$

Therefore, for any unbiased estimator $\hat{p}$,

$$\mathrm{Var}(\hat{p}) \geq \frac{p^2(1-p)}{n}.$$