Math 408 • Section 3963xD • Mathematical Statistics
Quiz # 5 Solution, April 2, 2013 Teaching Assistant: Greg Sokolov

This was a 20 minute quiz worth 10 points.

1.) (5 points) Suppose that X is a continuous random variable with p.d.f.

$$f_X(x) = \begin{cases} 1/4, & a-2 \le x < a \\ 1/2, & a \le x < a+1 \end{cases},$$

where a is unknown parameter. Find a 50% confidence interval for a if you observe $X = 4$.

Solution. From the p.d.f. of X it's easy to see that $P(a-1 \le X \le a+1/2) = 0.5$. Then

$$\begin{aligned} 0.5 &= P(a-1 \le X \le a+1/2) \\ &= P(-1 \le X - a \le 1/2) \\ &= P(-1 - X \le -a \le 1/2 - X) \\ &= P(X - 1/2 \le a \le X + 1). \end{aligned}$$

Thus $[X - 1/2, X + 1]$ is a 50% confidence interval for a. We observe $X = 4$, so this interval becomes $[3.5, 5]$.

2.) (5 points) Suppose that $X \sim \text{Poisson}(\lambda)$. That is, X has a p.m.f.

$$f_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \cdots.$$

Use the Taylor expansion for $e^x + e^{-x}$ at $x = \lambda$ to show that

$$P(X \text{ is even}) = \frac{1}{2}(1 + e^{-2\lambda}).$$

Solution. Since

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

for all $x \in \mathbb{R}$, we have that

$$e^{-\lambda} + e^{\lambda} = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} + \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \sum_{k=0}^{\infty} \left(\frac{(-\lambda)^k}{k!} + \frac{\lambda^k}{k!} \right) = \sum_{k \ge 0, \text{ even}} 2\frac{\lambda^k}{k!}.$$

Thus

$$\frac{e^{-\lambda} + e^{\lambda}}{2} = \sum_{k \ge 0, \text{ even}} \frac{\lambda^k}{k!},$$

and

$$\frac{1}{2}(1 + e^{-2\lambda}) = e^{-\lambda}\frac{e^{-\lambda} + e^{\lambda}}{2} = \sum_{k \ge 0, \text{ even}} e^{-\lambda}\frac{\lambda^k}{k!} = \sum_{k \ge 0, \text{ even}} P(X = k) = P(\text{X is even}).$$