Math 408 • Section 3963xD • Mathematical Statistics
Quiz # 4 Solution, March 5, 2013 Teaching Assistant: Greg Sokolov

This was a 20 minute quiz worth 10 points.

1.) (5 point) Suppose that $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ are independent. Use the fact that $X \sim \mathcal{N}(\mu, \sigma^2)$ has moment generating function

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

to derive the distribution of $X_1 + X_2$.

Solution. The moment generating function of $X_1 + X_2$ is

$$\begin{aligned}
M_{X_1+X_2}(t) &= E\left[e^{t(X_1+X_2)}\right] \\
&= E\left[e^{tX_1}\, e^{tX_2}\right] \\
&= E\left[e^{tX_1}\right]\, E\left[e^{tX_2}\right] \qquad \text{(here we used independence)} \\
&= M_{X_1}(t)\, M_{X_2}(t) \\
&= e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2} \\
&= e^{(\mu_1+\mu_2)t + \frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2},
\end{aligned}$$

which is the MGF of a $\mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ random variable. Since the distribution of a random variable is completely characterized by its moment generating function, it follows that

$$X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2).$$

2.) (5 point) A simple random sample of 400 people was taken from a city of 100 000 people. There were 320 males and 80 females in the sample. Estimate the population proportion p of males in the city, and use the result to approximate the standard deviation of your estimator.

Solution. Let $X_1, \ldots, X_n$ be Bernoulli random variables, where $n = 400$. Since it is a *simple random sample* from the population and because the population size is much larger than the sample size, X-s may be considered independent. Assume they also have a common distribution which is the true population distribution, and that n is large enough. Let $\hat{p}$ be the proportion estimator:

$$\hat{p} = \frac{X_1 + \cdots + X_n}{n}.$$

Then $E[\hat{p}] = p$ and

$$\mathrm{Var}[\hat{p}] = \frac{1}{n^2}\mathrm{Var}(X_1 + \cdots + X_n) = \frac{p(1-p)}{n}.$$

We observe $\hat{p} = 320/400 = 0.8$, hence our estimate of p is 0.8 and the standard deviation of our estimator can be approximated by $\sqrt{0.8 \cdot 0.2/400} = 0.02$.