Math 408 • Section 3963xD • Mathematical Statistics
Quiz # 3 Solution, February 19, 2013 Teaching Assistant: Greg Sokolov

This was a 20 minute quiz worth 10 points.

The Weak Law of Large Numbers says, that if $X_1, X_2, \cdots$ is a sequence of i.i.d. random variables with $E(|X_1|) < \infty$, then $\sum_{i=1}^n X_i/n$ converges in probability to $\mu = E(X_1)$.
Use Chebyshev's inequality to show that if $X_1, X_2, \cdots$ is a sequence of uncorrelated random variables, all having mean $E(X_i) = \mu$ and variance $\mathrm{Var}(X_i) = \sigma^2 < \infty$, then $\sum_{i=1}^n X_i/n$ converges in probability to μ.

Solution. Let
$$Y_n = \frac{X_1 + \cdots + X_n}{n}.$$
Then
$$EY_n = E\left(\frac{1}{n}\sum_{i=1}^n X_i\right) = \frac{1}{n}\sum_{i=1}^n EX_i = \frac{1}{n} n\, EX_1 = \mu,$$
and since $X_1, X_2, \cdots$ are uncorrelated, $\mathrm{Cov}(X_i, X_j) = 0$ for $i \neq j$, hence
$$\mathrm{Var}Y_n = E\left(\frac{1}{n}\sum_{i=1}^n X_i\right) = \frac{1}{n^2}\sum_{i=1}^n \mathrm{Var}X_i = \frac{1}{n^2} n\, \mathrm{Var}X_1 = \frac{\sigma^2}{n}.$$
According to Chebyshev's inequality, for any $\varepsilon > 0$
$$P(|Y_n - EY_n| \geq \varepsilon) \leq \frac{\mathrm{Var}Y_n}{\varepsilon^2} \quad \Leftrightarrow \quad P\left(\left|\frac{1}{n}\sum_{i=1}^n X_i - \mu\right| \geq \varepsilon\right) \leq \frac{\sigma^2}{n\varepsilon^2}.$$
Since probabilities are always nonnegative, taking the limit as $n \to \infty$, one gets
$$0 \leq \lim_{n\to\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^n X_i - \mu\right| \geq \varepsilon\right) \leq \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0.$$
It follows that
$$\lim_{n\to\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^n X_i - \mu\right| \geq \varepsilon\right) = 0.$$