Math 408 • Section 3963xD • Mathematical Statistics
Quiz # 2 Solution, February 5, 2013 Teaching Assistant: Greg Sokolov

This was a 20 minute quiz worth 10 points.

Let X be an exponentially distributed random variable with mean 1, i.e.

$$F_X(x) = \begin{cases} 1 - e^{-x}, & x \geq 0, \\ 0, & x < 0, \end{cases} \qquad f_X(x) = \begin{cases} e^{-x}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

where $F_X(x)$ is the c.d.f. of X and $f_X(x)$ is its p.d.f. Let $Y = e^{aX}$.

1.) (1 point) What is the support (the set of possible values) of Y?

Solution. Since X can have values in $[0, \infty)$, $Y = e^{aX}$ can have possible values in $[1, \infty)$.

2.) (3 points) What is the p.d.f. of Y?

Solution. First, let's find the c.d.f. of Y:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P\left(e^{aX} \leq y\right) \\ &= P\left(X \leq (\log y)/a\right) \\ &= 1 - e^{-(\log y)/a} = 1 - \frac{1}{y^{1/a}}, \quad y \geq 1. \end{aligned}$$

Then the p.d.f. of Y is

$$f_Y(y) = F_Y'(y) = \frac{1}{a} \frac{1}{y^{(1/a)+1}}, \quad y \geq 1.$$

3.) (3 points) What is the expected value of Y?

Solution.

$$\begin{aligned} E[Y] &= \int_1^\infty y f_Y(y)\, dy \\ &= \int_1^\infty \frac{1}{a} \frac{1}{y^{1/a}}\, dy = \begin{cases} \infty, & \text{if } a \geq 1, \\ \dfrac{a}{a-1}, & \text{if } a < 1. \end{cases} \end{aligned}$$

4.) (3 points) What is the joint c.d.f. of X and Y?

Solution. For $x \geq 0$ and $y \geq 1$,

$$\begin{aligned} P(X \leq x,\ Y \leq y) &= P\left(X \leq x,\ e^{aX} \leq y\right) \\ &= P\left(X \leq x,\ X \leq (\log y)/a\right) \\ &= P\left(X \leq \min(x, (\log y)/a)\right) \\ &= 1 - e^{-\min(x, (\log y)/a)} = 1 - \max\left(e^{-x}, \frac{1}{y^{1/a}}\right). \end{aligned}$$