Math 408 - Mathematical Statistics

Lecture 5. Joint Distributions

January 28, 2013

Agenda

- Bivariate Distributions
- Marginal Distributions
- Independent Random Variables
- Conditional Distributions
- Transformation of Several Random Variables
- Summary

Bivariate Distributions

- Discrete Case

Definition

Given a pair of discrete random variables X and Y, their **joint PMF** is defined by

$$f_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$$

- Continuous Case

Definition

A function $f_{X,Y}(x,y)$ is called the **joint PDF** of continuous random variables X and Y if

- $f_{X,Y}(x,y) \geq 0, \quad \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)dxdy = 1$
- For any set $A \subset \mathbb{R} \times \mathbb{R}$

$$\mathbb{P}((X,Y) \in A) = \int\int_A f_{X,Y}(x,y)dxdy$$

The **joint CDF** of X and Y is defined as $F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$

Marginal Distributions

- Discrete Case
  If X and Y have joint PMF $f_{X,Y}$, then the **marginal PMF** of X is

$$f_X(x) = \mathbb{P}(X = x) = \sum_y \mathbb{P}(X = x, Y = y) = \sum_y f_{X,Y}(x, y)$$

Similarly, the **marginal PMF** of Y is

$$f_Y(y) = \mathbb{P}(Y = y) = \sum_x \mathbb{P}(X = x, Y = y) = \sum_x f_{X,Y}(x, y)$$

- Continuous Case
  If X and Y have joint PDF $f_{X,Y}$, then the **marginal PDFs** of X and Y are

$$f_X(x) = \int f_{X,Y}(x, y)dy \quad \text{and} \quad f_Y(y) = \int f_{X,Y}(x, y)dx$$

Examples

- Suppose that the PMF f_{XY} is given in the following table:

	$Y=0$	$Y=1$
$X=0$	$1/10$	$2/10$
$X=1$	$3/10$	$4/10$

  Find the marginal PMF of X.
  Answer: $f_X(0)=3/10$, $f_X(1)=7/10$

- Suppose that

$$f_{X,Y}(x,y)=e^{-(x+y)}, \quad x,y\geq 0$$

  Find the marginal PDF of X.
  Answer: $f_X(x)=e^{-x}$, $x\geq 0$

Independent Random Variables

Definition

Two random variables X and Y are **independent** if, for every A and B

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$$

In principle, to check whether X and Y are independent, we need to check the above equation for all subsets A and B. Fortunately, we have the following result:

Theorem

Let X and Y have joint PDF/PMF $f_{X,Y}$. Then X and Y are independent if and only if

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

Example: Suppose that X and Y are independent and both have the same density

$$f(x) = \begin{cases} 2x, & x \in [0, 1] \\ 0, & x \notin [0, 1] \end{cases}$$

Find $\mathbb{P}(X + Y \leq 1)$. Answer: $1/6$

Conditional Distributions

- Discrete Case
  If X and Y are discrete, then we can compute the conditional probability of the event $\{X = x\}$ given that we have observed $\{Y = y\}$:

$$\mathbb{P}(X = x|Y = y) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)}$$

  This leads to the following definition of the **conditional PMF**:

$$f_{X|Y}(x|y) = \mathbb{P}(X = x|Y = y) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)} = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

- Continuous Case
  For continuous random variables, the **conditional PDF** is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

  Then,

$$\mathbb{P}(X \in A|Y = y) = \int_A f_{X|Y}(x|y)dx$$

Example

- Suppose that $X \sim U(0,1)$. After obtaining a value x of X, we generate $Y|X = x \sim U(x,1)$. What is the marginal distribution of Y?
  Answer:
$$f_Y(y) = -\ln(1-y) \quad y \in (0,1)$$

Transformation of Several Random Variables

In some cases we are interested in transformation of several random variables. For example, if X and Y are given random variables, we might want to know the distribution of X/Y, $X+Y$, $\max\{X, Y\}$, etc.

Let $Z = r(X, Y)$. The steps for finding f_Z are the following:

1. For each z, find the set $A_z = \{(x, y) : r(x, y) \leq z\}$
2. Find the CDF

$$F_Z(z) = \mathbb{P}(Z \leq z) = \mathbb{P}(r(X, Y) \leq z) = \mathbb{P}((X, Y) \in A_z) = \int\int_{A_z} f_{X,Y}(x, y)dxdy$$

3. Then PDF $f_Z(z) = F'_Z(z)$

Example: Let $X, Y \sim U[0, 1]$ be independent.
Find the density of $Z = X + Y$.

Answer:

$$f_Z(z) = \begin{cases} z, & 0 \leq z \leq 1 \\ 2 - z, & 1 < z \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Summary

- Joint Distributions:
  - Discrete case: $f_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$
  - Continuous case: $\mathbb{P}((X,Y) \in A) = \int\int_A f_{X,Y}(x,y)dxdy$
- Marginal Distributions
  - Discrete case: $f_X(x) = \sum_y f_{X,Y}(x,y)$
  - Continuous case: $f_X(x) = \int f_{X,Y}(x,y)dy$
- X and Y are independent if, for every A and B

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$$

  - X and Y are independent if and only if $\boxed{f_{X,Y}(x,y) = f_X(x)f_Y(y)}$
- Conditional PDF/PMF:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

- Algorithm for finding distribution of $Z = r(X,Y)$