Math 408 - Mathematical Statistics

Lecture 4. Continuous Random Variables and Transformations of Random Variables

January 25, 2013

Agenda

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- Important Examples
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  - Normal (Gaussian) Distribution
  - Exponential Distribution
  - Gamma Distribution
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Definition

Recall that a random variable is a (deterministic) map $X : \Omega \to \mathbb{R}$ that assigns a real number $X(\omega)$ to each (random) realization $\omega \in \Omega$.

Definition

A random variable is **continuous** if there exists a function f_X such that

- $f_X(x) \geq 0$ for all x
- $\int_{-\infty}^{+\infty} f_X(x)dx = 1$, and
- For every $a \leq b$

$$\boxed{P(a < X \leq b) = \int_a^b f_X(x)dx}$$

- The function $f_X(x)$ is called the probability density function (PDF)
- Relationship between the CDF $F_X(x)$ and PDF $f_X(x)$:

$$\boxed{F_X(x) = \int_{-\infty}^{x} f_X(t)dt} \qquad \boxed{f_X(x) = F'_X(x)}$$

Important Remarks

- If X is continuous then $\mathbb{P}(X = x) = 0$ for every x.
- Don't think of $f_X(x)$ as $\mathbb{P}(X = x)$. This is only true for discrete random variables.
- For continuous random variables, we get probabilities by integrating.
- A PDF can be bigger than 1 (unlike PMF!). For example:

$$f_X(x) = \begin{cases} 10, & x \in [0, 0.1] \\ 0, & x \notin [0, 0.1] \end{cases}$$

- Can a PDF be unbounded?
  Yes, of course! For instance

$$f_X(x) = \begin{cases} \frac{2}{3} x^{-1/3}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Important Examples

- **The Uniform Distribution**
  X has a uniform distribution on $[a, b]$, denoted $X \sim U[a, b]$, if

$$f(x) = \begin{cases} \frac{1}{b-a}, & x \in [a, b] \\ 0, & \text{otherwise} \end{cases}$$

- **Normal (Gaussian) Distribution**
  X has a Normal (or Gaussian) distribution with parameters μ and σ, denoted by $X \sim \mathcal{N}(\mu, \sigma^2)$, if

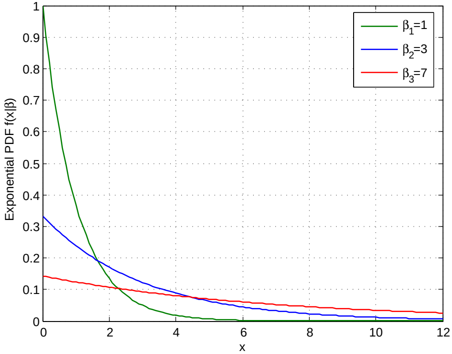
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}$$

  - Many phenomena in nature have approximately Normal distribution.
  - Distribution of a sum of random variables can be approximated by a Normal distribution (central limit theorem)

Important Examples

- **Exponential Distribution**
  X has an Exponential distribution with parameter $\beta > 0$, $X \sim \mathrm{Exp}(\beta)$, if

$$f(x) = \frac{1}{\beta} e^{-x/\beta}, \quad x > 0$$



The exponential distribution is used to model the life times of electronic components and the waiting times between rare events. β is a survival parameter: the expected duration of survival of the system is β units of time.

Important Examples

- **Gamma Distribution**
  X has a Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$, $X \sim \mathrm{Gamma}(\alpha, \beta)$, if

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, \quad x > 0$$

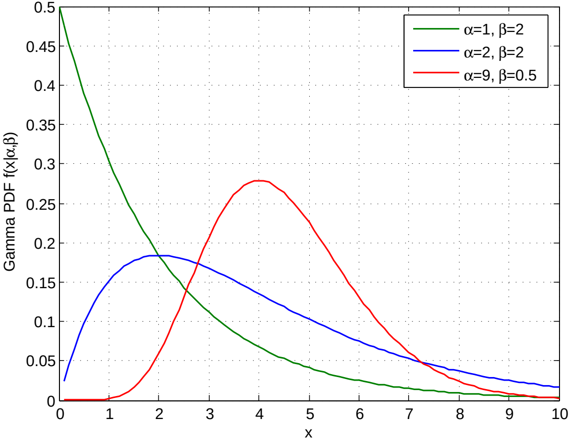
  - $\Gamma(\alpha)$ is the Gamma function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

  - The Gamma distribution is frequently used to model waiting times.
  - Exponential distribution is a special case of the Gamma distribution:

$$\mathrm{Gamma}(1, \beta) = \mathrm{Exp}(\beta)$$

Gamma Distribution



Important Examples

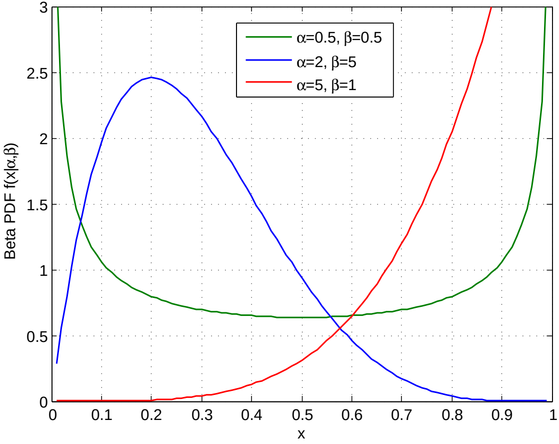
- **Beta Distribution**
  X has a Beta distribution with parameters $\alpha > 0$ and $\beta > 0$, $X \sim \mathrm{Beta}(\alpha, \beta)$, if

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1$$

  - The beta distribution is often used for modeling of proportions.
  - The beta distribution has an important application in the theory of order statistics. A basic result is that the distribution of the k^{th} largest $X_{(k)}$ of a sample of size n from a uniform distribution $X_1, \ldots, X_n \sim U(0,1)$ has a beta distribution:

$$X_{(k)} \sim \mathrm{Beta}(k, n-k+1)$$

Beta Distribution



Transformation of Random Variables

Suppose that X is a random variable with PDF/PMF (continuous/discrete) f_X and CDF F_X. Let $Y = r(X)$ be a function of X, for example, $Y = X^2$, $Y = e^X$.

Q: How to compute the PDF/PMF and CDF of Y?

In the discrete case, the answer is easy:

$$f_Y(y) = \mathbb{P}(Y = y) = \mathbb{P}(r(X) = y) = \mathbb{P}(\{x : r(x) = y\}) = \sum_{x_i : r(x_i) = y} f_X(x_i)$$

Example:

- $X \in \{-1, 0, 1\}$
- $\mathbb{P}(X = -1) = 1/4, \quad \mathbb{P}(X = 0) = 1/2, \quad \mathbb{P}(X = 1) = 1/4$
- $Y = X^2$
- Find PMF f_Y

Answer: $f_Y(0) = 1/2$ and $f_Y(1) = 1/2$.

Transformation of Random Variables: Continuous Case

The continuous case is harder.
These are the steps for finding the PDF f_Y:

1. For each y, let $A_y = \{x : r(x) \leq y\}$
2. Find the CDF $F_Y(y)$

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(r(X) \leq y) = \mathbb{P}(X \in A_y) = \int_{A_y} f_X(x)dx$$

3. The PDF is then $f_Y(y) = F'_Y(y)$

Example: Let $X \sim \mathrm{Exp}(1)$, and $Y = \ln X$. Find $f_Y(y)$.

Answer: $f_Y(y) = e^y e^{-e^y}$, $y \in \mathbb{R}$

Important Fact: When r is strictly monotonic, then r has an inverse $s = r^{-1}$ and

$$\boxed{f_Y(y) = f_X(s(y)) \left| \frac{ds(y)}{dy} \right|}$$

Summary

- A random variable is continuous if there exists a function f_X, called probability density function such that
  - $f_X(x) \geq 0$ for all x
  - $\int_{-\infty}^{+\infty} f_X(x)dx = 1$
  - For every $a \leq b$

$$\boxed{P(a < X \leq b) = \int_a^b f_X(x)dx}$$

- Relationship between the CDF $F_X(x)$ and PDF $f_X(x)$:

$$\boxed{F_X(x) = \int_{-\infty}^{x} f_X(t)dt} \qquad \boxed{f_X(x) = F'_X(x)}$$

- Important Examples: Uniform Distribution, Normal Distribution, Exponential Distribution, Gamma Distribution, Beta Distribution
- If $Y = r(X)$ and r is strictly monotonic, then

$$\boxed{f_Y(y) = f_X(s(y)) \left| \frac{ds(y)}{dy} \right|} \qquad s = r^{-1}$$