Math 408 - Mathematical Statistics

Lecture 23b. The Method of Moments

March 29, 2013

Method of Moments: Problem Formulation

Suppose that

$$X_1, \ldots, X_n \sim \pi(x|\theta)$$

where $\theta \in \Theta$, and we want to estimate θ based on the data $X_1, \ldots, X_n$.

The first method for constructing parametric estimators that we will study is called the method of moments.

- The estimators produced by this method are not optimal, but that are often easy to compute.
- They are also useful as starting values for other methods that require iterative numerical routines.

Method of Moments

Recall that the k^{th} moment of a probability distribution $\pi(x|\theta)$ is

$$\mu_k(\theta) = \mathbb{E}_\theta[X^k]$$

where $\mathbb{E}_\theta$ denotes expectation with respect to $\pi(x|\theta)$, i.e.

$$\mathbb{E}_\theta[f(X)] = \int f(x)\pi(x|\theta)dx$$

If $X_1, \ldots, X_n$ are i.i.d from $\pi(x|\theta)$, then the k^{th} sample moment if defined as

$$\hat{\mu}_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k$$

We can view $\hat{\mu}_k$ as an estimate of μ_k. Suppose that the parameter θ has k components:

$$\theta = (\theta_1, \ldots, \theta_k)$$

Method of Moments

Method of Moments

The **method of moments estimator** $\hat{\theta}$ *is defined to be the value of* θ *such that*

$$\begin{cases} \mu_1(\theta) = \hat{\mu}_1 \\ \mu_2(\theta) = \hat{\mu}_2 \\ \quad \ldots\ldots \\ \mu_k(\theta) = \hat{\mu}_k \end{cases} \tag{1}$$

- System (1) is a system of k equations with k unknowns: $\theta_1, \ldots, \theta_k$
- The solutions of this system $\hat{\theta}$ is the method of moments estimate of the parameter θ.

Example 1: Bernoulli

- Let $X_1, \ldots, X_n \sim \mathrm{Bernoulli}(p)$.
  Find the method of moments estimate of the parameter p.

Example 2: Normal

- Let $X_1, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$.
  Find the method of moments estimates of μ and σ^2.

Consistency of the MoM estimator

Question: How good is the estimator $\hat{\theta}$ obtained by the method of moments?

Definition

Let $\hat{\theta}_n$ be an estimate of a parameter θ based on a sample of size n. Then $\hat{\theta}_n$ is **consistent** if

$$\hat{\theta}_n \xrightarrow{\mathbb{P}} \theta$$

That is, for any $\varepsilon > 0$,

$$\mathbb{P}(|\hat{\theta}_n - \theta| \geq \varepsilon) \to 0 \text{ as } n \to \infty$$

Theorem

The method of moments estimate is consistent.

Summary

- If $X_1, \ldots, X_n \sim \pi(x|\theta)$, then the method of moments estimate $\hat{\theta}$ of $\theta = (\theta_1, \ldots, \theta_k)$ is the solution of

$$\begin{cases} \mu_1(\theta) = \hat{\mu}_1 \\ \mu_2(\theta) = \hat{\mu}_2 \\ \quad \ldots\ldots \\ \mu_k(\theta) = \hat{\mu}_k \end{cases}$$

  where

  - $\mu_k(\theta)$ is the k^{th} moment

$$\mu_k(\theta) = \mathbb{E}_\theta[X^k]$$

  - $\hat{\mu}_k$ is the k^{th} sample moment

$$\hat{\mu}_k = \frac{1}{n} \sum_{i=1}^{n} X_i^k$$

- The method of moments estimate $\hat{\theta}$ is a consistent estimate of θ.