Math 408 - Mathematical Statistics

Lecture 2. Conditional Probability

January 18, 2013

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- Properties of Conditional Probabilities
- Law of Total Probability
- Bayes' Theorem
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Motivation and Definition

Recall (see Lecture 1) that the sample space is the set of all possible outcomes of an experiment. Suppose we are interested only in part of the sample space, the part where we know some event – call it A – has happened, and we want to know how likely it is that various other events $(B, C, D \dots)$ have also happened.

What we want is the **conditional probability** of B given A.

Definition

If $\mathbb{P}(A) > 0$, then the conditional probability of B given A is

$$\boxed{\mathbb{P}(B|A) = \frac{\mathbb{P}(AB)}{\mathbb{P}(A)}}$$

Useful Interpretation:

Think of $\mathbb{P}(B|A)$ as the

fraction of times B occurs among those in which A occurs

Properties of Conditional Probabilities

Here are some facts about conditional probabilities:

1. For any fixed A such that $\mathbb{P}(A) > 0$, $\mathbb{P}(\cdot|A)$ is a probability, i.e. it satisfies the rules of probability:
   - $0 \leq \mathbb{P}(B|A) \leq 1$
   - $\mathbb{P}(\Omega|A) = 1$
   - $\mathbb{P}(\emptyset|A) = 0$
   - $\mathbb{P}(B|A) + \mathbb{P}(\bar{B}|A) = 1$
   - $\mathbb{P}(B + C|A) = \mathbb{P}(B|A) + \mathbb{P}(C|A) - \mathbb{P}(BC|A)$
2. Important: The rules of probability apply to events on the left of the bar.
3. In general

   $$\mathbb{P}(B|A) \neq \mathbb{P}(A|B)$$

   Example: the probability of spots given you have measles is 1 but the probability that you have measles given that you have spots is not 1.
4. What if A and B are independent? Then

   $$\mathbb{P}(B|A) = \frac{\mathbb{P}(AB)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(A)} = \mathbb{P}(B)$$

   Thus, another interpretation of independence is that knowing A does not change the probability of B.

Law of Total Probability

From the definition of conditional probability we can write

$$\mathbb{P}(AB) = \mathbb{P}(B|A)\mathbb{P}(A) \quad \text{and} \quad \mathbb{P}(AB) = \mathbb{P}(A|B)\mathbb{P}(B)$$

Often these formulae give us a convenient way to compute $\mathbb{P}(AB)$ when A and B are not independent.

A useful tool for computing probabilities is the following law.

Law of Total Probability

Let $A_1, \ldots, A_n$ be a partition of Ω, i.e.

- $\bigcup_{i=1}^{n} A_i = \Omega$ *($A_1, \ldots, A_k$ are jointly exhaustive events)*
- $A_i \bigcap A_j = \emptyset$ *for $i \neq j$ ($A_1, \ldots, A_k$ are mutually exclusive events)*
- $\mathbb{P}(A_i) > 0$

Then for any event B

$$\boxed{\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B|A_i)\mathbb{P}(A_i)}$$

Bayes' Theorem

Conditional probabilities can be inverted. That is,

$$\boxed{\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}}$$

This relationship is called **Bayes' Rule** after Thomas Bayes (1702-1761) who did not discover it (in this form, Bayes' Rule was proved by Laplace).

Example: False Positive Paradox

Problem

Suppose a rare disease infects one out of every 1000 people in a population. And suppose that there is a good, but not perfect, test for this disease: if a person has the disease, the test comes back positive 99% of the time. One the other hand, the test also produces some false positives. About 2% of uninfected patients also test positive. Suppose you just tested positive. What are your chances of having the disease?

Answer: the chances of having the disease is less than 5% !

Important Conclusion: When dealing with conditional probabilities:

don't trust your intuition, do computations!

Monty Hall problem

Problem

Suppose you are on a game show, and you are given the choice of three doors. A prize is placed at random between one of three doors. You pick a door, say door 1 (but the door is not opened), and the host, who knows what's behind the doors, opens another door which is empty. He then gives you the opportunity to keep your door 1 or switch to the other unopened door. Should you stay or switch?

Answer: You should switch!

Summary

- If $\mathbb{P}(A) > 0$, then

$$\boxed{\mathbb{P}(B|A) = \frac{\mathbb{P}(AB)}{\mathbb{P}(A)}}$$

- $\mathbb{P}(\cdot|A)$ satisfies the axioms of probability for fixed A. In general $\mathbb{P}(A|\cdot)$ does not satisfy the axioms of probability for fixed A.
- In general, $\mathbb{P}(B|A) \neq \mathbb{P}(A|B)$
- A and B are independent if and only if $\mathbb{P}(B|A) = \mathbb{P}(B)$
- Law of Total Probability: If $A_1, \ldots, A_n$ is a partition of Ω, then for any $B \subset \Omega$

$$\boxed{\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B|A_i)\mathbb{P}(A_i)}$$

- Bayes' Theorem

$$\boxed{\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}}$$