Math 408 - Mathematical Statistics

Lecture 1. ABC of Probability

January 16, 2013

Agenda

- Sample Spaces
- Realizations, Events
- Axioms of Probability
- Probability on Finite Sample Spaces
  - Example: B-day Problem
- Independent Events
- Summary

Sample Spaces, Realizations, Events

Probability Theory is the mathematical language for uncertainty quantification.

The starting point in developing the probability theory is to specify sample space = the set of possible outcomes.

Definition

- The **sample space** Ω is the set of possible outcomes of an "experiment"
- Points $\omega \in \Omega$ are called **realizations**
- **Events** are subsets of Ω

Next, to every event $A \subset \Omega$, we want to assign a real number $\mathbb{P}(A)$, called the probability of A. We call function $\mathbb{P} : \{\text{subsets of } \Omega\} \rightarrow \mathbb{R}$ a probability distribution.

We don't want $\mathbb{P}$ to be arbitrary, we want it to satisfy some natural properties (called axioms of probability):

1. $0 \leq \mathbb{P}(A) \leq 1$ (Events range from never happening to always happening)
2. $\mathbb{P}(\Omega) = 1$ (Something must happen)
3. $\mathbb{P}(\emptyset) = 0$ (Nothing never happens)
4. $\mathbb{P}(A) + \mathbb{P}(\bar{A}) = 1$ (A must either happen or not-happen)
5. $\mathbb{P}(A + B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(AB)$

Probability on Finite Sample Spaces

Suppose that the sample space is finite $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$.

Example:

If we toss a die twice, then Ω has $n = 36$ elements:

$$\Omega\{\ (i,j)\ :\ i,j = 1,2,3,4,5,6\}$$

If each outcome is equally likely, then $\mathbb{P}(A) = |A|/36$, where $|A|$ denotes the number of elements in A.
Test question: What is the probability that the sum of the dice is 11?
Answer: $2/36$, since the are two outcomes that correspond to this event: $(5,6)$ and $(6,5)$.

In general, if Ω is finite and if each outcome is equally likely, then

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|}$$

To compute the probability $\mathbb{P}(A)$, we need to count the number of points in an event A. Methods for counting points are called combinatorial methods.

Example: Birthday Problem

Suppose that a room of people contains n people.
What is the probability that at least two of them have a common birthday?

Assume that

- Every day of the year is equally likely to be a birthday
- There are 365 days in the year (disregard leap years)

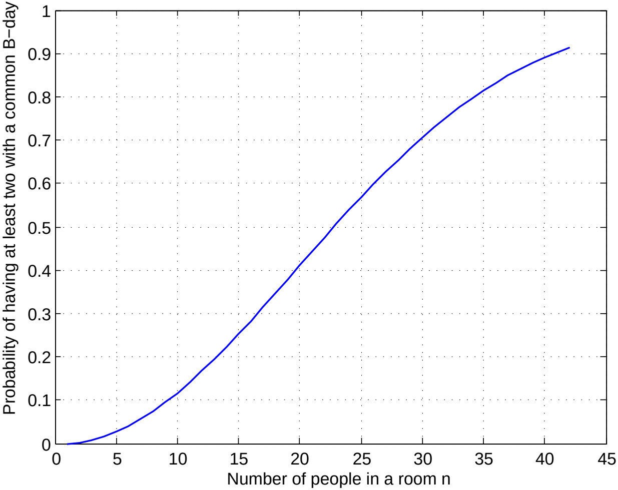
Then

- $\Omega = \{\omega = (x_1, \dots, x_n) \ : \ x_i = 1, 2, \dots, 365\}, \quad |\Omega| = 365^n$
- $A = \{\omega \in \Omega \ : \ x_i = x_j \text{ for some } i \neq j\}$
- $\bar{A} = \{\omega \in \Omega \ : \ x_i \neq x_j \text{ for all } i, j\}, \quad |\bar{A}| = 365 \times 364 \times \dots \times (365 - n + 1)$

$$\boxed{\mathbb{P}(A) = 1 - \frac{365 \times 364 \times \dots \times (365 - n + 1)}{365^n}}$$

n	$\mathbb{P}(A)$
4	0.016
23	0.507
32	0.753
42	0.91
56	0.988

Example: Birthday Problem



Independent Events

If we flip a fair coin twice, then the probability of two heads is $\frac{1}{2} \times \frac{1}{2}$. We multiply the probabilities because we regard the two tosses as independent. We can formalize this useful notion of independence as follows:

Definition

Two events A and B are **independent** if

$$\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B)$$

Independence can arise in two distinct ways:

1. We explicitly assume that two events are independent. For example, in tossing a coin twice, we usually assume that the tosses are independent which reflects the fact that the coin has no memory of the first toss.
2. We derive independence of A and B by verifying that $\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B)$. For example, in tossing a fair die, let $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 4\}$. Are A and B independent?
   Yes! Since $\mathbb{P}(A) = 1/2$, $\mathbb{P}(B) = 2/3$, $AB = \{2, 4\}$, $\mathbb{P}(AB) = 1/3 = (1/2) \times (2/3)$

Examples

- Suppose that A and B are disjoint events, each with positive probability. Can they be independent?
  Answer: No! $\mathbb{P}(AB) = \mathbb{P}(\emptyset) = 0$, but $\mathbb{P}(A)\mathbb{P}(B) > 0$
- Two people take turns trying to sink a basketball into a net.
  - Person 1 succeeds with probability $1/3$
  - Person 2 succeeds with probability $1/4$

  What is the probability that person 1 succeeds before person 2?
  Answer: $2/3$

Summary

- The sample space Ω is the set of possible outcomes of an "experiment"
- Points $\omega \in \Omega$ are called realizations
- Events are subsets of Ω
- Properties (axioms) of probability:
  - $0 \leq \mathbb{P}(A) \leq 1$ (Events range from never happening to always happening)
  - $\mathbb{P}(\Omega) = 1$ (Something must happen)
  - $\mathbb{P}(\emptyset) = 0$ (Nothing never happens)
  - $\mathbb{P}(A) + \mathbb{P}(\bar{A}) = 1$ (A must either happen or not-happen)
  - $\mathbb{P}(A + B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(AB)$
- A and B are independent if $\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B)$
- Independence is sometimes assumed and sometimes derived.
- Disjoint events with positive probability are not independent.