Homework 6

Chapter 9, #1

$X_1, \dots, X_n \sim \text{Bernoulli}(p)$, $n = 10$

$H_0 : p = \frac{1}{2}$ Test: Reject $H_0 \iff \sum_{i=1}^{n} X_i = 0$ or $\sum_{i=1}^{n} X_i = n$

$H_1 : p \neq \frac{1}{2}$

(a) What is the significance level of the test?

$$\alpha = \mathbb{P}(\text{type I error}) = \mathbb{P}(\text{Reject } H_0 \mid H_0) =$$

$$= \mathbb{P}\left(\sum_{i=1}^{n} X_i = 0 \,\middle|\, p = \tfrac{1}{2}\right) + \mathbb{P}\left(\sum_{i=1}^{n} X_i = n \,\middle|\, p = \tfrac{1}{2}\right)$$

$$= \binom{n}{0} p^0 (1-p)^{n-0} + \binom{n}{n} p^n (1-p)^{n-n} = \underline{0.02}$$

(b) If in fact the probability of heads is 0.1, what is the power of the test?

$$\text{power} = 1 - \beta$$

$$\beta = \mathbb{P}(\text{type II error}) = \mathbb{P}(\text{Accept } H_0 \mid H_1)$$

$$\text{power} = \mathbb{P}(\text{Reject } H_0 \mid H_1) = \mathbb{P}\left(\sum_{i=1}^{n} X_i = 0 \,\middle|\, p = 0.1\right) + \mathbb{P}\left(\sum_{i=1}^{n} X_i = n \,\middle|\, p = 0.1\right)$$

$$= (0.1)^0 \cdot (0.9)^{10} + (0.1)^{10} (0.9)^0 = \underline{0.349}$$

(#7) $X_1, \dots, X_n \sim \text{Poisson}(\lambda)$

$H_0: \lambda = \lambda_0$ v.s. $H_1: \lambda = \lambda_1 > \lambda_0$

Construct the likelihood ratio test at level α

<u>Solution</u> The LRT: Reject $H_0 \iff \dfrac{\mathcal{L}(\text{Data}\mid H_0)}{\mathcal{L}(\text{Data}\mid H_1)} < c$

1. The likelihood ratio:

$$\frac{\mathcal{L}(\text{Data}\mid H_0)}{\mathcal{L}(\text{Data}\mid H_1)} = \frac{\prod_{i=1}^{n} e^{-\lambda_0} \frac{\lambda_0^{x_i}}{x_i!}}{\prod_{i=1}^{n} e^{-\lambda_1} \frac{\lambda_1^{x_i}}{x_i!}} = e^{n(\lambda_1 - \lambda_0)} \left(\frac{\lambda_0}{\lambda_1}\right)^{\sum_{i=1}^{n} x_i}$$

2. The LRT

Reject $H_0 \iff$ $$e^{n(\lambda_1-\lambda_0)} \left(\frac{\lambda_0}{\lambda_1}\right)^{\sum_{i=1}^{n} x_i} < c$$

$$\left(\sum_{i=1}^{n} x_i\right) \cdot \underbrace{\ln \frac{\lambda_0}{\lambda_1}}_{<0} < \ln c - n(\lambda_1 - \lambda_0)$$

$$\sum_{i=1}^{n} x_i > \frac{\ln c - n(\lambda_1 - \lambda_0)}{\ln \lambda_0 - \ln \lambda_1} \equiv \xi \quad (\text{some constant})$$

Thus, $\boxed{\text{Reject } H_0 \iff \sum_{i=1}^{n} X_i > \xi}$

3. Significance level α

this is an equation on ξ.

$$\alpha = \mathbb{P}(\text{Reject } H_0 \mid H_0) = \mathbb{P}\left(\sum_{i=1}^{n} X_i > \xi \,\middle|\, \lambda = \lambda_0\right)$$

If $X_1 \sim \text{Poisson}(\lambda_1)$ and $X_2 \sim \text{Poisson}(\lambda_2)$, and X_1, X_2 are independent

$\Rightarrow (X_1 + X_2) \sim \text{Poisson}(\lambda_1 + \lambda_2)$. Thus $\underset{\text{under } H_0}{\sum X_i \sim} \text{Poisson}\Big(\underbrace{\sum_{i=1}^{n} \lambda_0}_{n \cdot \lambda_0}\Big)$

So, we have the following eq. on ξ

$$\alpha = \mathbb{P}(Y > \xi), \quad Y \sim \text{Poisson}(n\lambda_0)$$

$$\alpha = 1 - F_{n\lambda_0}(\xi) \quad \text{where } F_{n\lambda_0}(\cdot) \text{ is the CDF of } Y$$

$$\xi = F_{n\lambda_0}^{-1}(1-\alpha)$$

(#9) $X_1 \ldots X_n \sim N(\mu, \sigma^2)$, $n = 25$, $\sigma^2 = 100$

$H_0 : \underbrace{\mu = 0}_{\mu_0}$ v.s. $H_1 : \underbrace{\mu = 1.5}_{\mu_1}$

Construct a test at level $\alpha = 0.1$

What is the power of the test?

Repeat for $\alpha = 0.01$.

<u>Solution</u>

In Lecture 31-32, we constructed the LRT :

$$\text{Reject } H_0 \iff \overline{X}_n > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$

(1) If $\alpha = 0.1$, then

$$\text{Reject } H_0 \iff \overline{X}_n > z_{0.1} \frac{10}{5} = 2 z_{0.1} \approx \underline{\underline{2.56}}$$

The power :

$$1 - \beta = \mathbb{P}(\text{Reject } H_0 \mid H_1) = \mathbb{P}\left(\overline{X}_n > 2 z_{0.1} \mid \mu = \mu_1\right) \ominus$$

Under H_1 : $\overline{X}_n \sim N\left(\mu_1, \frac{\sigma^2}{n}\right) \implies \frac{\overline{X}_n - \mu_1}{\sigma}\sqrt{n} \sim N(0,1)$

$$\ominus\ \mathbb{P}\left(\frac{\overline{X}_n - \mu_1}{\sigma}\sqrt{n} > \frac{2 z_{0.1} - \mu_1}{\sigma}\sqrt{n}\right) = 1 - \Phi\left(\frac{2 z_{0.1} - \mu_1}{\sigma}\sqrt{n}\right) \approx \underline{\underline{0.3}}$$

(2) If $\alpha = 0.01$, then simularly

$$\text{Reject } H_0 \iff \overline{X}_n > 2 z_{0.01} \approx \underline{\underline{4.65}}$$

The power : $1 - \beta = 1 - \Phi\left(\frac{2 z_{0.01} - \mu_1}{\sigma}\sqrt{n}\right) \approx \underline{\underline{0.057}}$

#17 $X \sim N(0, \sigma^2)$

$H_0 : \sigma = \sigma_0$ v.s. $H_1 : \sigma = \sigma_1 > \sigma_0$

(a) LRT at level α

$$\frac{\mathcal{L}(\text{Data} \mid H_0)}{\mathcal{L}(\text{Data} \mid H_1)} = \frac{\frac{1}{\sqrt{2\pi}\,\sigma_0} e^{-\frac{x^2}{2\sigma_0^2}}}{\frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{x^2}{2\sigma_1^2}}} = \frac{\sigma_1}{\sigma_0} \exp\left(\frac{x^2}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\right)$$

Reject $H_0 \iff$ $$\frac{\sigma_1}{\sigma_0} \exp\left[\frac{x^2}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\right] < c$$

$$\frac{x^2}{2} \underbrace{\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)}_{<0} < \ln\left(c \frac{\sigma_0}{\sigma_1}\right)$$

$$x^2 > \frac{2}{\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}} \ln\left(c \frac{\sigma_0}{\sigma_1}\right) \equiv \xi$$

Thus the test rejects $H_0 \iff x^2 > \xi$

Level α: $\alpha = \mathbb{P}(\text{Reject } H_0 \mid H_0) = \mathbb{P}(x^2 > \xi \mid \sigma = \sigma_0) =$

$$= \mathbb{P}\Big(\underbrace{\left(\frac{x}{\sigma_0}\right)^2}_{\sim \chi_1^2} > \frac{\xi}{\sigma_0^2} \,\Big|\, \sigma = \sigma_0 \Big) \implies \frac{\xi}{\sigma_0^2} = \chi_1^2(\alpha)$$

Thus, Reject $H_0 \iff \boxed{x^2 > \sigma_0^2 \chi_1^2(\alpha)}$

(b) what if $X_1, \dots, X_n \sim N(0, \sigma^2)$?

$$\implies \frac{\mathcal{L}(\text{Data} \mid H_0)}{\mathcal{L}(\text{Data} \mid H_1)} = \left(\frac{\sigma_1}{\sigma_0}\right)^n \cdot \exp\left[\frac{\sum_{i=1}^n x_i^2}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\right]$$

$\implies$ Reject $H_0 \iff \sum_{i=1}^n x_i^2 > \xi$

$$\alpha = \mathbb{P}(\text{Reject } H_0 \mid H_0) = \mathbb{P}\left(\sum_{i=1}^n x_i^2 > \xi \,\Big|\, \sigma = \sigma_0\right) = \mathbb{P}\Big(\underbrace{\sum_{i=1}^n \left(\frac{x_i}{\sigma_0}\right)^2}_{\sim \chi_n^2} > \frac{\xi}{\sigma_0^2} \,\Big|\, \sigma = \sigma_0\Big)$$

$$\implies \frac{\xi}{\sigma_0^2} = \chi_n^2(\alpha)$$

Thus, Reject $H_0 \iff \boxed{\sum_{i=1}^n x_i^2 > \sigma_0^2 \chi_n^2(\alpha)}$