Chapter 8

⑤ X is a discrete random variable, $\begin{cases} \mathbb{P}(X=1) = \theta \\ \mathbb{P}(X=2) = 1-\theta \end{cases}$

Data: $x_1 = 1$, $x_2 = 2$, $x_3 = 2$

(a) Find the method of moments estimate of θ

$$\mu_1(\theta) = \hat{\mu}_1 \qquad \hat{\mu}_1 = \frac{1}{3}(x_1 + x_2 + x_3) = \frac{5}{3}$$

"theoretical" moment — sample moment

$$\mu_1(\theta) = \mathbb{E}_\theta[X] = 1 \cdot \theta + 2 \cdot (1-\theta) = 2 - \theta$$

Thus, we have: $2 - \theta = \frac{5}{3} \Rightarrow \hat{\theta}_{MoM} = 2 - \frac{5}{3} = \boxed{\frac{1}{3}}$

(b) What is the likelihood function?

$$\mathcal{L}(\theta) = \prod_{i=1}^{3} \pi(X = x_i \mid \theta) = \prod_{i=1}^{3} \mathbb{P}(X = x_i \mid \theta) = \theta \cdot (1-\theta) \cdot (1-\theta) = \boxed{\theta \cdot (1-\theta)^2}$$

(c) What is the MLE of θ?

$$\hat{\theta}_{MLE} = \arg\max_{\theta \in (0,1)} \mathcal{L}(\theta)$$

$$\mathcal{L}(\theta) = \theta^3 - 2\theta^2 + \theta \Rightarrow \mathcal{L}'(\theta) = 3\theta^2 - 4\theta + 1$$

$$\mathcal{L}'(\theta) = 0 \iff \theta = \frac{4 \pm \sqrt{16-12}}{6} = \frac{4 \pm 2}{6} \Rightarrow \begin{array}{l} \theta_1 = \frac{1}{3} \\ \theta_2 = 1 \end{array}$$

$$\boxed{\hat{\theta}_{MLE} = 1/3}$$

θ	0		1/3		1
sign $\mathcal{L}'(\theta)$		+		−	
$\mathcal{L}(\theta)$		$\nearrow$		$\searrow$	

Chapter 8

⑦ X follows a geometric distribution : $\mathbb{P}(X=k) = p(1-p)^{k-1}$, $k=1,2,\dots$

$X_1 \dots X_n$ is an i.i.d sample

(a) Find the MoM estimate of p.

$$\underbrace{\mu_1(p)}_{\frac{1}{p}} = \underbrace{\hat{\mu}_1}_{\overline{X}_n} \quad \Rightarrow \quad \boxed{\hat{p}_{MoM} = \frac{1}{\overline{X}_n}}$$

(b) Find the MLE of p.

The likelihood function is

$$\mathcal{L}(p) = \prod_{i=1}^{n} \pi(X_i|p) = \prod_{i=1}^{n} \mathbb{P}(X=X_i|p) = \prod_{i=1}^{n} p(1-p)^{X_i-1} = p^n (1-p)^{\sum_{i=1}^{n} X_i - n}$$

The log-likelihood :

$$\ell(p) = \log \mathcal{L}(p) = n \log p + \left(\sum_{i=1}^{n} X_i - n\right) \log(1-p)$$

$$\ell'(p) = \frac{n}{p} + \frac{\sum_{i=1}^{n} X_i - n}{p-1} \quad \Rightarrow \quad \ell'(p) = 0 \iff \frac{n}{p} = \frac{\sum_{i=1}^{n} X_i - n}{1-p}$$

$$\ell''(p) = -\frac{n}{p^2} - \frac{\sum_{1}^{n} X_i - n}{(p-1)^2} < 0$$

$$n - np = p \sum_{i=1}^{n} X_i - np$$

indeed maximum

$$p \cdot \sum_{i=1}^{n} X_i = n$$

$$\boxed{\hat{p}_{MLE} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} X_i} = \frac{1}{\overline{X}_n}}$$