

Chapter 8

- (5) X is a discrete random variable, $\begin{cases} \text{IP}(X=1) = \theta \\ \text{IP}(X=2) = 1-\theta \end{cases}$
Data: $x_1=1, x_2=2, x_3=2$

(a) Find the method of moments estimate of θ

$\mu_1(\theta) = \hat{\mu}_1$
"theoretical" moment sample moment

$$\hat{\mu}_1 = \frac{1}{3}(x_1 + x_2 + x_3) = \frac{5}{3}$$
$$\mu_1(\theta) = E_{\theta}[X] = 1 \cdot \theta + 2 \cdot (1-\theta) = 2-\theta$$

Thus, we have: $2-\theta = \frac{5}{3} \Rightarrow \hat{\theta}_{\text{MOM}} = 2 - \frac{5}{3} = \boxed{\frac{1}{3}}$

(b) What is the likelihood function?

$$\mathcal{L}(\theta) = \prod_{i=1}^3 \pi(X=x_i|\theta) = \prod_{i=1}^3 \text{IP}(X=x_i|\theta) = \theta \cdot (1-\theta) \cdot (1-\theta) = \boxed{\theta \cdot (1-\theta)^2}$$

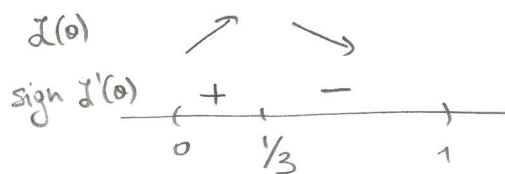
(c) What is the MLE of θ ?

$$\hat{\theta}_{\text{MLE}} = \arg \max_{\theta \in (0,1)} \mathcal{L}(\theta)$$

$$\mathcal{L}(\theta) = \theta^3 - 2\theta^2 + \theta \Rightarrow \mathcal{L}'(\theta) = 3\theta^2 - 4\theta + 1$$

$$\mathcal{L}'(\theta) = 0 \Leftrightarrow \theta = \frac{4 \pm \sqrt{16-12}}{6} = \frac{4 \pm 2}{6} \Rightarrow \begin{matrix} \theta_1 = \frac{1}{3} \\ \theta_2 = 1 \end{matrix}$$

$$\boxed{\hat{\theta}_{\text{MLE}} = 1/3}$$



Chapter 8

- ⑦ X follows a geometric distribution : $IP(X=k) = p(1-p)^{k-1}$, $k=1,2,\dots$
 $X_1, \dots, X_n$ is an i.i.d sample

(a) Find the MoM estimate of p .

$$\underbrace{\mu_1(p)}_{\frac{1}{p}} = \underbrace{\hat{\mu}_1}_{\bar{X}_n} \Rightarrow \boxed{\hat{p}_{\text{MoM}} = \frac{1}{\bar{X}_n}}$$

(b) Find the MLE of p .

The likelihood function is

$$\mathcal{L}(p) = \prod_{i=1}^n \pi(X_i|p) = \prod_{i=1}^n IP(X=X_i|p) = \prod_{i=1}^n p(1-p)^{X_i-1} = p^n (1-p)^{\sum_{i=1}^n X_i - n}$$

The log-likelihood :

$$\ell(p) = \log \mathcal{L}(p) = n \log p + \left(\sum_{i=1}^n X_i - n \right) \log(1-p)$$

$$\ell'(p) = \frac{n}{p} + \frac{\sum_{i=1}^n X_i - n}{p-1} \Rightarrow \ell'(p)=0 \Leftrightarrow \frac{n}{p} = \frac{\sum_{i=1}^n X_i - n}{1-p}$$

$$\ell''(p) = -\frac{n}{p^2} - \frac{\sum_{i=1}^n X_i - n}{(p-1)^2} < 0$$

indeed
maximum

$$\underbrace{n - np}_{=0} = p \sum_{i=1}^n X_i - \underbrace{np}_{=0}$$

$$p \cdot \sum_{i=1}^n X_i = n$$

$$\boxed{\hat{p}_{\text{MLE}} = \frac{1}{\frac{1}{n} \sum_{i=1}^n X_i} = \frac{1}{\bar{X}_n}}$$