Chapter 7

① Population : 1, 2, 2, 4, 8

- Population mean $\mu = \dfrac{1+2+2+4+8}{5} = \boxed{3.4}$
- Population variance $\sigma^2 = \frac{1}{5}\left((1-3.4)^2 + (2-3.4)^2 + (2-3.4)^2 + (4-3.4)^2 + (8-3.4)^2\right)$

$$\sigma^2 = \underline{6.24}$$

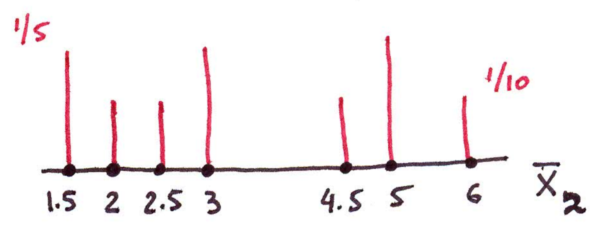
$\{X_1, X_2\}$ is a sample of size $n=2$ from the population

all possible samples of size $n=2$

$\{X_1, X_2\}$	$\bar{X}_2 = \frac{X_1+X_2}{2}$
$\{1,2\}$	1.5
$\{1,2\}$	1.5
$\{1,4\}$	2.5
$\{1,8\}$	4.5
$\{2,2\}$	2
$\{2,4\}$	3
$\{2,8\}$	5
$\{2,4\}$	3
$\{2,8\}$	5
$\{4,8\}$	6

$\Longrightarrow$ sampling distribution of $\bar{X}_2$ is

$$\mathbb{P}(\bar{X}_2 = 1.5) = \frac{2}{10} = \frac{1}{5}$$
$$\mathbb{P}(\bar{X}_2 = 2) = \frac{1}{10}$$
$$\mathbb{P}(\bar{X}_2 = 2.5) = \frac{1}{10}$$
$$\mathbb{P}(\bar{X}_2 = 3) = \frac{2}{10} = \frac{1}{5}$$
$$\mathbb{P}(\bar{X}_2 = 4.5) = \frac{1}{10}$$
$$\mathbb{P}(\bar{X}_2 = 5) = \frac{2}{10} = \frac{1}{5}$$
$$\mathbb{P}(\bar{X}_2 = 6) = \frac{1}{10}$$



Comparison with theoretical results

We know that $\mathbb{E}[\bar{X}_n] = \mu$

In this example $\mu = \mathbb{E}[\bar{X}_2] = 3.4$

$\mathbb{V}[\bar{X}_n] = \frac{\sigma^2}{n}\left(1 - \frac{n-1}{N-1}\right)$, in our case

$\mathbb{V}[\bar{X}_2] = \frac{(6.24)^2}{2}\left(1 - \frac{2-1}{5-1}\right) = \underline{2.34}$

$\Rightarrow \mathbb{E}[\bar{X}_2] = 1.5 \cdot \frac{1}{5} + 2 \cdot \frac{1}{10} + 2.5 \cdot \frac{1}{10} + 3 \cdot \frac{1}{5} + 4.5 \cdot \frac{1}{10} + 5 \cdot \frac{1}{5} + 6 \cdot \frac{1}{10} = \boxed{3.4}$

$\Rightarrow \mathbb{V}[\bar{X}_2] = (1.5)^2 \frac{1}{10} + 2^2 \cdot \frac{1}{10} + (2.5)^2 \frac{1}{10} + 3^2 \frac{1}{5} + (4.5)^2 \frac{1}{10} + 5^2 \frac{1}{5} + 6^2 \frac{1}{10} - (3.4)^2 = \underline{2.34}$

Chapter 7

(21) In order to halve the width of a 95% confidence interval for a mean, by what factor should the sample size be increased? Ignore the finite population correction.

Solution: 95% confidence interval for μ is

$$\left(\bar{X}_n - z_{\frac{\alpha}{2}} \sigma_{\bar{X}_n},\ \bar{X}_n + z_{\frac{\alpha}{2}} \sigma_{\bar{X}_n}\right), \quad \alpha = 0.05$$

Its width is then

$$\text{width}(n) = 2 z_{\frac{\alpha}{2}} \sigma_{\bar{X}_n} = 2 z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \sqrt{1 - \frac{n-1}{N-1}}$$

Ignoring the finite population correction,

$$\text{width}(n) = 2 z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

Thus, to halve the width: $2 z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \to z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$,

we need to quadruple the sample size: $\boxed{n \to 4n}$

(18) Population 1 has mean μ_1 $\xrightarrow{\text{survey 1}}$ 90% confidence interval I_1 (for μ_1)

Population 2 has mean μ_2 $\xrightarrow{\text{survey 2}}$ 90% confidence interval I_2 (for μ_2)

- Probability that neither interval contains the respective means

surveys are independent

$$\mathbb{P}(\mu_1 \notin I_1, \mu_2 \notin I_2) = \mathbb{P}(\mu_1 \notin I_1)\,\mathbb{P}(\mu_2 \notin I_2) =$$

$$= (1 - \mathbb{P}(\mu_1 \in I_1))\,\mathbb{P}(1 - \mathbb{P}(\mu_2 \in I_2))$$

$$= 0.1 \cdot 0.1 = \underline{0.01}$$

- Probability that both do

$$\mathbb{P}(\mu_1 \in I_1, \mu_2 \in I_2) = \mathbb{P}(\mu_1 \in I_1)\,\mathbb{P}(\mu_2 \in I_2) = 0.9 \cdot 0.9 = \underline{0.81}$$