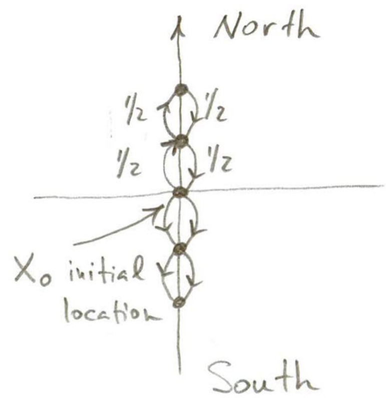
Homework 3

Chapter 5, # 13

A drunkard executes a "random walk" in the following way: each minute he takes a step north or south, with prob. ½ each, and his successive step directions are independent. His step length is 50 cm. Use the CLT to approximate the probability distribution of his location after 1 hour. Where is he most likely to be?



Solution Let $X_i = \begin{cases} +50 & \text{with prob. } 1/2 \\ -50 & \text{with prob } 1/2 \end{cases}$

Then the location of the drunkard after 1h is

$$Y = \sum_{i=1}^{n} X_i \quad , \text{ where } n = \underline{60} \left(\begin{matrix}\text{number} \\ \text{of steps}\end{matrix}\right)$$

According to the CLT, $nY = \overline{X}_n \mathrel{\dot\sim} N\left(\mu, \frac{\sigma^2}{n}\right)$,

where $\mu = \mathbb{E}[X_i] = 0$

$$\sigma^2 = \mathbb{V}[X_i] = \mathbb{E}[X_i^2] = 2500\tfrac{1}{2} + 2500\tfrac{1}{2} = 2500$$

Thus, $nY \mathrel{\dot\sim} N\left(0, \frac{2500}{n}\right) \Rightarrow Y \mathrel{\dot\sim} N(0, n \cdot 2500) = \boxed{N(0, 15 \cdot 10^4)}$

He is most likely to be at $\underset{x}{\arg\max}\, N(x \mid 0, 15 \cdot 10^4) = 0$

Thus, his is most likely to be where he started his walk.

Chapter 5, # 15

Suppose that you bet \$5 on each of a sequence of $n = 50$ independent fair games. Use the CLT to approximate the probability that you will lose more than \$75.

<u>Solution</u> Let $X_i = \begin{cases} 5 & \text{w.p. } 1/2 \\ -5 & \text{w.p } 1/2 \end{cases}$ $\quad i = 1, \dots, n$

The the total payoff is $T = \sum_{i=1}^{n} X_i$

$$\mathbb{P}(T < -75) = \mathbb{P}(n\bar{X}_n < -75) = \mathbb{P}\left(\bar{X}_n < -\frac{75}{n}\right) \circledcirc$$

$$\bar{X}_n \mathrel{\dot\sim} N\left(\mu, \frac{\sigma^2}{n}\right), \quad \mu = \mathbb{E}[X_i] = 0$$

$$\sigma^2 = \mathbb{V}[X_i] = 25 \cdot \tfrac{1}{2} + 25 \tfrac{1}{2} = 25$$

$$\circledcirc\ \mathbb{P}\left(\frac{(\bar{X}_n - \mu)\sqrt{n}}{\sigma} < \frac{\left(-\frac{75}{n} - \mu\right)\sqrt{n}}{\sigma}\right) \approx \Phi\left(-\frac{75}{\sigma\sqrt{n}}\right) = \Phi(-2.121) \approx \underline{\underline{0.017}}$$

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% Matlab source code.
% Problem #19, Chapter 5.
% Part (a): Use the Monte Carlo method with n=100 and n=1000 to
estimate
% int_0^1 cos(2pi*x)dx. Compare the estimates with the exact answer.
clear;
n1=100;          % number of Monte Carlo samples for the 1st estimate
n2=1000;         % number of Monte Carlo samples for the 2nd estimate
x1=rand(1,n1); % 1xn1 vector with components ~U[0,1]
x2=rand(1,n2); % 1xn2 vector with components ~U[0,1]
I_true=0;        % true value of the integral
I1=mean(cos(2*pi.*x1));   % 1st Monte Carlo estimate
I2=mean(cos(2*pi.*x2));   % 2nd Monte Carlo estimate

% Output: I1=-0.0255547,  I2=-0.007258.
% The 2nd estimate is more accurate

% Part (b): Use Monte Carlo to evaluate int_0^1 cos(2pi*x^2)dx
n=10^4;          % number of Monte Carlo samples
x=rand(1,n);   % 1xn vector with components ~U[0,1]
for i=1:n
    I(i)=mean(cos(2*pi.*(x(1:i).^2)));
    % I(i) is the Monte Carlo estimate that is based on i samples
end

% Output: the integral is approximately 0.24
% See also the Figure that show convergence of I(i) to 0.24 as i
increases
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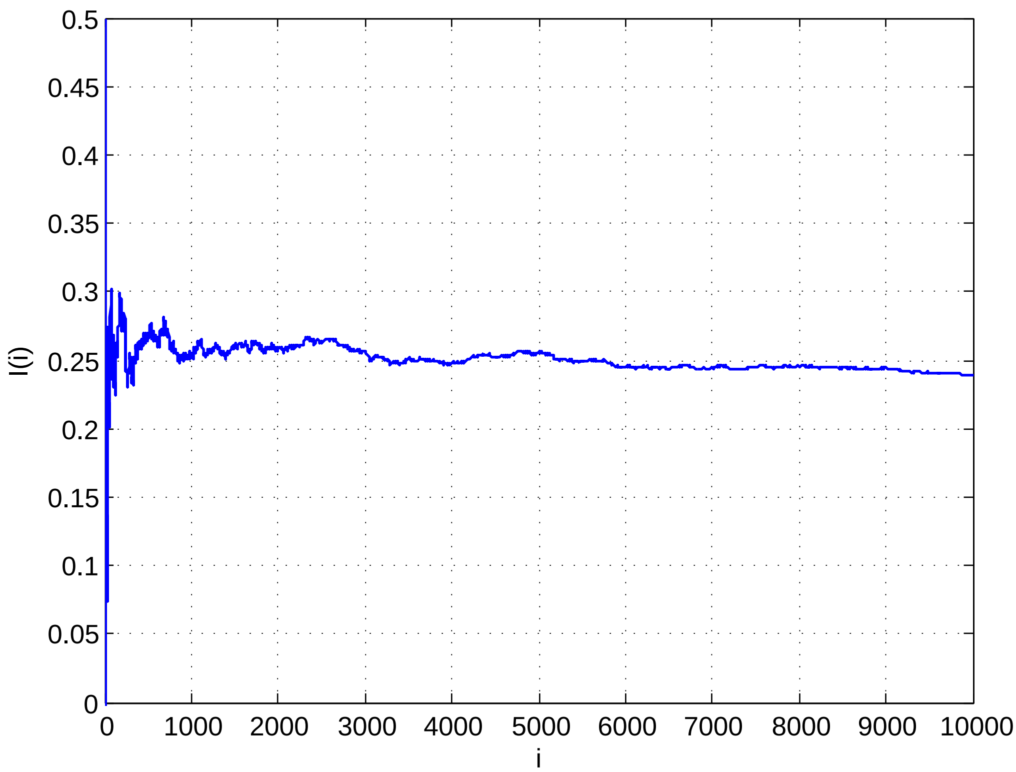

0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
I(i)
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000
i

- Chapter 6

③ $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $X_i \overset{iid}{\sim} N(0,1)$, $n=16$

Find c such that $\mathbb{P}(|\overline{X}_n| < c) = \frac{1}{2}$

Solution : Th.1: $\overline{X}_n \sim N(0, \frac{1}{n}) \Rightarrow Z = \sqrt{n}\,\overline{X}_n \sim N(0,1)$

$$\mathbb{P}(|\overline{X}_n| < c) = \mathbb{P}(|\sqrt{n}\,\overline{X}_n| < \sqrt{n}\cdot c) = \mathbb{P}(|Z| < \sqrt{n}\cdot c)$$

$$= \Phi(\sqrt{n}c) - \Phi(-\sqrt{n}c) = \Phi(\sqrt{n}c) - (1 - \Phi(\sqrt{n}c)) =$$

$$= 2\Phi(\sqrt{n}c) - 1$$

Therefore : $\mathbb{P}(|\overline{X}_n| < c) = \frac{1}{2} \Leftrightarrow 2\Phi(\sqrt{n}c) - 1 = \frac{1}{2}$

$$\sqrt{n}c = \Phi^{-1}\left(\frac{3}{4}\right)$$

$$\boxed{c = \frac{1}{4}\Phi^{-1}\left(\frac{3}{4}\right) \approx 0.17}$$