Homework # 2

Chapter 2. # 31

Phone calls are received at a certain residence as a Poisson process with parater $\lambda = 2$ per hour.

Let $X = \#$ calls per hour $\Rightarrow$ $X \sim \text{Poisson}(\lambda)$, $\lambda = 2 \frac{\text{calls}}{\text{hour}}$

(a) What is the probability that the phone rings during 10 min?

Solution $t = 10 \text{ min} = \frac{1}{6}$ hour. Let $\tilde{X} = \#$ calls per 10 min.

Then $\tilde{X} \sim \text{Poisson}(\tilde{\lambda})$, where $\tilde{\lambda} = 2 \frac{\text{calls}}{\text{hour}} = 2 \frac{\text{calls}}{6 \cdot 10\text{min}} = \frac{1}{3} \frac{\text{calls}}{10\text{min}}$

Thus, $\tilde{X} \sim \text{Poisson}(1/3)$.

$$\mathbb{P}(\text{Phone rings during } 10 \text{ min}) = \mathbb{P}(\tilde{X} > 0) = 1 - \mathbb{P}(\tilde{X} = 0)$$

$$= 1 - e^{-\tilde{\lambda}} = 1 - e^{-1/3} \approx \underline{\underline{0.28}}$$

(b) What is the time period during which the probability of receiving no phone calls is 1/2 ?

Solution We want to find t such that

$$\mathbb{P}(\text{No calls during } t \text{ hours}) = 1/2$$

Let $\hat{X} = \#$ calls per t hours $\Rightarrow$ $\hat{X} \sim \text{Poisson}(\hat{\lambda})$, where

$$\hat{\lambda} = 2 \frac{\text{calls}}{\text{hour}} = \underline{\underline{2t}} \frac{\text{calls}}{t \text{ hours}}$$

Thus, we obtain the following equation on t

$$\mathbb{P}(\hat{X} = 0) = 1/2$$

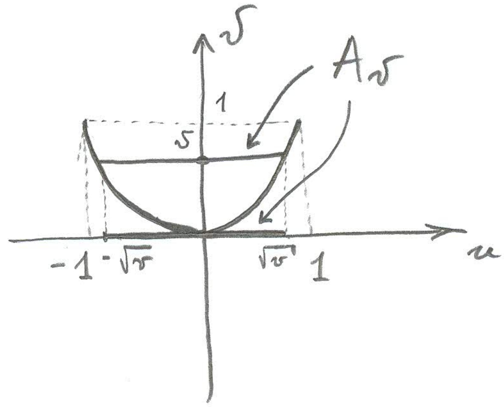
$$e^{-2t} = 1/2 \quad \Rightarrow \quad 2t = \log 2 \quad \Rightarrow \quad t = \frac{\log 2}{2} \approx 0.3466 \text{ hours}$$

or $t \approx \underline{\underline{20.79 \text{ min}}}$

Chapter 2. # 59

If $U \sim U[-1,1]$, find the PDF of $V = U^2$

Solution: Since $U \in [-1,1]$, $V \in [0,1]$



1. For each $v \in [0,1]$, let us find the set

$$A_v = \{u : r(u) \leq v\} = \{u : u^2 \leq v\}$$

$$A_v = \{u : -\sqrt{v} \leq u \leq \sqrt{v}\}$$

2. The CDF of V is then

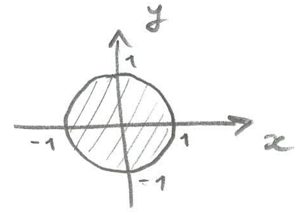
$$F_V(v) = \int_{A_v} f_U(u)\,du = \int_{-\sqrt{v}}^{\sqrt{v}} \frac{1}{2}\,du = \sqrt{v}, \quad v \in [0,1]$$

3. The PDF of V is then $f_V(v) = F_V'(v) = \dfrac{1}{2\sqrt{v}}$

Chapter 3. # 15

Suppose that X and Y have the joint PDF

$$f(x,y) = c \cdot \sqrt{1-x^2-y^2}, \quad x^2+y^2 \leq 1$$

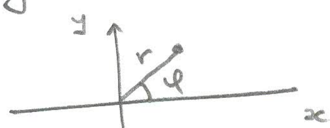


(a) Find c.

The constant c must be such that $\iint_{x^2+y^2\leq 1} f(x,y)\,dx\,dy = 1$

$\iint_{x^2+y^2\leq 1} \sqrt{1-x^2-y^2}\,dx\,dy$ is easier to compute using polar coordinates

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$



$$\iint_{x^2+y^2\leq 1} \sqrt{1-x^2-y^2}\,dx\,dy = \int_0^1 \int_0^{2\pi} \sqrt{1-r^2}\cdot \underset{\overset{\|}{|J|}}{r}\,d\varphi\,dr =$$

$$= 2\pi \int_0^1 \sqrt{1-r^2}\,r\,dr = \pi \int_0^1 \sqrt{1-r^2}\,dr^2 =$$

$$= \pi\, \frac{2}{3}\left(\sqrt{1-r^2}\right)^3\Big|_1^0 = \frac{2\pi}{3}$$

Thus, $\boxed{c = \dfrac{3}{2\pi}}$

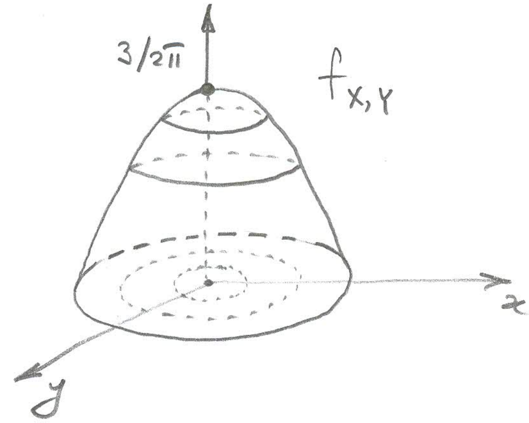
The Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{vmatrix}$$

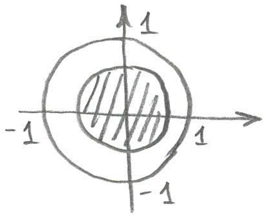
$$= r$$

(b) Sketch the joint density

$$f(x,y) = \frac{3}{2\pi}\sqrt{1-x^2-y^2} = \frac{3}{2\pi}\sqrt{1-r^2}$$



(c) Find $\mathbb{P}(X^2+Y^2 \le 1/2)$



$$\mathbb{P}(X^2+Y^2 \le 1/2) = \iint_{x^2+y^2\le 1/2} f_{X,Y}(x,y)\,dx\,dy \;\circeq$$

again it is more convenient to work in polar coordinates

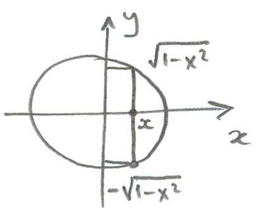
$$\circeq \int_0^{\sqrt{1/2}}\int_0^{2\pi} \frac{3}{2\pi}\sqrt{1-r^2}\cdot r\,d\varphi\,dr$$

$$= 3\int_0^{\sqrt{1/2}} \sqrt{1-r^2}\,r\,dr = \frac{3}{2}\int_0^{\sqrt{1/2}} \sqrt{1-r^2}\,dr^2 = \frac{3}{2}\cdot\frac{2}{3}(1-r^2)^{\frac{3}{2}}\Big|_{\sqrt{1/2}}^{0} = \underline{\underline{1-\left(\frac{1}{2}\right)^{3/2}}}$$

(d) Find the marginal densities of X and Y. Are X and Y independent?

By definition, $f_X(x) = \int f_{X,Y}(x,y)\,dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{3}{2\pi}\sqrt{1-x^2-y^2}\,dy \;\circeq$

$\uparrow\; a \equiv \sqrt{1-x^2}$



$$\circeq \int_{-a}^{a} \frac{3}{2\pi}\sqrt{a^2-y^2}\,dy = \frac{3}{2\pi}\underbrace{\int_{-a}^{a}\sqrt{a^2-y^2}\,dy}_{\text{this is a table integral}} =$$

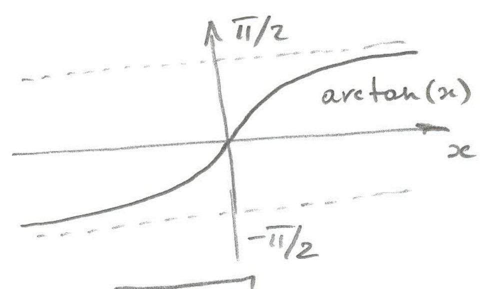
$$= \frac{3}{2\pi}\left[\frac{1}{2}y\sqrt{a^2-y^2} + \frac{1}{2}a^2\arctan\Big(\underbrace{\frac{y}{\sqrt{a^2-y^2}}}_{s}\Big)\right]\Big|_{-a}^{a} =$$

$$= \frac{3}{2\pi}\left[\frac{1}{2}a^2\lim_{s\to+\infty}\arctan(s) - \frac{1}{2}a^2\lim_{s\to-\infty}\arctan(s)\right] = \frac{3}{2\pi}\left[\frac{1}{2}a^2\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\right]$$

$$= \boxed{\frac{3}{4}(1-x^2), \quad x\in[-1,1]}$$

$f_X\cdot f_Y \neq f_{X,Y}$ thus are not independent

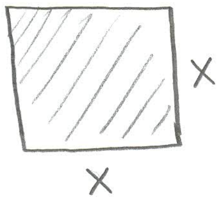
By symmetry, $\boxed{f_Y(y) = \frac{3}{4}(1-y^2), \quad y\in[-1,1]}$



(e) Find the conditional densities

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2\sqrt{1-x^2-y^2}}{\pi(1-y^2)}\;;\quad f_{Y|X}(y|x) = \frac{2\sqrt{1-x^2-y^2}}{\pi(1-x^2)}$$

Chapter 4. # 21



$X \sim U[0,1]$ Find the expected area of the square

Solution $\quad \text{Area} = X^2$

$$\mathbb{E}[\text{Area}] = \mathbb{E}[X^2] = \int_0^1 x^2\,dx = 1/3$$

Chapter 4. # 49

X and Y are independent measurements of a quantity μ.

$\mathbb{E}[X] = \mathbb{E}[Y] = \mu, \quad \sigma_X \neq \sigma_Y, \quad Z = \alpha X + (1-\alpha) Y, \quad \alpha \in [0,1]$

(a) Show that $\mathbb{E}[Z] = \mu$

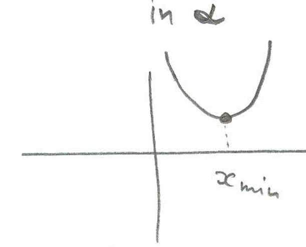
Easy : $\mathbb{E}[Z] = \mathbb{E}[\alpha X + (1-\alpha) Y] = \alpha \overbrace{\mathbb{E}[X]}^{\mu} + (1-\alpha) \overbrace{\mathbb{E}[Y]}^{\mu} = \mu$

(b) Find α in terms of σ_X and σ_Y to minimize $\mathbb{V}[Z]$.

$$\mathbb{V}[Z] = \mathbb{V}[\alpha X + (1-\alpha) Y] = \alpha^2 \sigma_X^2 + (1-\alpha)^2 \sigma_Y^2 = \underbrace{(\sigma_X^2 + \sigma_Y^2)\alpha^2 - 2\sigma_Y^2 \alpha + \sigma_Y^2}_{\text{quadratic in } \alpha}$$

since X and Y are independent

$$\Rightarrow \quad \alpha^* = -\frac{-2\sigma_Y^2}{2(\sigma_X^2 + \sigma_Y^2)} = \boxed{\frac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}}$$



$ax^2 + bx + c$

$x_{min} = -\frac{b}{2a}$

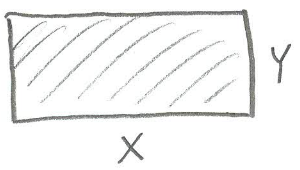
(c) Under what circumstances is it better to use the average $\frac{X+Y}{2}$ than either X or Y alone?

Let $Z = \frac{X+Y}{2} \Rightarrow \mathbb{V}[Z] = \frac{\sigma_X^2 + \sigma_Y^2}{4}$

It is better to use Z if

$$\begin{cases} \frac{\sigma_X^2 + \sigma_Y^2}{4} < \sigma_X^2 \\ \frac{\sigma_X^2 + \sigma_Y^2}{4} < \sigma_Y^2 \end{cases} \Leftrightarrow \begin{cases} \sigma_Y^2 < 3\sigma_X^2 \\ \sigma_X^2 < 3\sigma_Y^2 \end{cases} \Leftrightarrow \boxed{\frac{1}{3} < \frac{\sigma_X^2}{\sigma_Y^2} < 3}$$

Chapter 4. # 67



$X \sim U[0,1]$

$Y/X=x \sim U[0,x]$

Find the expected value of the perimeter and area of the rectangle.

Solution

(a) Perimeter $= 2X+2Y$

$\mathbb{E}[\text{Perimeter}] = 2\mathbb{E}[X] + 2\mathbb{E}[Y] \;=$

- $\mathbb{E}[X] = \frac{1}{2}$
- $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y/X]] = \mathbb{E}\left[\frac{X}{2}\right] = \frac{1}{4}$

$= \; 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = \frac{3}{2}$

(b) Area $= X \cdot Y$

$\mathbb{E}[\text{Area}] = \mathbb{E}[X \cdot Y] \neq \mathbb{E}[X] \cdot \mathbb{E}[Y]$ since X and Y are not independ.

$\mathbb{E}[XY] = \iint xy \, f_{X,Y}(x,y) \, dx \, dy$

$f_{X,Y}(x,y) = f_X(x) \cdot f_{Y/X}(y/x)$, $\quad f_X(x) = \begin{cases} 1 & , x \in [0,1] \\ 0 & , x \notin [0,1] \end{cases}$

$f_{Y/X}(y/x) = \begin{cases} \frac{1}{x} & \text{if } y \in (0,x] \\ 0 & \text{otherwise} \end{cases} \quad \Rightarrow \quad f_{X,Y}(x,y) = \begin{cases} \frac{1}{x} & \text{if } \begin{matrix} x \in [0,1] \\ y \in (0,x) \end{matrix} \\ 0 \end{cases}$

$\mathbb{E}[XY] = \int_0^1 \int_0^x \frac{xy}{x} \, dy \, dx = \int_0^1 1 \cdot \frac{y^2}{2}\Big|_0^x dx = \int_0^1 \frac{x^2}{2} \, dx = \frac{x^3}{6}\Big|_0^1 = \frac{1}{6}$