Homework 1

#7 Prove Bonferroni's inequality:

$$\mathbb{P}(AB) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1 \qquad (1)$$

Proof: We know that

$$\mathbb{P}(A+B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(AB)$$

Therefore, $\mathbb{P}(AB) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A+B)$.

Since $\mathbb{P}(A+B) \leq 1$, $\mathbb{P}(AB) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$ $\square$

P.S. In fact, the following is true:

$$\mathbb{P}(A_1 A_2 \dots A_n) \geq \mathbb{P}(A_1) + \mathbb{P}(A_2) + \dots + \mathbb{P}(A_n) - n + 1 \qquad (2)$$

(1) is a special case of (2), (2) can be proved by induction.

#45 Show that $\mathbb{P}(A_1 \dots A_n) = \mathbb{P}(A_1)\,\mathbb{P}(A_2|A_1)\,\mathbb{P}(A_3|A_1A_2) \dots \mathbb{P}(A_n|A_1 \dots A_{n-1})$

Proof Let us prove this by induction.

follows from the def. of the cond. prob

$n=1$: $\mathbb{P}(A_1) = \mathbb{P}(A_1)$, $n=2$: $\mathbb{P}(A_1A_2) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1)$

Assume that the statement is correct for $k = n-1$, i.e

$$\mathbb{P}(A_1 \dots A_{n-1}) = \mathbb{P}(A_1)\,\mathbb{P}(A_2|A_1) \dots \mathbb{P}(A_{n-1}|A_1 \dots A_{n-2}) \qquad (3)$$

Let us show that it is also correct for $k=n$.

$\mathbb{P}(AB) = \mathbb{P}(A)\,\mathbb{P}(B|A)$, where $A = A_1 \dots A_{n-1}$ and $B = A_n$

$$\mathbb{P}(A_1 \dots A_n) = \mathbb{P}(A_1 \dots A_{n-1}) \cdot \mathbb{P}(A_n|A_1 \dots A_{n-1})$$

$$= \mathbb{P}(A_1)\,\mathbb{P}(A_2|A_1) \dots \mathbb{P}(A_{n-1}|A_1 \dots A_{n-2})\,\mathbb{P}(A_n|A_1 \dots A_{n-1})$$

using (3)

$\square$

#65 Show that if A and B are independent, then

- A and $\overline{B}$ are independent
- $\overline{A}$ and $\overline{B}$ are independent

Proof: Events C and D are independent $\iff$ $\mathbb{P}(CD) = \mathbb{P}(C)\,\mathbb{P}(D)$

1. $$\mathbb{P}(A\overline{B}) = \mathbb{P}(A)\cdot\mathbb{P}(\overline{B}|A) = \mathbb{P}(A)\cdot\Big(1-\underbrace{\mathbb{P}(B|A)}_{=\mathbb{P}(B)}\Big) = \mathbb{P}(A)\big(1-\mathbb{P}(B)\big) = \mathbb{P}(A)\,\mathbb{P}(\overline{B})$$

from the def. of the conditional prob $\mathbb{P}(C|D) = \dfrac{\mathbb{P}(CD)}{\mathbb{P}(D)}$

$\mathbb{P}(B)$ since A and B are independent

2. $$\mathbb{P}(\overline{A}\,\overline{B}) = \mathbb{P}(\overline{A})\,\mathbb{P}(\overline{B}|\overline{A}) = \mathbb{P}(\overline{A})\cdot\Big(1-\underbrace{\mathbb{P}(B|\overline{A})}_{=\mathbb{P}(B)}\Big)$$

$$= \mathbb{P}(\overline{A})\big(1-\mathbb{P}(B)\big) = \mathbb{P}(\overline{A})\,\mathbb{P}(\overline{B}).$$

$\mathbb{P}(B)$ since B and $\overline{A}$ are independent (see 1.)

$\square$

#75 Let X_i be the number of population members at time $t=i$. Then

At time $t=1$: $X_1 = 1$ with probability 1

At time $t=2$: $X_2 = \begin{cases} 0 & \text{with prob } 1-p \\ 2 & \text{with prob. } p \end{cases}$

At time $t=3$: $X_3 \in \{0, 2, 4\}$

$$\underbrace{\mathbb{P}(X_3=0)}_{\text{probability that there are no members in the 3rd generation}} = \mathbb{P}(X_3=0|X_2=0)\cdot\mathbb{P}(X_2=0) + \mathbb{P}(X_3=0|X_2=2)\,\mathbb{P}(X_2=2)$$

law of total prob

$\mathbb{P}(X_3=0|X_2=0) = 1$, $\mathbb{P}(X_2=0) = 1-p$

$\mathbb{P}(X_3=0|X_2=2) = (1-p)^2$, $\mathbb{P}(X_2=2) = p$

Thus

$$\mathbb{P}(X_3=0) = 1-p+(1-p)^2\cdot p = p^3-2p^2+p-p+1 = \boxed{p^3-2p+1}$$

#77 A_k = "player hits the bull's-eye on trial k"

$A_1, A_2, \dots$ are independent

$\mathbb{P}(A_k) = p = 0.05$

How many times should he throw so that his probability of hitting the bull's-eye at least once is $1/2$?

Solution Let B_n = "after n trials, the player hit the bull's-eye at least once".

$$\mathbb{P}(B_n) = 1 - \mathbb{P}(\bar{B}_n) = 1 - \prod_{i=1}^{n} \mathbb{P}(\bar{A}_i) = 1 - (1-p)^n$$

$\bar{B}_n$: did not hit the target even once

Thus, we have an equation: $1 - (1-p)^n = 1/2$

$$(1-p)^n = 1/2$$

$$n = -\frac{1}{\log_2(1-p)} \approx 13.5$$

Since n is an integer, the answer is

$$\boxed{n = 14 \text{ times}}$$