Problem 1. Determine the longest interval in which the given initial value problem is certain to have a unique solution:

$$(\cos^2 x - 1)y'' + y' \ln|x-4| + \sin x = 1, \qquad y(5) = 0, \quad y'(5) = 1.$$

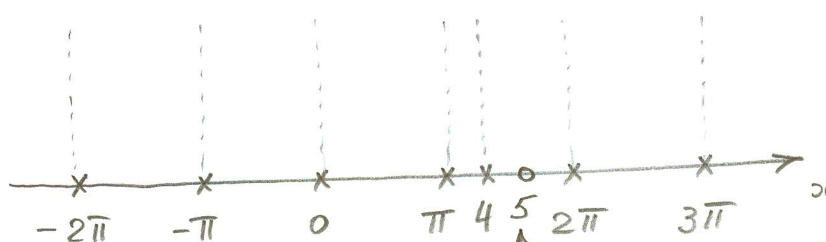
First, let us write the equation in the standard form:

$$y'' + y' \cdot \frac{\ln|x-4|}{\cos^2 x - 1} + \frac{\sin x}{\cos^2 x - 1} = \frac{1}{\cos^2 x - 1}$$

Points of discontinuity of the coefficients

$$\frac{\ln|x-4|}{\cos^2 x - 1}, \quad \frac{\sin x}{\cos^2 x - 1}, \quad \text{and} \quad \frac{1}{\cos^2 x - 1}$$

are: $\quad x \neq 4 \quad$ and $\quad \cos x \neq \pm 1 \iff x \neq 0, \pm\pi, \pm 2\pi, \ldots$



initial conditions $\begin{cases} y(5) = 0 \\ y'(5) = 1 \end{cases}$

$\Rightarrow$ the solution is certain to exist on $(4, 2\pi)$

$$\boxed{(4, 2\pi)}$$

Problem 2. Solve the given initial value problem

$$y'' - 2y' + 10y = 0, \quad y(0) = 1, \quad y'(0) = -2.$$

The characteristic equation is $\lambda^2 - 2\lambda + 10 = 0$

$$\Rightarrow \quad \lambda_{1,2} = \frac{2 \pm \sqrt{4-40}}{2} = 1 \pm 3i$$

The general solution is then

$$y(t) = C_1 e^t \cos 3t + C_2 e^t \sin 3t$$

Let us find C_1 and C_2 from the initial conditions

$$y(0) = 1 \quad \Rightarrow \quad C_1 = 1$$

$$y'(t) = C_1 e^t \cos 3t - 3C_1 e^t \sin 3t + C_2 e^t \sin 3t + 3C_2 e^t \cos 3t$$

$$y'(0) = -2 \quad \Rightarrow \quad C_1 + 3C_2 = -2 \quad \Rightarrow \quad C_2 = -1$$

Thus, the solution of the IVP is

$$y(t) = e^t \cos 3t - e^t \sin 3t$$

$$\boxed{e^t \cos 3t - e^t \sin 3t}$$

Problem 3. Find a particular solution of the following 2nd order ODE using the method of undetermined coefficients:

$$y'' + 2y' + 2y = 10 + e^{-t}\cos t$$

1) $y'' + 2y' + 2y = 10$ A particular solution of this equation is $\boxed{Y_1(t) = 5}$

2) $y'' + 2y' + 2y = e^{-t}\cos t$

Let us look for a solution in the form $Y_2(t) = t^s\left[Ae^{-t}\cos t + Be^{-t}\sin t\right]$

where $s = \#$ of times $\lambda = -1+i$ is a root of the characteristic eq.

Thus, $s = 1$, and

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

- $Y_2(t) = Ate^{-t}\cos t + Bte^{-t}\sin t$
- $Y_2'(t) = Ae^{-t}\cos t - Ate^{-t}\cos t - Ate^{-t}\sin t + Be^{-t}\sin t - Bte^{-t}\sin t + Bte^{-t}\cos t$
  $= Ae^{-t}\cos t + Be^{-t}\sin t + (B-A)te^{-t}\cos t - (A+B)te^{-t}\sin t$
- $Y_2''(t) = -Ae^{-t}\cos t - Ae^{-t}\sin t - Be^{-t}\sin t + Be^{-t}\cos t + (B-A)e^{-t}\cos t - (B-A)te^{-t}\cos t$
  $- (B-A)te^{-t}\sin t - (A+B)e^{-t}\sin t + (A+B)te^{-t}\sin t - (A+B)te^{-t}\cos t =$
  $= 2(B-A)e^{-t}\cos t - 2(A+B)e^{-t}\sin t - 2Bte^{-t}\cos t + 2Ate^{-t}\sin t$

So ☺

$$2(B-A)e^{-t}\cos t - 2(A+B)e^{-t}\sin t - 2Bte^{-t}\cos t + 2Ate^{-t}\sin t$$
$$+ 2Ae^{-t}\cos t + 2Be^{-t}\sin t + 2(B-A)te^{-t}\cos t - 2(A+B)te^{-t}\sin t$$
$$+ 2Ate^{-t}\cos t + 2Bte^{-t}\sin t = e^{-t}\cos t$$

$$2Be^{-t}\cos t - 2Ae^{-t}\sin t = e^{-t}\cos t$$

$$\begin{cases} A = 0 \\ B = \frac{1}{2} \end{cases}$$

$$Y_2(t) = \frac{1}{2}te^{-t}\sin t$$

$$Y = Y_1 + Y_2 = 5 + \frac{1}{2}te^{-t}\sin t$$

$$\boxed{Y(t) = 5 + \frac{1}{2}te^{-t}\sin t}$$

Problem 4. Find the general solution of the given differential equation

$$y'' - 2y' + y = e^x \ln x, \quad x > 0$$

1. Fundamental set of solutions for the homogeneous equation

$$y'' - 2y' + y = 0 \qquad \lambda^2 - 2\lambda + 1 = 0 \Rightarrow \underline{\lambda = 1}$$

$$\Rightarrow \begin{cases} y_1(x) = e^x \\ y_2(x) = x \cdot e^x \end{cases}$$

2. The Wronskian $W[y_1 y_2](x) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$

3. A particular solution of the nonhomogeneous equation :

$$Y(t) = y_2(x) \int \frac{y_1(x) g(x)}{W(x)} dx - y_1(x) \int \frac{y_2 g(x)}{W(x)} dx$$

- $\displaystyle\int \frac{y_1(x) g(x)}{W(x)} dx = \int \frac{e^x e^x \ln x}{e^{2x}} dx = \int \ln x \, dx = x \ln x - x$

- $\displaystyle\int \frac{y_2(x) g(x)}{W(x)} dx = \int \frac{x e^x e^x \ln x}{e^{2x}} dx = \int x \ln x \, dx = \int \ln x \, d\frac{x^2}{2}$

$$= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x^2 \cdot \frac{1}{x} dx = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 = \frac{1}{4} x^2 (2 \ln x - 1)$$

$$Y(t) = e^x \cdot x^2 (\ln x - 1) - \frac{1}{4} e^x x^2 (2 \ln x - 1) = x^2 e^x \left(\ln x - 1 - \frac{1}{2} \ln x + \frac{1}{4} \right)$$

$$= \frac{1}{4} x^2 e^x (2 \ln x - 3)$$

4. The general solution is

$$y(t) = c_1 e^x + c_2 x \cdot e^x + \frac{1}{4} x^2 e^x (2 \ln x - 3)$$

$$\boxed{c_1 e^x + c_2 x e^x + \frac{1}{4} x^2 e^x (2 \ln x - 3)}$$

Problem 5. Find the inverse Laplace transform of the following function

$$F(s) = \frac{8s^2 - 4s + 18}{s(s^2+9)}$$

1. $$F(s) = \frac{8s^2-4s+18}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}$$

$$8s^2 - 4s + 18 = A(s^2+9) + (Bs+c)\cdot s$$

$$\Rightarrow \begin{cases} A + B = 8 \\ C = -4 \\ 9A = 18 \end{cases} \Rightarrow \begin{cases} A = 2 \\ B = 6 \\ C = -4 \end{cases}$$

2. $$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{6s-4}{s^2+9}\right\} =$$

$$= 2 + \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}\cdot 6 - 4\,\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}\frac{1}{3} =$$

$$= 2 + 6\cos 3t - \frac{4}{3}\sin 3t$$

$$\boxed{f(t) = 2 + 6\cos 3t - \frac{4}{3}\sin 3t}$$