Problem 1. Find the solution of the following initial value problem

$$xy' + (x+1)y = \sin x\,, \quad y\left(\frac{\pi}{2}\right) = \frac{1}{\pi}$$

$$y' + \frac{x+1}{x} y = \frac{1}{x}\sin x$$

$$y'\cdot\mu + \underbrace{\frac{x+1}{x}\mu}_{\mu'} y = \frac{1}{x}\sin x\ \mu$$

$$\mu' = \frac{x+1}{x}\mu \Rightarrow \frac{d\mu}{\mu} = \left(1+\frac{1}{x}\right)dx \Rightarrow \ln|\mu| = \int 1 + \frac{1}{x}\,dx =$$

$$= x + \ln|x| + C$$

$$|\mu| = e^{x + \ln|x| + C} = \hat{C} e^{x}|x|$$

$$\underbrace{\mu = x\cdot e^{x}}_{\uparrow \text{ integrating factor}} \text{ is a solution}$$

$$(y\cdot\mu)' = \frac{1}{x}\sin x\cdot\mu$$

$$(y\cdot x\cdot e^{x})' = e^{x}\sin x$$

$$y\cdot x e^{x} = \underbrace{\int e^{x}\sin x\,dx}_{I}$$

Let us compute integral I : $I = \int e^{x}\sin x\,dx = \int \sin x\,de^{x} =$

$$= e^{x}\cdot\sin x - \int e^{x}\cos x\,dx = e^{x}\sin x - \int \cos x\,de^{x} = e^{x}\sin x - e^{x}\cos x + \int e^{x}d\cos x$$

$$= e^{x}(\sin x - \cos x) - \underbrace{\int e^{x}\sin x\,dx}_{I} \Rightarrow I = \frac{1}{2}e^{x}(\sin x - \cos x) + C$$

Thus, the general solution is $y = x^{-1}e^{-x}\cdot\left(\frac{1}{2}e^{x}(\sin x - \cos x) + C\right)$, or

$$y = \frac{\sin x - \cos x}{2x} + C\frac{e^{-x}}{x}$$

$$y\left(\frac{\pi}{2}\right) = \frac{1}{\pi} \Rightarrow \frac{1}{\pi} = \frac{1-0}{\pi} + C\frac{e^{-\pi/2}}{\pi/2} \Rightarrow C = 0$$

$$y = \frac{\sin x - \cos x}{2x}$$

$$\boxed{y = \frac{\sin x - \cos x}{2x}}$$

Problem 2. Find an integrating factor and solve the equation

$$y + (2xy - e^{-2y})y' = 0$$

This equation is not exact since $\frac{\partial}{\partial y} y = 1$ and $\frac{\partial}{\partial x}(2xy - e^{-2y}) = 2y$

Let us find an integrating factor.

$$y \cdot \mu + (2xy - e^{-2y})\mu y' = 0 \quad , \quad \mu = \mu(x,y)$$

We want to find μ such that $\frac{\partial}{\partial y}(y\mu) = \frac{\partial}{\partial x}\left[(2xy - e^{-2y})\mu\right]$

$$\mu + y\mu_y' = 2y \cdot \mu + (2xy - e^{-2y})\mu_x'$$

Assume that μ does not depend on x, in other words $\mu = \mu(y)$ and $\mu_x' = 0$, then

$$y\mu' = (2y - 1)\mu$$

$$\frac{d\mu}{\mu} = \left(2 - \frac{1}{y}\right) dy$$

$$\ln|\mu| = 2y - \ln|y| + c$$

$$|\mu| = \hat{c} \cdot e^{2y} \frac{1}{|y|}$$

For example, $\underbrace{\mu = \frac{e^{2y}}{y}}_{\text{integrating factor}}$ is a solution

With $\mu = \frac{e^{2y}}{y}$, we have

$$e^{2y} + \left(2xe^{2y} - \frac{1}{y}\right)y' = 0$$

This equation is exact.

$$\begin{cases} \frac{\partial \psi}{\partial x} = e^{2y} \Rightarrow \psi = x \cdot e^{2y} + \xi(y) \\ \frac{\partial \psi}{\partial y} = 2xe^{2y} - \frac{1}{y} \end{cases}$$

From the 2^{nd} equation we obtain:

$$\frac{\partial}{\partial y}\left(x \cdot e^{2y} + \xi(y)\right) = 2xe^{2y} - \frac{1}{y}$$

$$2xe^{2y} + \xi' = 2xe^{2y} - \frac{1}{y}$$

$$\xi' = -\frac{1}{y}$$

$$\xi = -\ln|y|$$

Thus, the solution is

$$xe^{2y} - \ln|y| = C$$

$$\boxed{xe^{2y} - \ln|y| = C}$$

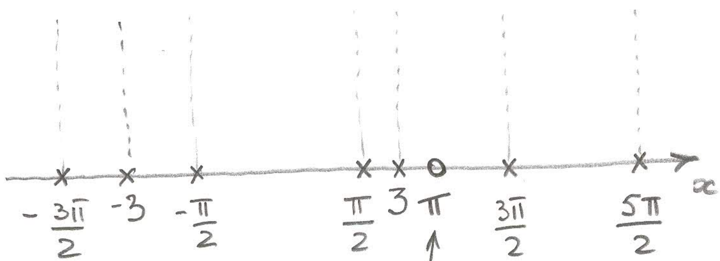
Problem 3. Determine (without solving the problem) an interval in which the solution of the initial value problem is certain to exist.

$$(9 - x^2)y' + (\tan x)\, y = \sin x\,, \qquad y(\pi) = 0$$

$$y' + \frac{\sin x}{\cos x}\,\frac{1}{9-x^2}\,y = \frac{\sin x}{9-x^2}$$

Points of discontinuity of the coefficients $\dfrac{\sin x}{(9-x^2)\cos x}$ and $\dfrac{\sin x}{9-x^2}$

are : $x = \pm 3$ and $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$



initial condition $y(\pi)=0$

$\Rightarrow$ the solution is certain to exist on $\left(3, \frac{3\pi}{2}\right)$

$$\boxed{\left(3, \frac{3\pi}{2}\right)}$$

Problem 4. Find the solution of the initial value problem

$$x' = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

1. Eigenvalues

$$\begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = 0$$

$$(1+\lambda)^2 + 1 = 0$$

$$1 + \lambda = \pm i$$

$$\begin{cases} \lambda_1 = -1+i \\ \lambda_2 = -1-i \end{cases}$$

2. Eigenvector.

$$\lambda_1 = -1 + i \qquad (\alpha = -1,\ \beta = 1)$$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\mathbb{V} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\mathbb{a}} + i \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\mathbb{b}}$$

3. General Solution

$$\mathbb{x}(t) = C_1 e^{-t} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t \right] +$$

$$+ C_2 e^{-t} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t \right]$$

4. Initial condition

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Thus, the solution is

$$\mathbb{x}(t) = e^{-t} \begin{pmatrix} \cos t + \sin t \\ \cos t - \sin t \end{pmatrix}$$

$$\boxed{\mathbb{x}(t) = e^{-t} \begin{pmatrix} \cos t + \sin t \\ \cos t - \sin t \end{pmatrix}}$$

Problem 5.

Find the general solution of the given system of ODEs.

Describe how the solution behave as $t \to \infty$

$$x' = \begin{pmatrix} -4 & 1 \\ -1 & -2 \end{pmatrix} x$$

1. Eigenvalues

$$\begin{vmatrix} -4-\lambda & 1 \\ -1 & -2-\lambda \end{vmatrix} = 0$$

$$(\lambda+4)(\lambda+2)+1=0$$

$$\lambda^2+6\lambda+9=0$$

$$(\lambda+3)^2=0$$

$$\underline{\underline{\lambda=-3}}$$

2. Eigenvector

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \quad , \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

3. Generalized eigenvector

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad , \quad w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

4. General Solution

$$x(t) = C_1 e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-3t}\left[t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$

$$\lim_{t\to\infty} x(t) = 0$$

$$\boxed{\begin{array}{l} x(t) = C_1 e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-3t}\left[t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \\ \lim_{t\to\infty} x(t) = 0 \end{array}}$$