Math 245 - Mathematics of Physics and Engineering I

Lecture 41. Review of the Course

April 25, 2012

Agenda

- First Order ODEs
- Systems of Two Linear First Order ODEs
- Second Order Linear ODEs
- The Laplace Transform
- Systems of n Linear First Order ODEs
- Nonlinear ODEs and Stability

First Order ODEs

We studied several types of first order ODEs:

- Linear equations: $\boxed{y'(t) + p(t)y = g(t)}$

  Method of integrating factors:

$$y(t) = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)dt\right), \qquad \mu(t) = e^{\int p(t)dt}$$

- Exact equations: $\boxed{M(x,y) + N(x,y)y' = 0}$ and $\exists\psi(x,y) : \psi'_x = M, \psi'_y = N$

  Solutions are given implicitly by

$$\psi(x,y) = C$$

  - Separable equations: $\boxed{h(y)y' = g(x)}$ are a special case of exact $(\psi = H - G)$

    Solutions are given implicitly:

$$H(y) = G(x) + C$$

    where H and G are antiderivatives of h and g, $\frac{dH(y)}{dy} = h(y)$, $\frac{dG(x)}{dx} = g(x)$

    - ⋆ Autonomous equations: $\boxed{y' = f(y)}$ are a special case of separable $(h = 1/f, g = 1)$. Used in population dynamics.

Criterion of Exactness

Q: How to systematically determine whether a given ODE is exact?

$$M(x,y) + N(x,y)y' = 0 \tag{1}$$

Theorem

Let $M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ be continuous in the region $R : x \in (\alpha, \beta)$, $y \in (\gamma, \delta)$. Then equation (1) is an exact ODE in R if and only if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \tag{2}$$

A function ψ satisfying $\frac{\partial \psi}{\partial x} = M(x,y)$ and $\frac{\partial \psi}{\partial y} = N(x,y)$ exists if and only if (2).

"Almost exact equations": It is sometimes possible to convert a differential equation that is not exact into an exact equation by a suitable integrating factor μ. Equation for μ is

$$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} + \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu = 0$$

Existence and Uniqueness of Solutions

Theorem

Consider the following first order **linear** *ODE:*

$$y' + p(t)y = g(t)$$

If $p(t)$ and $g(t)$ are continuous on an interval (α, β) containing $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies this ODE for each $t \in (\alpha, \beta)$, and that also satisfies the initial condition $y(t_0) = y_0$ for any y_0.

Theorem

Consider the following first order **nonlinear** *ODE:*

$$y' = f(t, y)$$

Let the functions f and $\partial f/\partial y$ be continuous in some open rectangle $t \in (\alpha, \beta)$, $y \in (y_1, y_2)$ containing the point (t_0, y_0). Then, in some interval $t \in (t_0 - h, t_0 + h) \subset (\alpha, \beta)$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0$$

Systems of Two Linear First Order ODEs

We studied homogeneous autonomous system:

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

The Eigenvalue Method:

- If $\lambda_1 \neq \lambda_2$ are two different real eigenvalues of $\mathbf{A}$, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are the corresponding eigenvectors, then a fundamental set of solutions is

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1, \qquad \mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2$$

- If $\lambda_{1,2} = \alpha \pm i\beta$ is a pair of complex eigenvalues of $\mathbf{A}$, and $\mathbf{v}_{1,2} = \mathbf{a} \pm i\mathbf{b}$ are the corresponding eigenvectors, then a fundamental set of solutions is

$$\mathbf{x}_1 = e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t), \qquad \mathbf{x}_2 = e^{\alpha t}(\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t)$$

- If $\lambda_1 = \lambda_2 = \lambda$ is a repeated eigenvalue of $\mathbf{A}$ and $\mathbf{A}$ is nondiagonal, then a fundamental set of solution is

$$\mathbf{x}_1 = e^{\lambda t}\mathbf{v}, \qquad \mathbf{x}_2 = te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}$$

  - $\mathbf{v}$ is the only independent eigenvector corresponding to λ
  - $\mathbf{w}$ is the generalized eigenvector corresponding to λ, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v}$

Nonhomogeneous Systems

If $\mathbf{A}$ is nonsingular, then it is possible to reduce a nonhomogeneous system to a homogeneous one. If $\tilde{\mathbf{x}}$ is a solution of the homogeneous system

$$\frac{d\tilde{\mathbf{x}}}{dt} = \mathbf{A}\tilde{\mathbf{x}}$$

then the solution of the nonhomogeneous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{b}$$

is given by

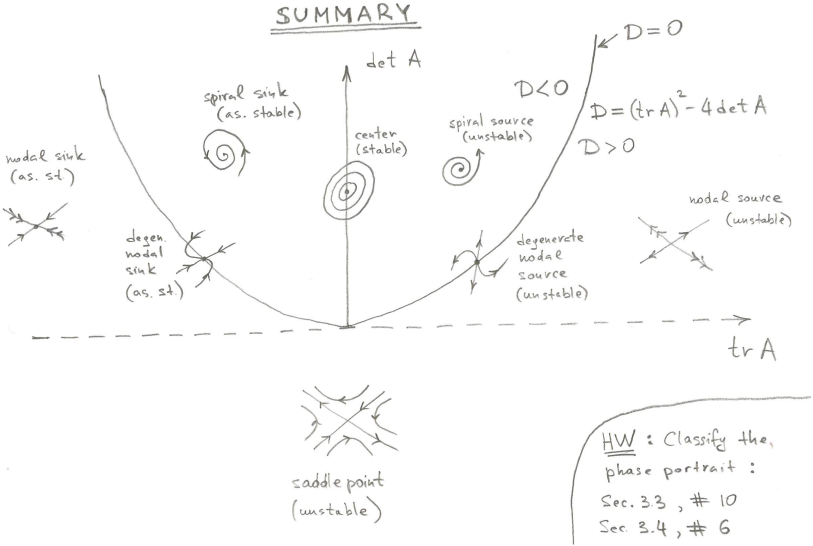
$$\boxed{\mathbf{x} = \tilde{\mathbf{x}} + \mathbf{x}_{\mathrm{eq}} = \tilde{\mathbf{x}} - \mathbf{A}^{-1}\mathbf{b}}$$

Thus, to solve a nonhomogeneous autonomous system (with nonsingular $\mathbf{A}$), we need

- Find its equilibrium solution (linear algebra problem)
- Solve the corresponding homogeneous system

Classification of Phase Portraits

If we assume that $\det \mathbf{A} \neq 0$, then $\mathbf{x} = \mathbf{0}$ is the only critical point of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.



Existence and Uniqueness of Solutions

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$$

Theorem

Let

- $\mathbf{P}(t)$ *and* $\mathbf{g}(t)$ *be continuous on an open interval* $I = (\alpha, \beta)$
- $t_0 \in I$
- $\mathbf{x}_0$ *be any given vector*

Then there **exists a unique solution** *of the initial value problem*

$$\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

on the interval I.

Second order Linear ODEs

Theory of 2nd order linear ODEs follows from the theory of systems of two linear first order ODEs, since, by introducing the state variables

$$x_1 = y, \qquad x_2 = y'$$

we can convert any second order equation into a system of first order equations:

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t), \\ y(t_0) = y_0, \\ y'(t_0) = y_1. \end{cases}$$

is equivalent to

$$\begin{cases} \mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix}, \\ \mathbf{x}(t_0) = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \end{cases}$$

Second Order ODEs with Constant Coefficients

The general solution of the ODE

$$ay'' + by' + cy = 0$$

is

- Distinct Real Roots, $\lambda_1 \neq \lambda_2$, $b^2 - 4ac > 0$

$$\boxed{y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}$$

- Repeated Roots, $\lambda_1 = \lambda_2 = \lambda$, $b^2 - 4ac = 0$

$$\boxed{y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}}$$

- Complex Conjugate Roots, $\lambda = \alpha \pm i\beta$, $b^2 - 4ac > 0$

$$\boxed{y(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t}$$

Nonhomogeneous Equations

General Strategy for Solving $ay'' + by' + cy = g(t)$:

1. Find the general solution $c_1 y_1 + c_2 y_2$ of the corresponding homogeneous equation $ay'' + by' + cy = 0$. This solution is called the complementary solution.
2. Find some single solution Y of the nonhomogeneous equation. Often this solution is referred to as a particular solution.
3. The general solution of $ay'' + by' + cy = g(t)$ is then $\boxed{y = c_1 y_1 + c_2 y_2 + Y}$.

Question: How to find a particular solution Y?

There are two methods:

- Method of Undetermined Coefficients
  - Advantage: easy to use
  - Disadvantage: sometimes does not work
- Method of Variation of Parameters
  - Advantage: general method (always works)
  - Disadvantage: computationally difficult

Method of Undetermined Coefficients

To find a particular solution of a nonhomogeneous equation

$$ay'' + by' + cy = g(t)$$

do the following:

1. Make sure that $g(t)$ involves nothing more than exponential functions $e^{\alpha t}$, sines $\sin \beta t$, cosines $\cos \beta t$, polynomials $P_n(t) = a_0 t^n + a_1 t^{n-1} + \ldots + a_n$, or sums or products of such functions. If this is not the case, use the method of variation of parameters.
2. If $g(t) = g_1(t) + g_2(t) + \ldots + g_n(t)$, then the original problem beaks down to n subproblems: the i^{th} subproblem is to find a particular solution $Y_i(t)$ of
$$ay'' + by' + cy = g_i(t)$$
3. Find $Y_i(t)$ using the table on the next slide.
4. $Y(t) = Y_1(t) + \ldots + Y_n(t)$ is a particular solution of the original nonhomogeneous equation.

Table

The particular solution of $ay'' + by' + cy = g(t)$

	$g(t)$	$Y(t)$
1	$P_n(t)$	$t^s G_n(t)$
2	$P_n(t)e^{\alpha t}$	$t^s G_n(t)e^{\alpha t}$
3	$P_n(t)e^{\alpha t}\sin\beta t$	$t^s\left[G_n(t)e^{\alpha t}\cos\beta t + H_n(t)e^{\alpha t}\sin\beta t\right]$
4	$P_n(t)e^{\alpha t}\cos\beta t$	$t^s\left[G_n(t)e^{\alpha t}\cos\beta t + H_n(t)e^{\alpha t}\sin\beta t\right]$

- $P_n(t), G_n(t), H_n(t)$ are polynomials of degree n
- $s = 0, 1, 2$ is the smallest integer that will ensure that no term in $Y(t)$ is a solution of the corresponding homogeneous equation:
  - Case 1: $s = \#$ times 0 is a root of the characteristic equation
  - Case 2: $s = \#$ times α is a root of the characteristic equation
  - Cases 3,4: $s = \#$ times $\alpha + i\beta$ is a root of the characteristic equation

Variation of Parameters for Equations

How to find a particular solution of

$$y'' + by' + cy = g(t)$$

1. Find a fundamental set of solution $y_1(t)$ and $y_2(t)$ of the corresponding homogeneous equation
2. A particular solution is then

$$\boxed{Y(t) = y_2(t)\int \frac{y_1(t)g(t)}{W[y_1, y_2](t)}dt - y_1(t)\int \frac{y_2(t)g(t)}{W[y_1, y_2](t)}dt}$$

where W is the Wronskian

$$W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

The Laplace Transform

- Laplace transform: $f(t) \mapsto F(s)$

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt}$$

  - $f(t)$ is a signal in the t-domain
  - $F(s)$ is its representation in the s-domain

- Laplace transform is linear:

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\}$$

- $f(t)$ is of exponential order (as $t \to \infty$) if for some constants t_0, M, and a

$$|f(t)| \leq M e^{at}, \quad \text{for } t \geq t_0$$

- Laplace transform $\mathcal{L}\{f\}$ exists if $f(t)$ is a piecewise continuous function of exponential order.

Properties of the Laplace Transform

- If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a$, and c is a constant, then

$$\boxed{\mathcal{L}\{e^{ct}f(t)\} = F(s-c)} \quad s > a + c$$

- If
  - $f, f', \ldots, f^{(n-1)}$ are continuous
  - $f^{(n)}$ is piecewise continuous on the interval $0 \leq t \leq T$, for any T.
  - $f, f', \ldots, f^{(n)}$ are of exponential order: $|f^{(i)}(t)| \leq Me^{at}$.

  Then

$$\boxed{\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)} \quad s > a$$

- If
  - f is piecewise continuous on the interval $0 \leq t \leq T$
  - f is of exponential order: $|f(t)| \leq Me^{at}$

  Then for any positive integer n

$$\boxed{\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)} \quad s > a$$

- For any positive integer n,

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}} \quad s > 0$$

The Inverse Laplace Transform

Definition

If $f(t)$ is piecewise continuous and of exponential order on $[0, \infty)$ and $\mathcal{L}\{f(t)\} = F(s)$, then we call $f(t)$ the **inverse Laplace transform** of $F(s)$, and denote it by

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}f(t)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a} \quad s > a$
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$t^p,\ p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2} \quad s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2} \quad s > a$
$t^n e^{at},\ n \in \mathbb{N}$	$\frac{n!}{(s-a)^{n+1}} \quad s > a$
$e^{at} f(t)$	$F(s-a)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$

Partial Fraction Decomposition

To find $\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\}$, use Partial Fraction Decomposition.

Partial Fraction Decomposition:

- If $Q(s) = (s - s_1)(s - s_2)\dots(s - s_n)$, where all s_i are distinct, then

$$Y(s) = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \ldots + \frac{A_n}{s - s_n}$$

- If any root s_j of $Q(s)$ is of multiplicity k, i.e. $Q(s) = \dots(s - s_j)^k \dots$, then the j^{th} term must be changed to

$$\frac{A_j}{s - s_j} \rightsquigarrow \frac{A_{j_1}}{s - s_j} + \frac{A_{j_2}}{(s - s_j)^2} + \ldots + \frac{A_{j_k}}{(s - s_j)^k}$$

- If $Q(s)$ has a pair of complex conjugate roots $\alpha \pm i\beta$, then the factorization of $Q(s)$ contains factor $(s - \alpha)^2 + \beta^2$. If roots $\alpha \pm i\beta$ have multiplicity k, then the partial fraction expansion of $Y(s)$ must include the term

$$\frac{A_1 s + B_1}{(s - \alpha)^2 + \beta^2} + \frac{A_2 s + B_2}{[(s - \alpha)^2 + \beta^2]^2} + \ldots + \frac{A_k s + B_k}{[(s - \alpha)^2 + \beta^2]^k}$$

Solving Initial Value Problems with Laplace Transforms

General Scheme:

1. Using table of Laplace transforms and properties of the Laplace transform $\mathcal{L}$
   - linearity
   - $\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$
   - $\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$
   - etc.

   transform the IVP for a **linear** ODE with **constant coefficients** into an algebraic equation in the $s-$domain.
2. Find the Laplace transform $Y(s)$ of the solution by solving this algebraic equation.
3. Find the solution of the IVP $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ using partial fraction decompositions, the linearity of $\mathcal{L}^{-1}$, and a table of Laplace transforms.

Discontinuous and Periodic Functions

- The unit step function (or Heaviside function) and its translation:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \qquad u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

- The Laplace transform of u_c with $c \geq 0$ is

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s} \quad s > 0$$

- The Laplace transform of the shifted function

$$\mathcal{L}\{f_c(t)\} = \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

- If f is periodic with period T and is piecewise on $[0, T]$, then

$$\mathcal{L}\{f(t)\} = \frac{F_T(s)}{1 - e^{-sT}}, \qquad F_T(s) = \mathcal{L}\{f_T\} = \int_0^T e^{-st} f(t) dt$$

Impulse Functions

- Unit Impulse Function or Dirac Delta Function:

$$\delta(t-t_0) \text{``="} \begin{cases} +\infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases}$$

- For any continuous function on an interval $a \leq t_0 \leq b$,

$$\int_a^b f(t)\delta(t-t_0)dt = f(t_0)$$

- The Laplace transform:

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty e^{-st}\delta(t-t_0)dt = e^{-st_0}$$

- The delta function is the derivative of the unit step function:

$$\delta(t-t_0) = u'(t-t_0)$$

Systems of n Linear First Order ODEs

This is a straightforward generalization of a 2-dim case.
Essentially, the only new notion here is the matrix exponential function:

Definition

Let $\mathbf{A}$ be an $n \times n$ constant matrix. The **matrix exponential function** is defined as follows:

$$\boxed{e^{\mathbf{A}t} = \mathbf{I}_n + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \frac{1}{3!}\mathbf{A}^3 t^3 + \ldots = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!}}$$

Theorem

Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ constant matrices, and t and τ be real or complex numbers. Then

- $e^{\mathbf{A}(t+\tau)} = e^{\mathbf{A}t} e^{\mathbf{A}\tau}$
- $\mathbf{A}$ *commutes with* $e^{\mathbf{A}t}$, *that is,* $\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$
- $(e^{\mathbf{A}t})^{-1} = e^{-\mathbf{A}t}$
- *If* $\mathbf{AB} = \mathbf{BA}$, *then* $e^{(\mathbf{A}+\mathbf{B})t} = e^{\mathbf{A}t} e^{\mathbf{B}t}$

How to construct $e^{\mathbf{A}t}$?

- If $\mathbf{\Phi}(t)$ is the special fundamental matrix ($\mathbf{\Phi}(0) = \mathbf{I}_n$), then

$$e^{\mathbf{A}t} = \mathbf{\Phi}(t)$$

- If $\mathbf{X}(t)$ is any fundamental matrix for $\mathbf{x}' = \mathbf{A}\mathbf{x}$, then

$$e^{\mathbf{A}t} = \mathbf{X}(t)\mathbf{X}^{-1}(0)$$

- If $\mathbf{A}$ has n linearly independent eigenvectors, then

$$e^{\mathbf{A}t} = \mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1}$$

- Using the inverse Laplace transform:

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\left\{(s\mathbf{I}_n - \mathbf{A})^{-1}\right\}$$

- Solution of the initial value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}, \ \ \mathbf{x}(0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

Autonomous Nonlinear Systems and Stability

$$\frac{dx}{dt} = F(x,y), \qquad \frac{dy}{dt} = G(x,y)$$

- Existence and Uniqueness of Solutions: If F, G and $\partial F/\partial x, \partial F/\partial y, \partial G/\partial x$, $\partial G/\partial y$ are continuous in some domain $\mathcal{D}$ that contains (x_0, y_0). Then there is an interval $t \in (t_0 - h, t_0 + h)$ in which there exists a unique solution of

$$\frac{dx}{dt} = F(x,y), \qquad \frac{dy}{dt} = G(x,y)$$

  that also satisfies the initial conditions $x(t_0) = x_0, \; x(y_0) = y_0$
- Stability and Instability:
  - A critical point is **stable** if all trajectories that start close to the critical point remain close to it for all future time.
  - A critical point is **asymptotically stable** if all close trajectories not only remain close but approach the critical point as $t \to \infty$.
  - A critical point is **unstable** if at least some nearby trajectories do not remain close the critical point as t increases.

Almost Linear Systems

- System $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x})$ is called an almost linear system in the neighborhood of $\mathbf{x} = \mathbf{0}$ if
  - $\mathbf{g}(\mathbf{x})$ has continuous partial derivatives
  - $\frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}\|} \to 0$, as $\mathbf{x} \to \mathbf{0}$
- The system $x' = F(x, y),\ y' = G(x, y)$ is almost linear in the neighborhood of (x_0, y_0) whenever the functions F and G are twice differentiable.The corresponding linear system is

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad u_1 = x - x_0,\ u_2 = y - y_0$$

- Relationship between types and stability properties of almost linear systems and their linearizations is given by the following theorem.

Phase Portraits

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x}) \tag{3}$$

Theorem

Let λ_1 and λ_2 be the eigenvalues of the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

- *If $\lambda_{1,2} = \pm i\beta$ (stable center), then the type and stability of $\mathbf{x} = \mathbf{0}$ for (3) are*
  - *Type: Center or Spiral Sink or Spiral Source*
  - *Stability: Undetermined*
- *If $\lambda_1 = \lambda_2 > 0$ (unstable degenerate nodal source), then for (3) $\mathbf{x} = \mathbf{0}$ is*
  - *Type: Spiral Source or Nodal Source*
  - *Stability: Unstable*
- *If $\lambda_1 = \lambda_2 < 0$ (as. stable degenerate nodal sink), then for (3) $\mathbf{x} = \mathbf{0}$ is*
  - *Type: Spiral Sink or Nodal Sink*
  - *Stability: Asymptotically Stable*
- *In all other cases, the type and stability of $\mathbf{x} = \mathbf{0}$ for the nonlinear system and its linearization are the same.*

Periodic Solutions and Limiting Cycles

- A periodic solution $\mathbf{x}(t)$ is a solution that satisfies the relation $\mathbf{x}(t+T) = \mathbf{x}(t)$ for some constant $T > 0$ that is called the period.
- The trajectories of periodic solutions are closed curves in the phase plane.
- For a linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$
  - If $\lambda_{1,2} = \pm i\beta$, then all solutions are periodic.
  - Otherwise, there are no periodic solutions (except for $\mathbf{x} = \mathbf{0}$)
- A closed trajectory that attracts other trajectories is called a **limit cycle**.

General Theorems

$$x' = F(x,y), \quad y' = G(x,y)$$

- Let F and G have continuous first partial derivatives in a domain $D \subset \mathbb{R}^2$.
  - A closed trajectory must enclose at least one critical point.
  - If it encloses only one critical point, the critical point cannot be a saddle point.
  - If D is simply connected and $F_x + G_x$ has the same sign throughout D, then there is no closed trajectory lying entirely in D.

Thank you for attention and good luck on the final!