Math 245 - Mathematics of Physics and Engineering I

Lecture 38. Almost Linear Systems

April 20, 2012

Agenda

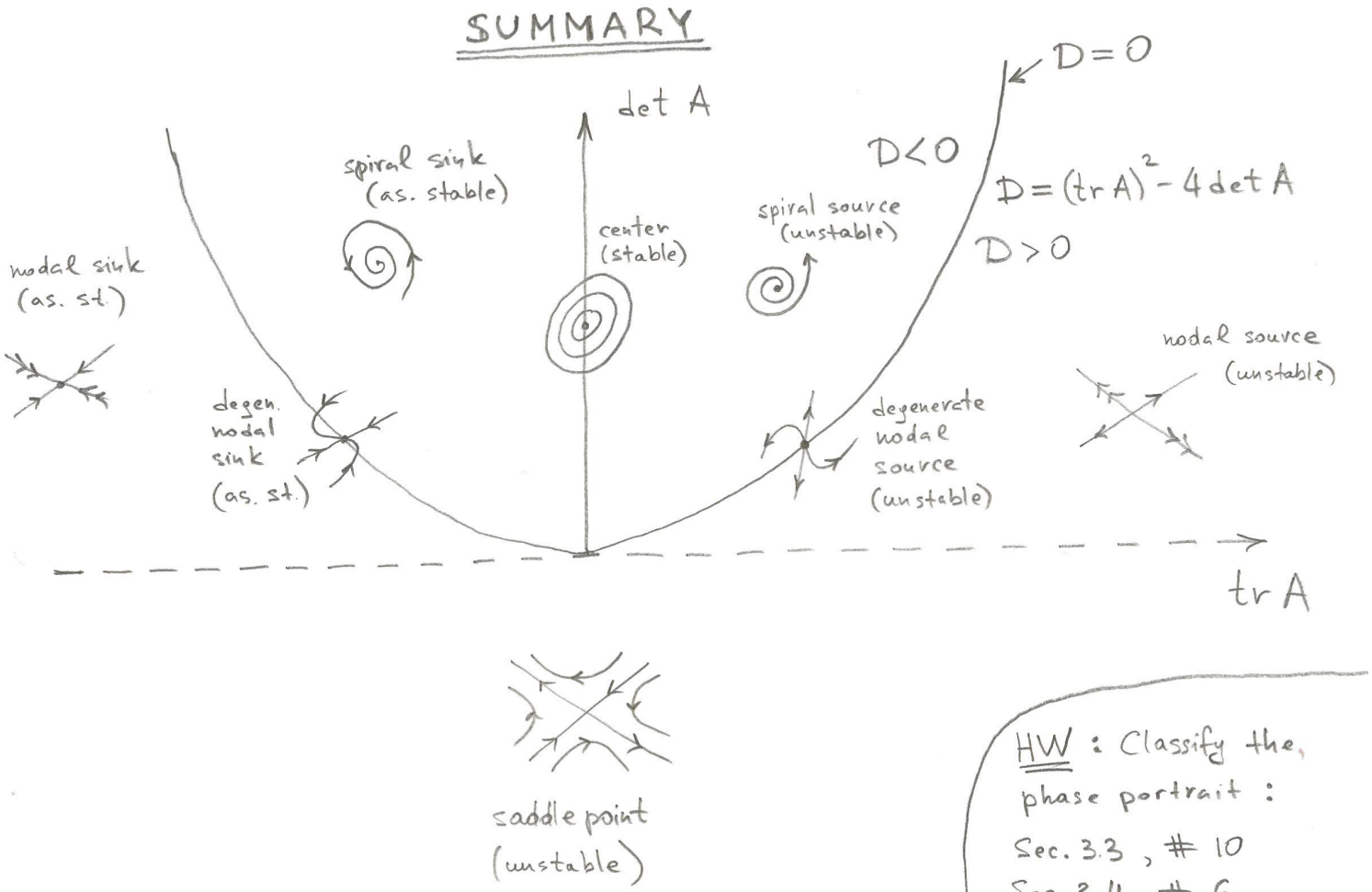
- Stability Properties of Linear Systems
- Effect of Small Perturbations
- Linear Approximations to Nonlinear Systems
- Stability of Phase Portraits
- Summary and Homework

Linear Systems

In Lecture 14, we studied stability properties of linear systems

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

If we assume that $\det \mathbf{A} \neq 0$, then $\mathbf{x} = \mathbf{0}$ is the only critical point of (1).



Linear Systems

Theorem

The critical point $\mathbf{x} = \mathbf{0}$ *of the linear system* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *is*

- **stable**, *but not asymptotically stable, if* λ_1 *and* λ_2 *are purely imaginary*
- **asymptotically stable**, *if* λ_1 *and* λ_2 *are*
  - *real and negative or*
  - *complex with negative real part*
- **unstable**, *if* λ_1 *and* λ_2 *are*
  - *real and at least one of them is positive or*
  - *complex with positive real part*

Thus, both the type and stability properties are determined by the eigenvalues of the coefficient matrix $\mathbf{A}$. In turn, the values of λ_1 and λ_2 depend on the coefficients, i.e. entities of $\mathbf{A}$. In applications, these coefficients usually are obtained from measurements of certain physical quantities. Such measurements are often subject to small errors. Therefore, it is interesting to investigate the following question:

can small perturbations of coefficients affect the stability of a critical point?

Small Perturbations

The eigenvalues λ_1 and λ_2 are the roots of the polynomial equation

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{1}$$

It is possible to show that small perturbations of coefficients are reflected in small perturbations of the eigenvalues. There are only two sensitive situations:

1. **Most sensitive case**: $\lambda_{1,2} = \pm i\beta$ (the critical point is a stable center)

$$\begin{aligned} \lambda_1 = i\beta \quad &\rightsquigarrow \quad \hat{\lambda}_1 = \hat{\alpha} + i\hat{\beta} \\ \lambda_2 = -i\beta \quad &\rightsquigarrow \quad \hat{\lambda}_1 = \hat{\alpha} - i\hat{\beta} \end{aligned} \tag{2}$$

   - If $\hat{\alpha} = 0$ (very unlikely), then the critical point remains a stable center
   - If $\hat{\alpha} > 0$, then the critical point becomes unstable spiral source
   - If $\hat{\alpha} < 0$, then the critical point becomes asymptotically stable spiral sink

2. **Less sensitive case**: $\lambda_1 = \lambda_2 = \lambda$, the critical point is a degenerate nodal sink ($\lambda < 0$) or a degenerate nodal source ($\lambda > 0$).

$$\lambda_1 = \lambda_2 \quad \rightsquigarrow \quad \hat{\lambda}_1 \neq \hat{\lambda}_2 \tag{3}$$

   - If $\hat{\lambda}_{1,2}$ are real, then the critical node becomes a nodal sink/source
   - If $\hat{\lambda}_{1,2}$ are complex, then the critical node becomes a spiral sink/source

In Case 1, both stability and trajectories are affected.

In Case 2, only trajectories are affected.

Linear Approximations to Nonlinear Systems

Our next goal: to investigate the behavior of trajectories of a nonlinear system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ near a critical point

Strategy: approximate the nonlinear system by a linear system (trajectories of linear systems are easy to describe)

Question: What does it mean for a nonlinear system to be approximately linear?

Consider the following system:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x}) \tag{4}$$

and suppose that $\mathbf{x} = \mathbf{0}$ is an isolated critical point of (4). For system (4) to be close to the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we must assume that $\mathbf{g}(\mathbf{x})$ is small. More precisely,

Definition

System (4) is called an **almost linear system** in the neighborhood of $\mathbf{x} = \mathbf{0}$ if

- $\mathbf{g}(\mathbf{x})$ has continuous partial derivatives
- As $\mathbf{x} \to \mathbf{0}$,

$$\frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}\|} \to 0 \tag{5}$$

Example

- Determine whether the following system is almost linear in the neighborhood of the origin.

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x^2 - xy \\ -0.75xy - 0.25y^2 \end{pmatrix}$$

Linearization

Let us return to the general nonlinear system:

$$x' = F(x, y), \qquad y' = G(x, y) \tag{6}$$

Assume that (x_0, y_0) is an isolated critical point.

Theorem

The system (6) is almost linear in the neighborhood of (x_0, y_0) whenever the functions F and G have continuous partial derivatives up to order 2.

Linearization using Taylor expansions about (x_0, y_0):

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad u_1 = x - x_0,\ u_2 = y - y_0 \tag{7}$$

Remarks:

- If F and G are twice differentiable, then (6) is almost linear
- (7) is the linear system that corresponds to (6) near (x_0, y_0)
- Matrix in (7) is called Jacobian matrix

Example

- Find the linear system corresponding to the pendulum equations near the origin and near the critical point $(\pi, 0)$.

$$x' = y, \qquad y' = -\omega^2 \sin x - \gamma y$$

<u>Answer:</u>

- Near origin:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- Near $(\pi, 0)$:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad u_1 = x - \pi, \ u_2 = y$$

Phase Portraits

Consider the almost linear system in the neighborhood of $\mathbf{x} = \mathbf{0}$

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x}) \tag{8}$$

Since $\mathbf{g}(\mathbf{x}) \ll \mathbf{A}\mathbf{x}$ when $\mathbf{x}$ is small, it is reasonable to expect that the trajectories of the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are good approximations of trajectories of (8) near the origin.This is **true for many (but not all) cases**.

Theorem

Let λ_1 and λ_2 be the eigenvalues of the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

- *If $\lambda_{1,2} = \pm i\beta$ (stable center), then the type and stability of $\mathbf{x} = \mathbf{0}$ for (8) are*
  - *Type: Center or Spiral Sink or Spiral Source*
  - *Stability: Undetermined*
- *If $\lambda_1 = \lambda_2 > 0$ (unstable degenerate nodal source), then for (8) $\mathbf{x} = \mathbf{0}$ is*
  - *Type: Spiral Source or Nodal Source*
  - *Stability: Unstable*
- *If $\lambda_1 = \lambda_2 < 0$ (as. stable degenerate nodal sink), then for (8) $\mathbf{x} = \mathbf{0}$ is*
  - *Type: Spiral Sink or Nodal Sink*
  - *Stability: Asymptotically Stable*
- *In all other cases, the type and stability of $\mathbf{x} = \mathbf{0}$ for the nonlinear system and its linearization are the same.*

Summary

- System $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x})$ is called an almost linear system in the neighborhood of $\mathbf{x} = \mathbf{0}$ if
  - $\mathbf{g}(\mathbf{x})$ has continuous partial derivatives
  - $\frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}\|} \to 0$, as $\mathbf{x} \to \mathbf{0}$
- The system $x' = F(x, y),\ y' = G(x, y)$ is almost linear in the neighborhood of (x_0, y_0) whenever the functions F and G are twice differentiable.The corresponding linear system is

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad u_1 = x - x_0,\ u_2 = y - y_0$$

- Relationship between types and stability properties of almost linear systems and their linearizations is given by the theorem on Slide 10.

<u>Homework:</u>

- Section 7.2
  - 5(a,b,c), 7(a,b,c), 13(a,b,c), 19(a,b,c)