Math 245 - Mathematics of Physics and Engineering I

Lecture 27. Discontinuous Functions and Periodic Functions

March 23, 2012

ODE with Discontinuous or Periodic Forcing Functions

In Lecture 26, we discussed the general procedure used for solving initial value problems be means of the Laplace transform.

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

1. Transform the IVP into an algebraic equation in the $s-$domain.
2. Find the Laplace transform $Y(s)$ nof the solution.
3. Find the solution of the IVP $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

In many applications, the nonhomogeneous term $g(t)$, also called forcing function, is modeled by a discontinuous function or by a periodic function

Examples:

- In the actual physical system the forcing function is continuous, but it sometimes changes rapidly over a very short period of time.
- Engineering systems are often tested by subjecting them to discontinuous forcing functions.
- Vibrations of mechanical systems.

Goal: to develop properties of the Laplace transform of discontinuous and periodic functions.

The Unit Step Function

To deal effectively with functions having jump discontinuities, it is helpful to introduce the **unit step function** or **Heaviside function**:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \tag{1}$$

- In applications, the Heaviside function often represents a force or signal that turned on at time $t = 0$.

Translation of the Heaviside function:

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases} \tag{2}$$

The signal that is turned on at time $t = c$ and then turned off at time $t = d > c$ can be modeled by an **indicator function**:

$$u_{cd}(t) = u_c(t) - u_d(t) = \begin{cases} 0, & t < c \text{ or } t \geq d \\ 1, & c \leq t < d \end{cases} \tag{3}$$

Laplace Transform of the Unit Step Function

Theorem

- *The Laplace transform of* u_c *with* $c \geq 0$ *is*

$$\boxed{\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}} \quad s > 0$$

- *The Laplace transform of* $u_{cd} = u_c - u_d$ *with* $0 \leq c \leq d$ *is*

$$\boxed{\mathcal{L}\{u_{cd}(t)\} = \frac{e^{-cs} - e^{-ds}}{s}} \quad s > 0$$

Time-Shifted Functions

For a given function $f(t)$ defined for $t \geq 0$, define

$$f_c(t) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases} \tag{4}$$

In terms of the unit step function, $f_c(t)$ can be written as follows:

$$f_c(t) = u_c(t)f(t-c) \tag{5}$$

Theorem

If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a$ and $c > 0$, then

$$\boxed{\mathcal{L}\{f_c(t)\} = \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)} \quad s > a$$

Example: Find the Laplace transform of the function

$$f(t) = \begin{cases} t, & 0 < t < 2 \\ 1, & 2 \leq t < 3 \\ e^{-2t}, & 3 \leq t \end{cases}$$

Periodic Functions

Definition

A function f is said to be **periodic with period** $T > 0$ if

$$f(t+T) = f(t)$$

for all t is the domain of f.

In discussing a periodic function $f(t)$ it is convenient to introduce a window function $f_T(t)$ defined by

$$f_T(t) = \begin{cases} f(t), & t \in [0, T) \\ 0, & \text{otherwise} \end{cases} = f(t)(u(t) - u_T(t)) \tag{6}$$

The entire periodic function is then can be written in terms of its window function as follows:

$$f(t) = \sum_{n=0}^{\infty} f_T(t - nT) u_{nT}(t) \tag{7}$$

Laplace Transform of a Periodic Function

Theorem

If f is periodic with period T and is piecewise on $[0, T]$, then

$$\boxed{\mathcal{L}\{f(t)\} = \frac{F_T(s)}{1 - e^{-sT}}}$$

where

$$F_T(s) = \mathcal{L}\{f_T\} = \int_0^T e^{-st} f(t) dt$$

Examples

- Find the Laplace transform of the following periodic function with period $T = 2$

$$f_T(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

<u>Answer:</u>

$$F(s) = \frac{1 - e^{-s}}{s^2(1 - e^{-2s})} - \frac{e^{-s}}{s(1 - e^{-2s})}$$

- Find the inverse Laplace transform of

$$F(s) = \frac{1 - e^{-s}}{s(1 - e^{-2s})}$$

<u>Answer:</u> $f(t)$ has period $T = 2$

$$f_T(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & 1 \leq t < 2 \end{cases}$$

Summary

- The unit step function (or Heaviside function) and its translation:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \qquad u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

- The Laplace transform of u_c with $c \geq 0$ is

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s} \quad s > 0$$

- The Laplace transform of the shifted function

$$\mathcal{L}\{f_c(t)\} = \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

- If f is periodic with period T and is piecewise on $[0, T]$, then

$$\mathcal{L}\{f(t)\} = \frac{F_T(s)}{1 - e^{-sT}}, \qquad F_T(s) = \mathcal{L}\{f_T\} = \int_0^T e^{-st} f(t)dt$$

Homework

Homework:

- Section 5.5
  - 5, 9, 13, 23