Math 245 - Mathematics of Physics and Engineering I

Lecture 26. Solving Initial Value Problems with Laplace Transforms

March 21, 2012

Agenda

- General Scheme
- Examples

General Scheme

1. Using table of Laplace transforms and properties of the Laplace transform $\mathcal{L}$
   - linearity
   - $\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$
   - $\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$
   - etc.

   transform the IVP for a **linear** ODE with **constant coefficients** into an algebraic equation in the $s-$domain.
2. Find the Laplace transform $Y(s)$ of the solution by solving this algebraic equation.
3. Find the solution of the IVP $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ using partial fraction decompositions, the linearity of $\mathcal{L}^{-1}$, and a table of Laplace transforms.

Example

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}f(t)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a} \quad s > a$
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$t^p,\ p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2} \quad s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2} \quad s > a$
$t^n e^{at},\ n \in \mathbb{N}$	$\frac{n!}{(s-a)^{n+1}} \quad s > a$
$e^{at} f(t)$	$F(s-a)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$

Example: Find the solution of the IVP using Laplace transform:

$$y' + 2y = \sin 4t, \qquad y(0) = 1$$

Answer:

$$y(t) = \frac{6}{5}e^{-2t} - \frac{1}{5}\cos 4t + \frac{1}{10}\sin 4t$$

Example

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}f(t)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a} \quad s > a$
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Example: Find the solution of the IVP using Laplace transform:

$$y^{(4)} + 2y^{(2)} + y = 0, \quad y(0) = 1, \ y'(0) = -1, \ y''(0) = 0, \ y'''(0) = 2$$

Answer:

$$y(t) = (1 - \frac{1}{2}t) \cos t + \frac{1}{2}(t-1) \sin t$$

Example

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}f(t)$
e^{at}	$\frac{1}{s-a} \quad s > a$
$t^p,\ p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
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The Laplace transform method easily extends to systems of **linear** ODEs with **constant coefficients**.

Example: Solve the IVP:

$$\begin{cases} x'' + y' + 2x = 0, \\ 2x' - y' = \cos t. \end{cases} \qquad x(0) = 0, \quad x'(0) = 0, \quad y(0) = 0$$

Answer:

$$x(t) = \frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{-t}\cos t - \frac{3}{5}e^{-t}\sin t$$

$$y(t) = 2x(t) - \sin t$$

Homework

Homework:

- Section 5.4
  - 3, 11, 15