Lecture 14 : Classification of Phase Portraits

We study homogeneous autonomous systems : $x' = Ax$ (#)

Assume that A is nonsingular ($\det A \neq 0$) $\Rightarrow$ $x = 0$ is the unique equilibrium solution (critical point)

The characteristic equation of matrix A is

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \boxed{\lambda^2 - \operatorname{tr} A \cdot \lambda + \det A = 0} \qquad (*)$$

- trace $\operatorname{tr} A = a_{11} + a_{12}$
- $\det A = a_{11} a_{22} - a_{12} \cdot a_{21}$

The discriminant of (*) is

$$D = (\operatorname{tr} A)^2 - 4 \cdot \det A$$

Let us consider <u>3 cases</u> : $D > 0$, $D < 0$, $D = 0$

I. $D > 0$ $\Rightarrow$ A has two different real eigenvalues $\lambda_{1,2} = \dfrac{\operatorname{tr} A \pm \sqrt{D}}{2}$

The general solution of (#) is then $x(t) = C_1 e^{\lambda_1 t} \vartheta_1 + C_2 e^{\lambda_2 t} \vartheta_2$

eigenvector corresp. to λ_1 ; eigenvec. corresp. to λ_2

Let us consider 3 subcases :

(i) $\underline{0 < \lambda_1 < \lambda_2}$; $0 \quad \lambda_1 \quad \lambda_2$

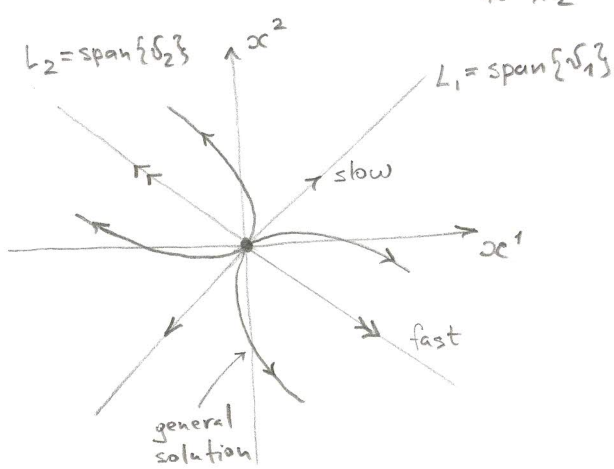
$\operatorname{tr} A > 0$
$\det A > 0$

$C_1 > 0 \Rightarrow x_1 = C_1 e^{\lambda_1 t} \vartheta_1 \to \infty$ in the direction on ϑ_1

$C_1 < 0 \Rightarrow x_1 = C_1 e^{\lambda_1 t} \vartheta_1 \to \infty$ in the direction of $-\vartheta_1$

$C_2 > 0 \Rightarrow x_2 = C_2 e^{\lambda_2 t} \vartheta_2 \to \infty$ in the direction of ϑ_2

$C_2 < 0 \Rightarrow x_2 = C_2 e^{\lambda_2 t} \vartheta_2 \to \infty$ in the direction of ϑ_2



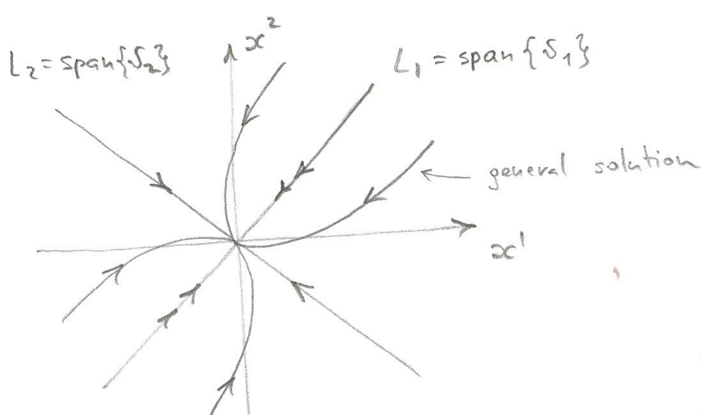
λ_1 → means "slow" escape to ∞
$\wedge$
λ_2 ↠ means "fast" escape to ∞

In this case, $x = 0$ is called <u>nodal source</u> (it is unstable)

(ii) $\underline{\lambda_1 < \lambda_2 < 0}$; $\lambda_1 \quad \lambda_2 \quad 0$

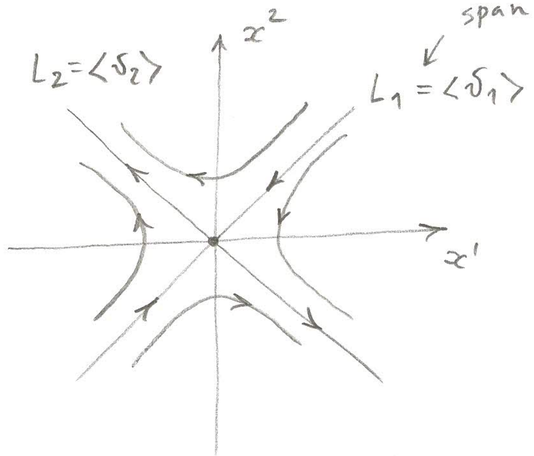
$\operatorname{tr} A < 0$
$\det A > 0$

In this case, $x = 0$ is called <u>nodal sink</u> (it is asymptotically stable)



→ "slow" convergence to 0
↠ "fast" convergence to 0

(iii) $\lambda_1 < 0 < \lambda_2$



det $A<0$

$\lambda_1 < 0 \Rightarrow x_1 = C_1 e^{\lambda_1 t} v_1 \to 0$

$\lambda_2 > 0 \Rightarrow x_2 = C_2 e^{\lambda_2 t} v_2 \to \infty$
in the direction $\pm v_2$

This is saddle point (unstable)

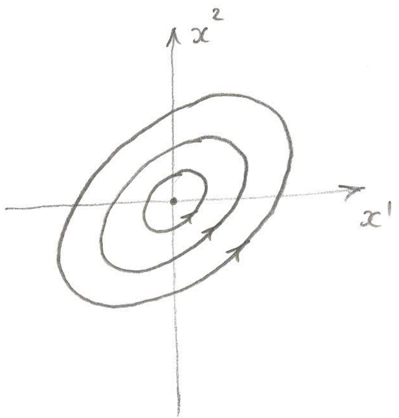
II. $D < 0 \Rightarrow$ A has two complex conjugate eigenvalues

$$\lambda_{1,2} = \frac{\operatorname{tr} A \pm \sqrt{D}}{2} = \underbrace{\frac{\operatorname{tr} A}{2}}_{\alpha} \pm i \underbrace{\sqrt{-D}}_{\beta} = \alpha \pm i\beta$$

If $v = a + ib$ is an eigenvector corresponding to $(\alpha + i\beta)$, then

the general solution: $x(t) = e^{\alpha t}\left[c_1(a\cos\beta t - b\sin\beta t) + c_2(a\sin\beta t + b\cos\beta t)\right]$

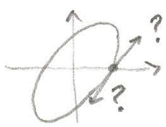
(i) $\alpha = 0 \Rightarrow$ all trajectories are periodic $\Leftrightarrow \operatorname{tr} A = 0$



This case is called center (stable)

How to find the direction of rotation?

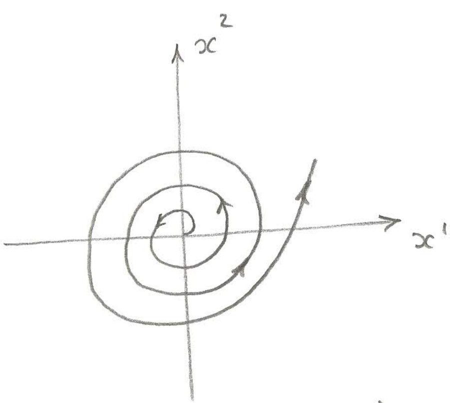
Easy: $\dot{x} = Ax \underset{\text{at } \binom{1}{0}}{\Rightarrow} \dot{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$



if $a_{21} > 0 \Rightarrow$  counterclockwise

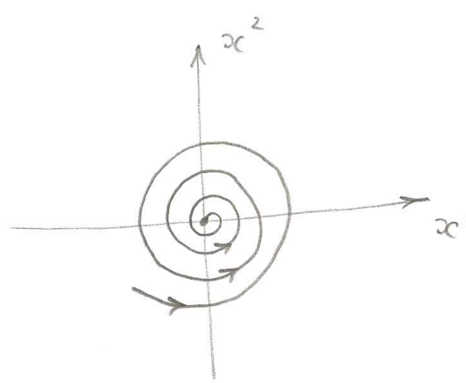
if $a_{21} < 0 \Rightarrow$  clockwise

(ii) $\alpha > 0 \Leftrightarrow \operatorname{tr} A > 0$



spiral source (unstable)

(iii) $\alpha < 0 \Leftrightarrow \operatorname{tr} A < 0$



spiral sink (asymptotically stable)

III. $D = 0 \Rightarrow A$ has a repeated real eigenvalue $\lambda = \frac{\operatorname{tr} A}{2}$

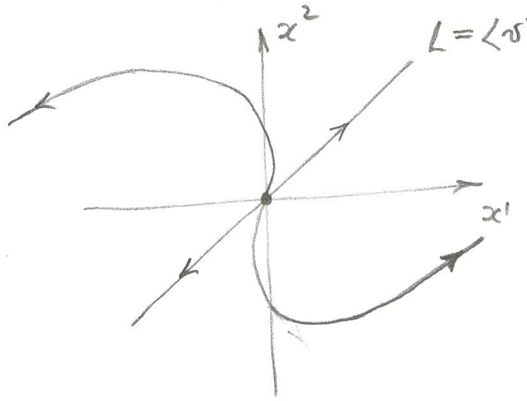
Consider the case when there is only one eigenvector v corresponding to λ.

The general solution in this case is (i.e. A is not diagonal)

$$x(t) = c_1 e^{\lambda t} v + c_2 e^{\lambda t}(w + tv)$$

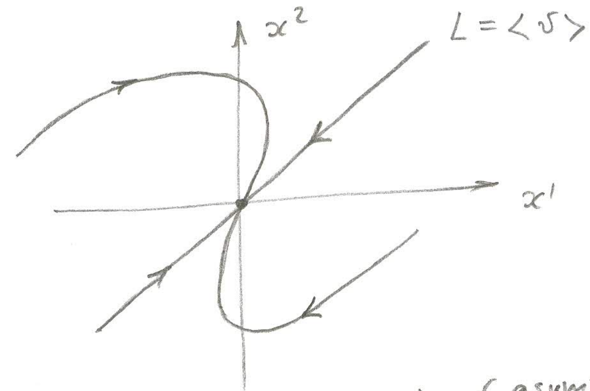
generalized eigenvector: $(A - \lambda I)w = v$

(i) $\lambda > 0 \iff \operatorname{tr} A > 0$



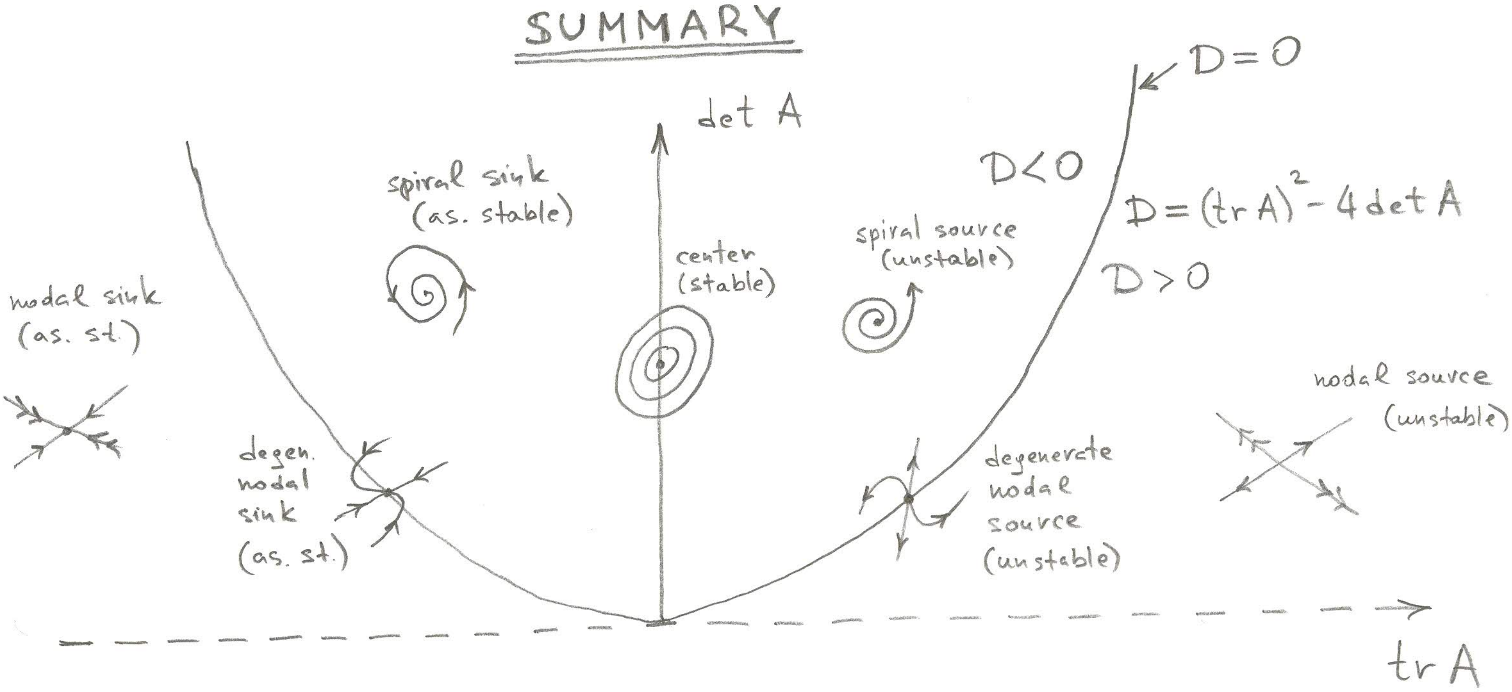
degenerate nodal source (unstable)

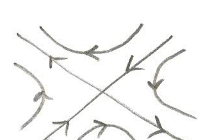
(ii) $\lambda < 0 \iff \operatorname{tr} A < 0$



degenerate nodal sink (asympt. stable)

SUMMARY





saddle point (unstable)

HW: Classify the phase portrait:

Sec. 3.3, # 10

Sec. 3.4, # 6

Sec. 3.5, # 4