Math 245 - Mathematics of Physics and Engineering I

Lecture 13. Homogeneous Autonomous Systems: Repeated Eigenvalues

February 8, 2012

Repeated Eigenvalues

We study homogeneous autonomous system:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

In Lecture 10 and 11, we learn how to solve this system when eigenvalues λ_1 and λ_2 of matrix $\mathbf{A}$ are real and different and complex conjugate, respectively.

The last (third) possibility for λ_1 and λ_2 is to be real and equal

$$\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$$

In this case there are two different possibilities for the corresponding eigenvectors:

- $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, i.e λ has two independent eigenvectors.
- $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent, i.e λ has only one independent eigenvector.

λ has 2 independent eigenvectors

It is easy to show that $\mathbf{A}$ has a repeated eigenvalue λ and two independent eigenvectors if and only if

$$\mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

In this case the general solution of

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \mathbf{x}$$

is given by

$$\mathbf{x} = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda t} \\ c_2 e^{\lambda t} \end{pmatrix}$$

λ has only one independent eigenvector

This is a more common case: matrix $\mathbf{A}$ is nondiagonal.
Let $\mathbf{v}$ be the only independent eigenvector that corresponds to λ. Then

$$\mathbf{x}_1 = e^{\lambda t}\mathbf{v} \tag{1}$$

is a solution of the system $d\mathbf{x}/dt = \mathbf{A}\mathbf{x}$. To find a fundamental set of solutions, we must find an additional solution. Let us look for another solution in the following form:

$$\mathbf{x}_2 = te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}, \tag{2}$$

where $\mathbf{w}$ is a vector to be determined. If (2) is a solution of the system, then $\mathbf{w}$ must satisfy

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v} \tag{3}$$

Definition

The vector $\mathbf{w}$ is called a generalized eigenvector corresponding to the eigenvalue λ.

- Linear algebra: (3) can be always solved for $\mathbf{w}$
- Wronskian $W[\mathbf{x}_1, \mathbf{x}_2] \neq 0 \Rightarrow \mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental set
- The general solution is then $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$

Examples

Find the general solution of the system

- $$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

- $$\mathbf{x}' = \begin{pmatrix} -1/2 & 1 \\ 0 & -1/2 \end{pmatrix} \mathbf{x}$$

Summary

- We study homogeneous autonomous system:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

with repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda$.

  - If $\mathbf{A}$ is diagonal, $\mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, then the general solution is given by

$$\mathbf{x} = \begin{pmatrix} c_1 e^{\lambda t} \\ c_2 e^{\lambda t} \end{pmatrix}$$

  - If $\mathbf{A}$ is nondiagonal, then a fundamental set of solution is forme by

$$\begin{aligned} \mathbf{x}_1 &= e^{\lambda t}\mathbf{v} \\ \mathbf{x}_2 &= te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w} \end{aligned}$$

    where

    - $\mathbf{v}$ is the only independent eigenvector corresponding to λ
    - $\mathbf{w}$ is the generalized eigenvector corresponding to λ, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v}$

Homework

<u>Homework:</u>

- Section 3.5
  - Find the general solution: 3, 5
  - Find the solution of the initial value problem: 9, 11