MINIMAL DEFINABLE GRAPHS WITH NO DEFINABLE TWO-COLORINGS

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ABSTRACT. We sketch our results about the structure of Borel graphs with Borel chromatic number at least three ordered by the relation of injective Borel homomorphism.

In [CMSV19] we have shown that a Borel graph has no Borel two-coloring if and only if it contains a continuous homomorphic copy of a graph called $L_0$. Now, we would like to consider the case when the homomorphism is required to be injective.

For all $n \in \mathbb{N}$, let $L_n$ denote the graph on $\{(0), \ldots, (n)\}$ with respect to which $(i)$ and $(j)$ are neighbors if and only if $|i-j| = 1$. Given $b \in \mathbb{N}^\mathbb{N}$ and $\bar{s} = (s_i,n)_{(i,n)\in 2\times \mathbb{N}}$ such that $s_i,n \in \bigcup_{m\leq n} \{0, \ldots, b(m)\} \times 2^{n-m}$ for all $(i,n) \in 2 \times \mathbb{N}$, define graphs $H_{b,\bar{s},n}$ on $\bigcup_{m\leq n} \{0, \ldots, b(m)\} \times 2^{n-m}$ by setting $H_{b,\bar{s},0} = L_{b(0)}$ and letting $H_{b,\bar{s},n+1}$ be the acyclic connected graph containing $\{(s_i \vee (j))_{i<2} \mid j < 2 \text{ and } (s_i)_{i<2} \in H_{b,\bar{s},n}\}$ and $L_{b(n+1)}$ in which $(s_0,n,0)$ is a neighbor of $(0)$, and $(b(n+1))$ is a neighbor of $(s_1,n,1)$. Set $X_b = \{(c,k,n) \in 2^\mathbb{N} \times \mathbb{N} \times \mathbb{N} \mid k \leq b(n)\}$, define $\pi_{b,n} \colon X_b \cap (2^\mathbb{N} \times \mathbb{N} \times \{0, \ldots, n\}) \rightarrow \bigcup_{m\leq n} \{0, \ldots, b(m)\} \times 2^{n-m}$ by $\pi_{b,n}(c,k,m) = (k) \vee c \upharpoonright (n-m)$ for all $n \in \mathbb{N}$, and let $\mathbb{H}_{b,\bar{s}}$ be the digraph on $X_b$ consisting of all pairs of the form $((c_i,k_i,n_i))_{i<2}$ such that $\pi_{b,n}(c_i,k_i,n_i))_{i<2} \in H_{b,\bar{s},n}$ and $\forall m \geq n \ a_0(m) = c_1(m)$, where $n = \max(n_0,n_1)$.

A tower over the canonical undirectable forest of lines is a graph of the form $L_b = \mathbb{H}_{b,\bar{s}}$, where $b \in \{0\} \times \mathbb{N}^\mathbb{N}$ and $\bar{s} = (s_i,n)_{(i,n)\in 2\times \mathbb{N}}$ is given by $s_{i,0} = (0,i)$ and $s_{i,n} = (0)^n \vee (1) \vee (i)$ for all $i < 2$ and $n > 0$. A straightforward Baire category argument shows that if $b \in \{0\} \times (2\mathbb{N}+1)^\mathbb{N}$, then $L_b$ does not have a Borel two-coloring. Note that $L_0$ is the graph $L_b$, where $b(n) = 2n-1$, for $n > 0$.

In order to give basis results for quasi-orders substantially stronger than homomorphism, we must introduce two more types of graphs.

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The symmetrization of the graph of a tower over the odometer is a graph of the form $\Sigma_b = \mathbb{H}_{b,\bar{s}}$, where $b \in \{0\} \times \mathbb{N}$ and $\bar{s} = (s_{i,n})_{(i,n)\in2\times\mathbb{N}}$ is given by $s_{0,n} = (0) \ast (1)^n \ast (0)$ and $s_{1,n} = (0) \ast (0)^n \ast (1)$. A straightforward Baire category argument shows that if $b$ has alternating parity, then $\Sigma_b$ does not have a Borel two-coloring.

The symmetrization of the graph of a tower over the shift on $[\mathbb{N}]^{\aleph_0}$ is a graph of the form $S_{b,c} = \mathbb{H}_{b,\bar{s}}$, where $b \in \{0\} \times \mathbb{N}$, $c \in \prod_{n \in \mathbb{N}} \{0, \ldots, b(n)\}$, and $\bar{s} = (s_{i,n})_{(i,n)\in2\times\mathbb{N}}$ is given by $s_{i,0} = (0, i)$ and $s_{i,n+1} = (c(n), i)$ for all $i < 2$ and $n \in \mathbb{N}$. A straightforward Baire category argument shows again that if $b \in \{0\} \times (2\mathbb{N} + 1)^\mathbb{N}$, then $\Sigma_b$ does not have a Borel two-coloring.

**Theorem 1.** Suppose that $X$ is a Hausdorff space and $G$ is an analytic graph on $X$ with no Borel two-coloring. Then at least one of the following holds:

1. The graph $G$ contains an odd cycle.
2. There exists $b \in \{0\} \times \mathbb{N}$ of alternating parity for which there is an injective continuous homomorphism from $\Sigma_b$ into $G$.
3. There exists $b \in \{0\} \times (2\mathbb{N} + 1)^\mathbb{N}$ for which there is an injective continuous homomorphism from $\mathbb{L}_b$ into $G$.
4. There exist $b \in \{0\} \times (2\mathbb{N} + 1)^\mathbb{N}$ and $c \in \prod_{n \in \mathbb{N}} \{0, \ldots, b(n)\}$ for which there is an injective continuous homomorphism from $S_{b,c}$ into $G$.

Moreover, if $G$ is acyclic, then injective continuous homomorphism can be strengthened to continuous embedding.

**Proof (Sketch).** Using category- and measure-theoretic arguments, find a locally countable acyclic Borel subgraph of $G$ with Borel chromatic number at least three for which there is a Borel way of selecting one or two ends from each connected component (see e.g. [Mil09]). A case-by-case analysis of the subgraph then yields the theorem. \hfill \Box

This result is sharp, in the sense that there are continuum-sized families of graphs of each of the three types (i.e., $\Sigma_b$, $\mathbb{L}_b$, and $S_{b,c}$) such that none has a Borel two-coloring, but every analytic graph on a Hausdorff space that admits an injective Borel homomorphism into at least two of them has a Borel two-coloring.

**References**

[CMSV19] Raphael Carroy, Benjamin D. Miller, David Schrittesser, and Zoltan Vidnyanszky, *Minimal definable graphs of definable chromatic number at least three*, Available at


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