Practice Problems
MATH 3260

Definitions. planar graph, homeomorphic graphs, contractibility, dual graph, the Ramsey numbers $R(s,t)$, chromatic polynomials; vertex, edge and face colorability, chromatic numbers, χ and χ'; cubic graph, complete matching, partial matching, Hall's condition, augmenting path.

Theorems. Euler's formula, Kuratowski's theorem, estimation on the number of edges of a connected planar graphs and planar graphs without triangles, duality: relation between the graph and its dual, Ramsey's theorem (finite and infinite version), characterization of 2-colorability-(f) and 3-colorability-(f) of cubic graphs; König's theorem on edge colorings; Hall's theorem; Megner's theorem (if covered on Thursday).

1. Calculate the chromatic polynomial of the following graphs:

   (a) the complete graph K_6,

   (b) the complete bipartite graph $K_{1,6}$.

   Solution. We have proved on the lecture that the chromatic polynomial of a complete graph on n vertices is $k(k-1)\cdot \ldots \cdot (k-n+1)$, thus the chromatic polynomial of K_6 is $k(k-1)(k-2)(k-3)(k-4)(k-5)$.
   Note that $K_{1,6}$ is a tree on 7 vertices. Hence its chromatic polynomial is $k(k-1)^6$.

2. Three mathematicians, A, B, C got 4 cakes with numbers $2, 3, 7, 11$. A likes the numbers 2 and 3, B's favourites are $2, 7$ and C's favourites are $2, 3$ and 11. Is it possible to give each person a cake with a number the person likes? Draw the corresponding bipartite graph, give an example of a complete matching from the set of the mathematicians to the cakes and check Hall's condition!

   Solution. In order to check Hall's condition, we have to compare the size of every subset of $\{A, B, C\}$ to the size of the set of its neighbors.

$$N_G(\{A\}) = \{2,3\}, \text{so } |N_G(\{A\})| = 2 \geq 1 = |\{A\}|$$
$$N_G(\{B\}) = \{2,7\}, \text{so } |N_G(\{B\})| = 2 \geq 1 = |\{B\}|$$
$$N_G(\{C\}) = \{2,3,11\}, \text{so } |N_G(\{C\})| = 3 \geq 1 = |\{C\}|$$
$$N_G(\{A,B\}) = \{2,3,7\}, \text{so } |N_G(\{A,B\})| = 3 \geq 2 = |\{A,B\}|$$
$$N_G(\{A,C\}) = \{2,3,11\}, \text{so } |N_G(\{A,C\})| = 3 \geq 2 = |\{A,C\}|$$
$$N_G(\{B,C\}) = \{2,3,7,11\}, \text{so } |N_G(\{B,C\})| = 4 \geq 2 = |\{B,C\}|$$
$$N_G(\{A,B,C\}) = \{2,3,7,11\}, \text{so } |N_G(\{A,B,C\})| = 4 \geq 3 = |\{A,B,C\}|$$

   Which shows that Hall's condition holds. A possible complete matching is A to 2, B to 7 and C to 11.

3. Let G be a graph with components $C_1, \ldots, C_k$. Calculate $P_G(n)$ from the polynomials $P_{C_i}(n)$ for $1 \leq i \leq k$.

   Solution. Since the colors of the different components can be assigned independently, the chromatic $P_G(n)$ is just the product of the polynomials $P_{C_i}(n)$.

4. Is it possible to give

   (a) five

   (b) six

integers, such that among any three of them there are two with greatest common divisor greater than 1 and there are two which are co-prime (i. e. their greatest common divisor is 1)?

Solution. It is not hard to check that the numbers $6, 15, 35, 77$ and 22 answer the first question affirmatively.

The answer to the second question is no: suppose it is, and label the edges of the complete graph on 6 vertices with the numbers. Color an edge red if the corresponding numbers are co-prime, and color an edge blue otherwise. By the fact that $R(3,3) = 6$ we get that this graph must contain a monochromatic triangle. This contradicts our assumption.

5. Let G be a simple graph with n vertices and m edges. Show by induction that the coefficient of k^{n-1} is $-m$.

   Solution. We show by induction on the number of edges m that the chromatic polynomial of a graph G with n vertices and m edges has the form $k^n - mk^{n-1} + \dots$.

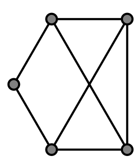
   If $m = 0$ then G has no edges, so its chromatic polynomial is k^n, which shows the statement.

   Suppose that we have proved the statement for every graph with at most m edges. Let G be a graph with $m+1$ edges and n vertices, and let e be any edge of G. We have to prove that the chromatic polynomial of G has the form $k^n - (m+1)k^{n-1} + \dots$. If G has only one vertex, the statement is trivial. Otherwise, the graph $G \setminus e$ has $n-1$ vertices and m edges, while the graph $G - e$ has n vertices and m edges. By the formula proved on the lecture and the inductive hypothesis used for $G \setminus e$ and $G - e$ we have

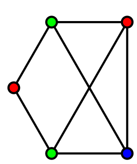
   $$P_G(k) = P_{G-e}(k) - P_{G\setminus e}(k) = k^n - mk^{n-1} + \dots - (k^{n-1} - mk^{n-2} + \dots) = k^n - (m+1)k^{n-1} + \dots.$$

   This shows the statement.

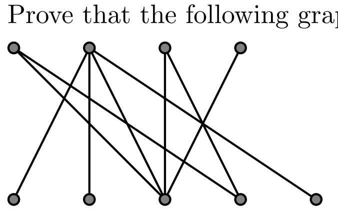
6. Find the edge chromatic number of the following graph:

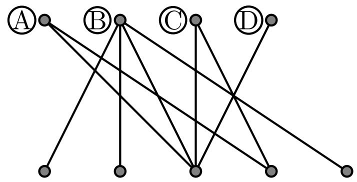
   Solution. One can easily check that the below assignment of 3 colors is a good coloring, hence the chromatic number of the graph is ≤ 3. However, the graph contains a triangle, thus it is impossible to color it with to colors. Consequently, the chromatic number of the graph is 3.

7. Prove that the following graph has no complete matching from the upper part to the lower part.

   Solution. Consider the below labelling of the vertices.



$|N_G(\{A, C, D\})| = 2 < 3 = |\{A, C, D\}|$, so the set $\{A, C, D\}$ violates Hall's condition. Thus, by Hall's theorem there is no complete matching in the graph.

8. In a company there are workers and jobs to do. Some workers are able to do some jobs, we know that each job requires at least two workers, and one worker can be assigned only to at most one job. Give a necessary and sufficient condition on when is it possible to assign all the jobs!

   Solution. We claim that there exists such an assignment if and only if every set of jobs can be performed by at least twice as many workers as the size of the set, or more formally, in the bipartite graph $G(V_1, V_2)$ corresponding to the jobs V_1, and the workers V_2, with two vertices adjacent if the corresponding worker is able to do the corresponding job, we have that for every $X \subset V_1$ the condition $|N_G(X)| \geq 2|X|$ holds.

   ($\Rightarrow$) Suppose that there is such an assignment. For every $X \subset V_1$ we need at least $2|X|$ many workers, hence necessarily $|N_G(X)| \geq 2|X|$.

   ($\Leftarrow$) Now suppose that for every $X \subset V_1$ the condition $|N_G(X)| \geq 2|X|$ holds. Duplicate every vertex in V_1 and connect the duplicated vertices to the neighbors of the original vertex. Denote the new graph by $G' = G'(V_1', V_2)$. Clearly, as the size of every subset of V_1 has been duplicated, we obtain that our condition implies that for every $X \subset V_1'$ the inequality $|N_{G'}(X)| \geq |X|$ holds. In other words, G' satisfies Hall's condition, so by Hall's theorem it has a complete matching from V_1' to V_2. It is not hard to see that "gluing back together" the duplicates, the complete matching will now turn to an assignment of 2 vertices of V_2 to each vertex of V_1, showing our claim.

9. Prove that the chromatic polynomial of an n-long cycle is $(k-1)^n + (-1)^n(k-1)$.

   Solution. The claim can be proved by induction on the length of the cycle. If the cycle has length 3, that is, it is a triangle, we have proved the statement.

   Now, suppose that we have the claim for the cylces of length at most n and consider a cycle C_{n+1} of length C_{n+1}. Erasing any edge from the cycle, using the fact that the chromatic polynomial of a tree on n vertices is $k(k-1)^{n-1}$, and noting that $C_{n+1} \setminus e = C_n$ we get

$$P_{C_{n+1}}(k) = P_{C_{n+1}-e}(k) - P_{C_n}(k) = k(k-1)^n - ((k-1)^n + (-1)^n(k-1)) =$$
$$= (k-1)^{n+1} + (-1)^{n+1}(k-1),$$

   which shows that the induction can be carried out.

10. Suppose that the edges of the graph K_{17} are colored with 3 colors. Show that there exists a monochromatic triangle!

    Solution. Suppose that we color with colors R, B and G. Pick any vertex. As this vertex has 16 edges incident to it, there will be 6 having the same color (say, color R). Consider the 6 vertices corresponding to these edges. If any two of them are connected with an edge colored R, then graph contains an R triangle. If not, then all the edges between them are colored with B or G. Thus, we have a coloring of K_6 with 2 colors. We have shown that for any such a coloring, K_6 contains a monochromatic triangle, which is a monochromatic triangle in the original graph as well.

11. A graph is called k-critical if $\chi(G) = k$ and the deletion of any vertex decreases the chromatic number of G.

(a) Find all 2-critical and 3-critical graphs.

(b) Prove that if G is k-critical then every vertex of G has degree at least $k-1$.

Solution.

(a) If G is a graph note that $\chi(G) = 1$ if and only if G has no edges, while $\chi(G) = 2$ if and only if G is bipartite.

Hence, if we delete any vertex from a 2-critical graph, we obtain a graph without edges. This is only possible if the graph consists of only two vertices and one edge, so the only 2-critical graph is K_2.

Now, if G is 3-critical then G is not bipartite, but deleting any vertex it becomes a bipartite graph. In other words, for every vertex v of G the graph $G \setminus \{v\}$ contains no odd cycles, but G contains one. Consider an odd cycle C_k of G of the smallest length. Any vertex of G is on this odd cycle (otherwise, deleting the vertex, the cycle would remain in the graph.) Finally, note that if any two non-consequtive were adjacent in C_k, then the graph G would contain an odd cycle of a shorter length, contradicting the choice of C_k. Thus $G = C_k$.

(b) Suppose that a vertex v of G has degree $\leq k-2$. Since G is k-critical, deleting v results in a graph with chromatic number less than k. Thus, $G \setminus \{v\}$ has a $k-1$ coloring, so color the vertices of $G \setminus \{v\}$ with $k-1$ colors. Since v has at most $k-2$ many neighbors, at least one color doesn't appear as the color of the neighbors of v. Coloring v with one of the missing colors, we obtain a $k-1$ coloring of G, contradicting the assumption that $\chi(G) = k$.

Problem. (Discussed on the lecture) Suppose that a cubic planar graph has fewer than 12 many faces. Show that one of the faces has at most four edges in its boundary.

Solution. (By Jordan Teitelbaum) We argue by contradiction. Let n, m and f be the number of vertices, edges and faces of such a graph and let f_k be the number of faces with exactly k edges in their boundary.

Clearly $f = f_5 + f_6 + \dots$. Moreover,

$$2m = 5f_5 + 6f_6 + 7f_7 + \dots,$$

and by the graph being cubic, we get $3n = 2m$. Using Euler's formula, we get

$$2 = n - m + f = \frac{1}{3}(5f_5 + 6f_6 + 7f_7 + \dots) - \frac{1}{2}(5f_5 + 6f_6 + 7f_7 + \dots) + f_5 + f_6 + \dots,$$

so

$$\begin{aligned} 12 &= 2(5f_5 + 6f_6 + 7f_7 + \dots) - 3(5f_5 + 6f_6 + 7f_7 + \dots) + 6(f_5 + f_6 + f_7 + \dots) = \\ &= f_5 - f_7 - 2f_8 - \dots, \end{aligned}$$

which shows that $f_5 \geq 12$, contradicting our assumption.

Remark. If we allow exactly 12 faces (as I posed the problem), the statement becomes false: the stereographic projection of a dodecahedron is a cubic planar graph with 12 faces, all faces having exactly 5 edges in their boundary.