Practice Problems
MATH 3260

Definitions. graph, digraph, simple graph, subgraph, isomorphism, tournament, adjacency and incidence matrix, connected graph, connected component, bridge (cut edge), cut set, walk, trail, path, cycle, degree, in-degree, out-degree, end vertex, isolated vertex, bipartite graph, K_n, $K_{n,m}$, Eulerian walk, Eulerian graph, semi-Eulerian graph, Hamiltonian path, Hamiltonian graph, tree, spanning tree, Prüfer code, planar graph;

Theorems. Handshaking lemma/dilemma, characterization of bipartite graphs, König's lemma, characterization of Eulerian graphs, Ore's theorem on Hamiltonian graphs, Dijkstra's algorithm, equivalent descriptions of trees, Cayley's theorem, Euler's formula, Kuratowski's theorem.

1. (a) Draw a graph and calculate its incidence and adjacency matrices.

   (b) Draw the graph with adjacency matrix

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   How many connected components does this graph have? Is it bipartite? Is it Eulerian?

2. Show that the graph in the second part of the previous problem is planar by drawing it in the plane. Verify Euler's formula for the drawing.

3. Is the Petersen graph planar?

4. Give an example of a locally finite, connected graph with infinitely many end vertices.

5. Let T be a tree. We know that each vertex of T has degree a or degree b. We also know that 9 vertices have degree a, while 92 vertices have degree b. Calculate a and b.

6. Suppose that G is a connected planar graph with n vertices and $m \geq 5$ edges such that every cycle in G has at least 5 edges. Show that $m \leq \frac{5}{3}(n-2)$.

7. Draw the labelled tree with Prüfer code $(3, 3, 4, 8, 6, 8)$.

8. Let G be the graph with vertices $\{1, \ldots, 100\}$, distinct vertices i and j are connected if $|i-j|$ is a multiple of 3. Is G connected? If not, how many componets are there?

9. Suppose that G is a simple graph on n vertices such that

   (a) the degree of each vertex is at least $\frac{n-1}{2}$

   (b) for any two non-adjacent vertices v and w we have $deg(v) + deg(w) \geq n-1$. Show that G is connected.

10. Suppose that G is a simple graph on n vertices such that for any two non-adjacent vertices v and w we have $deg(v) + deg(w) \geq n-1$. Show that G is connected.

11. Let G be a connected simple graph with at least two vertices, and v be a vertex of G. Let G' be the graph obtained from G by deleting the vertex v (and all edges incident with it). Show that if G' is connected then G has a spanning tree T such that v is an end vertex of T.

12. In a gathering of six people some of them shake hands with each other. Prove that either there are three people none of whom shook hands or three people all of whom shook hands with each other.

13. In a tournament 8 teams participated. Each team played against each other team once and there were no draws. Is it possible that 3 teams won twice, 4 teams won 3 times and one team won 7 times?