MATH3260 Introduction to Graph Theory
Midterm Examination
Solutions

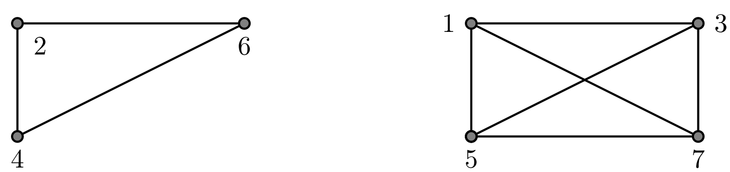
1. Define the notion of a

   (a) Hamiltonian graph
   Answer. A Hamiltonian graph is a graph which has a Hamiltonian cycle, that is, a cycle that contains each vertex exactly once.

   (b) degree of a vertex
   Answer. The degree of a vertex v is the number of edges adjacent to v, loops are counted twice.

   (c) tournament
   Answer. A tournament is a digraph, where every pair of distinct vertices is connected by exactly one directed edge (a tournament contains no loops).

2. Let G be the graph with vertices labelled by $\{1, 2, 3, \ldots, 7\}$, two distinct vertices i and j are adjacent if $|i-j|$ is even. Draw the graph G and give the adjacency matrix of G. How many connected components does G have?
   Solution.



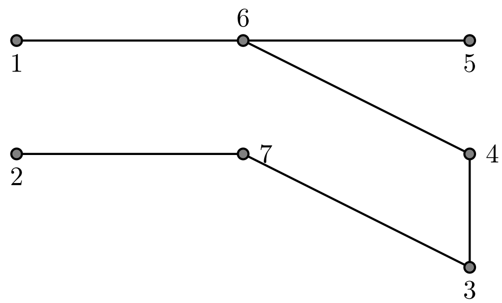
As the drawing shows, two distinct vertices are adjacent if and only if the parity of their labels is the same. Hence there are two connected components, namely the vertices with even and the vertices with odd labels.

The adjacency matrix of the graph (where the nth row and coloumn corresponds to the vertex labelled by n) is

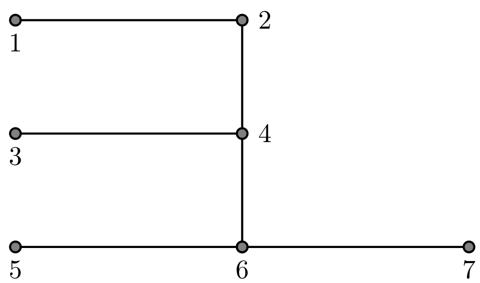
$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Remark. Since the wording of the problem was somewhat misleading, instead of the above graph (and matrix) the same graph with a loop at each vertex (and its adjacency matrix) is also accepted as a correct solution.

3. (a) Draw the labelled tree which has the Prüfer code $(6, 7, 6, 4, 3)$.
   Solution. We follow the algorithm outlined in the lecture:
   The minimal label not present in the list is 1, hence 1 is adjacent to 6. We erase 6 and 1 is not considered anymore.
   The minimal label not present in the list is and still to be considered is 2, hence 2 is adjacent to 7. We erase 7 and 2 is not considered anymore.
   Similarly, 5 is adjacent to 6, 6 is adjacent to 4, 4 is adjacent to 3 and finally 3 is adjacent to 7.

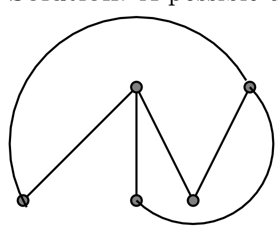


(b) Give the Prüfer code of the following tree:



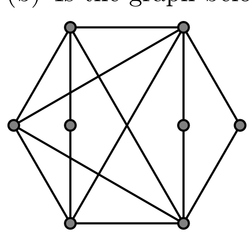
Solution. The end vertex with the minimal label is 1, it is adjacent to 2, so the code starts with 2. We erase the vertex labelled by 1.
The end vertex with the minimal label is 2, it is adjacent to 4, so the code continues with 4. We erase 2.
Etc., this yields the code $(2, 4, 4, 6, 6)$.

4. (a) Show that the graph $K_{3,2}$ is planar by drawing it in the plane without intersecting the edges. Verify Euler's formula for the drawing.
Solution. A possible drawing of $K_{3,2}$ is the following:

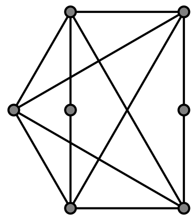


The drawing has 5 vertices, 6 edges and 3 faces, so Euler's formula is indeed true in this case: $5 - 6 + 3 = 2$.

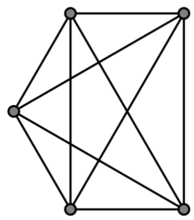
(b) Is the graph below planar?



Solution. The graph contains a subgraph of the following form:



Removing two vertices from this graph, we get a copy of K_5:



Thus, the original graph contains a homeomorphic copy of K_5, so by Kuratowski's theorem it can't be planar.

5. Let G be a connected graph such that the longest path in G contains n vertices. Suppose that P_1 and P_2 are paths of maximal length (i. e., of length n). Show that P_1 and P_2 intersect, that is, they contain a common vertex!
Solution. Suppose that P_1 and P_2 does not intersect. Let $v_1^1, \dots, v_n^1$ be the vertices of P_1 and $v_1^2, \dots, v_n^2$ be the vertices of P_2. By the connectivity of the graph there exists a path P_3 of minimal length, which connects a vertex of P_1 (say, v_i^1) to a vertex of P_2 (say, v_j^1).

   Suppose that there are at least as many edges in P_1 before v_i^1 and in P_2 before v_j^2 as after them (the other cases are similar). By the minimality of P_3 it intersects P_1 in only one vertex (and similarly for P_2). Then the path obtained from gluing the first part of P_1, P_3 and the first part of P_3, that is, the path $v_1^1, \dots, v_i^1, P_3, v_j^2, v_{j-1}^2, \dots, v_1^2$ has at least one more edge then P_1 (or P_2) contradicting the assumption on the maximality of P_1.

6. Let G be a connected simple graph and e an edge of G. Suppose that every spanning tree of G contains e. Show that e is a bridge.
Solution. Suppose that it is not the case. Then, since e is not a bridge after erasing e the graph (denote it by G') remains connected. We proved that every connected graph contains a spanning tree, so let T be a spanning tree of G'. Clearly, T is also a spanning tree of G. But T does not contain e, which contradicts the assumption.

7. Is it possible to draw 9 rectangles in the plane such that each of them intersects either 1 or 3 other rectangles?
Solution. No. Suppose the contrary, i. e., there exists such a drawing. Consider the simple graph where each rectangle corresponds to a vertex and two vertices are adjacent if the corresponding rectangles intersect. By the assumption, in this graph every vertex has degree either 1 or 3. But there are nine vertices and according to a consequence of the Handshaking Lemma (proved in the lecture) in a simple graph the number of vertices with odd degrees is even, a contradiction.