MATH3260 Introduction to Graph Theory
Midterm Examination
Solutions

1. Define the notion of a

   (a) complete matching from V_1 to V_2 in a bipartite graph $G(V_1, V_2)$,

   (b) Ramsey number $R(s,t)$,

   (c) (vertex) chromatic number of a simple graph.

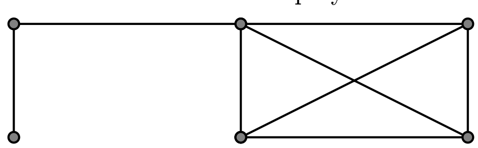
   Solution.

   (a) A complete matching from V_1 to V_2 is a one-to-one correspondence between all vertices of V_1 and some vertices of V_2 such that the corresponding vertices are adjacent.

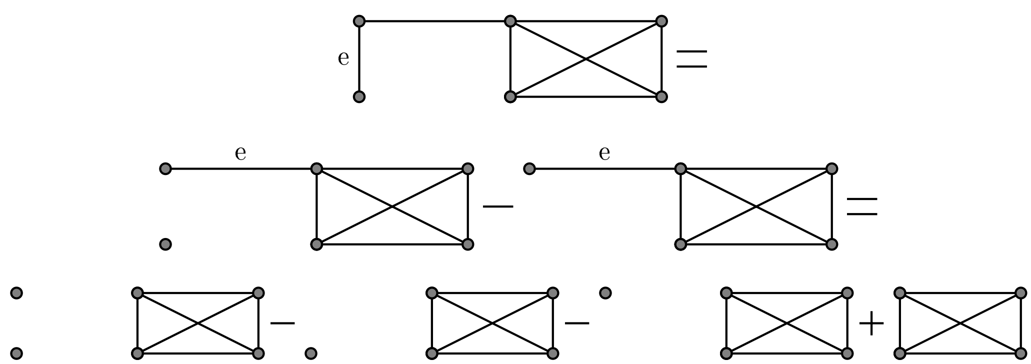
   (b) The Ramsey number $R(s,t)$ (let us denote it by n) is the minimal number n such that for every colouring of the edges of the graph K_n with two colours, K_n will contain a subgraph K_s with all edges coloured with the first colour or a subgraph K_t with all edges coloured with the second colour.

   (c) The chromatic number of a graph G is the minimal number k such that the vertices of G can be coloured with k colours in such a way that no two adjacent vertices have the same colour.

2. Calculate the chromatic polynomial of the below graph:



Solution. Using the formula on the chromatic polynomial of and the usual conventions, one gets the following "equations":



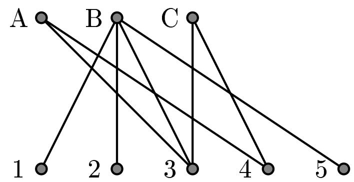
From which, using the fact that the components of the above graphs are complete graphs or 1 or 4 vertices we obtain that the chromatic polynomial is:

$$k\cdot k\cdot k(k-1)(k-2)(k-3) - k\cdot k(k-1)(k-2)(k-3) - k\cdot k(k-1)(k-2)(k-3) + k(k-1)(k-2)(k-3) =$$

$$= k(k-1)(k-2)(k-3)\cdot(k^2-2k+1) = k(k-1)^3(k-2)(k-3).$$

Alternatively, one can deduce the same result by calculating the polynomial directly, from the number of the possible colours.

3. Check whether Hall's condition holds for the below graph (by checking it for every subset of the upper part). In case it does, find a complete matching from the upper part to the lower part.



Solution. In order to check Hall's condition, we have to compare the size of every subset of $\{A, B, C\}$ to the size of the set of its neighbours.

$$N_G(\{A\}) = \{3, 4\}, \text{ so } |N_G(\{A\})| = 2 \geq 1 = |\{A\}|$$
$$N_G(\{B\}) = \{1, 2, 3, 5\}, \text{ so } |N_G(\{B\})| = 4 \geq 1 = |\{B\}|$$
$$N_G(\{C\}) = \{3, 4\}, \text{ so } |N_G(\{C\})| = 2 \geq 1 = |\{C\}|$$
$$N_G(\{A, B\}) = \{1, 2, 3, 4, 5\}, \text{ so } |N_G(\{A, B\})| = 5 \geq 2 = |\{A, B\}|$$
$$N_G(\{A, C\}) = \{3, 4, 5\}, \text{so } |N_G(\{A, C\})| = 2 \geq 2 = |\{A, C\}|$$
$$N_G(\{B, C\}) = \{1, 2, 3, 4, 5\}, \text{so } |N_G(\{B, C\})| = 5 \geq 2 = |\{B, C\}|$$
$$N_G(\{A, B, C\}) = \{1, 2, 3, 4, 5\}, \text{so } |N_G(\{A, B, C\})| = 5 \geq 3 = |\{A, B, C\}|$$

Which shows that Hall's condition holds. A possible complete matching is A to 3, B to 4 and C to 1.

4. Ladies and gentlemen participate on a ball. Some ladies wish to waltz with some men. We know that no lady wishes to dance with more than k many gentlemen and no gentleman is liked by more than k many ladies. Show that the pairs of dancers can be arranged in such a way that during k waltzes every lady dances with every gentleman whom she likes.

   Solution. Let $G = G(V_1, V_2)$ be the bipartite graph where every vertex in V_1 represents a lady, while every vertex in V_2 represents a gentleman, and two vertices are adjacent if the corresponding lady wishes to dance with the corresponding gentleman. Note that the assumption of the problem means that in this graph every degree is $\leq k$. By König's theorem such graph's have edge chromatic number $\leq k$. A good edge colouring with $\leq k$ many colours gives an arrangement: the pairs of the ith waltz should be the participants whose corresponding vertices are adjacent and coloured by the ith colour. Since at every vertex the edges have different colours, this is a good arrangement.

5. Show that if $k \geq 1$ and $G(V_1, V_2)$ is a k-regular bipartite graph (that is, every vertex in G has degree exactly k) then there exists a complete matching from V_1 to V_2.

   Solution. Suppose the contrary. Since there is no complete matching, Hall's condition must fail for the graph, that is, there exists a set $A \subset V_1$ such that $|N_G(A)| < |A|$.

   By the k-regularity, the number of edges with one endpoint in A is exactly $k|A|$. Since the other endpoint of these edges are in $N_G(A)$, the number of edges with one endpoint in $N_G(A)$ is at least $k|A|$. But, on the one hand $k|A| > k|N_G(A)|$, on the other hand, by the k-regularity the total number of edges leaving $|N_G(A)|$ is $k|N_G(A)|$ so $k|A| \leq k|N_G(A)|$, a contradiction.

6. Show that the edge chromatic number, χ', of a Hamiltonian cubic graph is 3.

   Solution. Let G be such a graph. Clearly, $\chi'(G) \geq 3$, as every vertex is incident with 3 edges and the colours of those must be different.

   Now we give a good 3-colouring of the edges of G. First note that, by the Handshaking Lemma, a cubic graph has evenly many vertices. Our graph is Hamiltonian, it contains a Hamiltonian cycle C and, since the number of the edges of the graph is even, the length of C is even as well.

Colour the edges of the cycle with red and blue alternatingly and erase the coloured edges. In the remaining graph every degree is exactly one, hence the graph consists of non-incident edges. Colour those edges with green. Now consider the original graph, G. Clearly, at every vertex of the graph two edges of C and one edge outside of C meet. By the fact that the length of C is even we obtain that at every vertex the edges have pairwise different colours.

7. Prove that among any six irrational numbers there exist 3 such that the sum of any two of them is irrational.

   Solution. Label the edges of the complete graph on 6 vertices with the numbers. Color an edge red if the sum of the corresponding numbers is rational, and colour an edge blue otherwise. By the fact that $R(3,3) = 6$ we get that this graph must contain a monochromatic triangle. If the triangle is blue, we are done.

   So suppose that the graph contains a red triangle. This means, that we have 3 irrational numbers a, b, c such that $a + b, a + c$ and $b + c$ are all rational. But then $a + b + a + c + b + c = 2(a + b + c)$ is rational and consequently $a + b + c$ is rational as well. So $a + b + c - (b + c) = a$ is also rational, contradicting our assumption. Thus, the graph cannot contain a red triangle.