On Homomorphism Graphs

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Abstract

We introduce a new type of examples of bounded degree acyclic Borel graphs and study their combinatorial properties in the context of descriptive combinatorics, using a generalization of the determinacy method of Marks [Mar16]. The motivation for the construction comes from the adaptation of this method to the LOCAL model of distributed computing [BCG+21]. Our approach unifies the previous results in the area, as well as produces new ones. In particular, we show that for $\Delta > 2$ it is impossible to give a simple characterization of acyclic Δ-regular Borel graphs with Borel chromatic number at most Δ: such graphs form a $\mathbf{\Sigma}^1_2$-complete set. This implies a strong failure of Brooks'-like theorems in the Borel context.

1 Introduction

Descriptive combinatorics is an area concerned with the investigation of combinatorial problems on infinite graphs that satisfy additional regularity properties (see, e.g., [Pik21, KM20] for surveys of the most important results). In recent years, the study of such problems revealed a deep connection to other areas of mathematics and computer science. The most relevant to our study are the connections with the so-called LOCAL model from the area of distributed computing. There are several recent results that use distributed computing techniques in order to get results either in descriptive combinatorics [Ber20, Ber21, BCG+21, Ele18, GR21a], or in the theory of random processes [HSW17, GR21b].

The starting point of our work was the investigation of the opposite direction. Namely, our aim was to adapt the celebrated *determinacy technique* of Marks [Mar16] to the LOCAL model of distributed computing. In order to perform the adaptation (which is indeed possible, see our conference paper [BCG+21][1]), we had to circumvent several technical hurdles that, rather surprisingly, lead to the main objects that we study in this paper, *homomorphism graphs* (defined in Section 3).

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[1] The connection between the current paper and the conference paper [BCG+21] is the following. The latter paper builds a theory of local problems on trees from several perspectives and aims to a broader audience. The original

We refer the reader to [BCG$^+$21] for a detailed discussion of the concepts and their connections to the LOCAL model.

Before we state our results, we recall several basic notions and facts. A *graph* G on a set X is a symmetric subset of $X^2 \setminus \{(x,x) : x \in X\}$. We will refer to X as the *vertex set of* G, in symbols $V(G)$, and to G as the *edge set.* If $n \in \{1, 2, \dots, \aleph_0\}$, an *$n$-coloring of* G is a mapping $c : V(G) \to n$ such that $(x,y) \in G \implies c(x) \neq c(y)$. The chromatic number of G, $\chi(G)$ is the minimal n for which an n-coloring exists. If G and H are graphs, a *homomorphism* from G to H is a mapping $c : V(G) \to V(H)$ that preserves edges. Note that $\chi(G) \leq n$ if and only if G admits a homomorphism to the complete graph on n vertices, K_n. We denote by $\Delta(G)$ the supremum of the vertex degrees of G. In what follows, we will only consider graphs with degrees bounded by a finite number, unless explicitly stated otherwise. A graph is called *Δ-regular* if every vertex has degree Δ. It is easy to see that $\chi(G) \leq \Delta(G) + 1$. Moreover, Brooks' theorem states that this inequality is sharp only in trivial situations: if $\Delta(G) > 2$, it happens if and only if G contains a complete graph on $\Delta(G) + 1$ vertices, and if $\Delta(G) = 2$, it happens if and only if G contains an odd cycle.

We say that G is a *Borel* graph if $V(G)$ is a standard Borel space, see [Kec95], and the set of edges of G is a Borel subset of $V(G) \times V(G)$ endowed with the product Borel structure. The *Borel chromatic number,* $\chi_B(G)$, of G is defined as the minimal n for which a Borel n-coloring exists, here we endow n with the trivial Borel structure. Similar concepts are studied when we relax the notion of Borel measurable to merely *measurable* with respect to some probability measure, or *Baire* measurable with respect to some compatible Polish topology.

It has been shown by Kechris-Solecki-Todorčević [KST99] that $\chi_B(G) \leq \Delta(G) + 1$, and it was a long standing open problem, whether Brooks' theorem has a literal extension to the Borel context, at least in the case $\Delta(G) > 2$. For example, it has been proved by Conley-Marks-Tucker-Drob [CMTD16] that in the measurable or Baire measurable setting the answer is affirmative. Eventually, this problem has been solved by Marks [Mar16], who showed the existence of Δ-regular acyclic Borel graphs with Borel chromatic number $\Delta + 1$. Remarkably, this result relies on Martin's Borel determinacy theorem, one of the cornerstones of modern descriptive set theory.

Results

First let us give a high-level overview of the strategy (for the precise definition of the notions discussed below see Section 3). Fix a $\Delta > 2$. To a given Borel graph $\mathcal{H}$ we will associate an acyclic Borel graph $\mathbf{Hom}^{ac}(T_\Delta, \mathcal{H})$ of degrees bounded by Δ. Roughly speaking, the vertex set of the graph will be a collection of pairs (x, h), where $x \in V(\mathcal{H})$ and h is a homomorphism from the Δ-regular infinite rooted tree T_Δ to $\mathcal{H}$ that maps the root to x, and (x, h) is adjacent to (x', h') if h' is obtained from h by moving the root to a neighbouring vertex.

The main idea is that the we can use the combinatorial properties of $\mathcal{H}$ to control the properties of $\mathbf{Hom}^{ac}(T_\Delta, \mathcal{H})$. Most importantly, we will argue that from a Borel Δ-coloring c of $\mathbf{Hom}^{ac}(T_\Delta, \mathcal{H})$ we can construct a Δ-coloring of $\mathcal{H}$: to each x we associate games analogous to the ones developed by Marks, in order to select the "largest" set among the sets $\{h : c(x, h) = i\}$ for $i \leq \Delta$, and color x with the appropriate i. As this selection will be based on the existence of winning strategies, the coloring of $\mathcal{H}$ will not be Borel. However, it will still be in a class that has all the usual regularity properties (we call this class weakly provably $\mathbf{\Delta}^1_2$, see Section 2 for the definition of this class and the corresponding chromatic number, $\chi_{wpr-\mathbf{\Delta}^1_2}$). Thus we will be able to prove the following.

version of this paper should have been a journal version of some results from [BCG$^+$21] aiming to people working in descriptive combinatorics. In the end, we added several new applications of our method that cannot be found in [BCG$^+$21].

Theorem 1.1. *Let $\mathcal{H}$ be a locally countable Borel graph. Then we have*

$$\chi_{wpr-\mathbf{\Delta}^1_2}(\mathcal{H}) > \Delta \;\Rightarrow\; \chi_B(\mathbf{Hom}^{ac}(T_\Delta, \mathcal{H})) > \Delta.$$

In particular, $\chi_{wpr-\mathbf{\Delta}^1_2}(\mathcal{H}) > \Delta$ holds if the Ramsey measurable (if $V(\mathcal{H}) = [\mathbb{N}]^{\mathbb{N}}$), Baire measurable, or μ measurable chromatic number of $\mathcal{H}$ is $> \Delta$.

Next we list the applications. In each instance we use a version of Theorem 1.1 for a carefully chosen target graph $\mathcal{H}$. These graphs come from well-studied contexts of descriptive combinatorics, namely, *Ramsey property* and *Baire category*.

a) Complexity result. We apply homomorphism graphs in connection to projective complexity and Brooks' theorem. One might conjecture that the right generalization of Brooks' theorem to the Borel context is that Marks' examples serve as the analogues of the complete graph, i.e., whenever G is a Borel graph with $\chi_B(G) = \Delta(G) + 1$, then G must contain a Borel homomorphic copy of the corresponding example of Marks. Note that in the case $\Delta(G) = 2$ this is the situation, as there is a Borel analogue of odd cycles that admits a homomorphism into each Borel graph G with $\chi_B(G) > 2$ (see [CMSV21]).

In [TV21] it has been shown that it is impossible to give a simple characterization of acyclic Borel graphs with Borel chromatic number ≤ 3. The construction there was based on a Ramsey theoretic statement, the Galvin-Prikry theorem [GP73]. An important weakness of that proof is that it uses graphs of finite but unbounded degrees. Using the homomorphism graph combined with the method developed in [TV21] and Marks technique, we obtain the analogous result for bounded degree graphs.

Theorem 1.2. *For each $\Delta > 2$ the family of Δ-regular acyclic Borel graphs with Borel chromatic number $\leq \Delta$ has no simple characterization, namely, it is $\mathbf{\Sigma}^1_2$-complete.*

From this we deduce a strong negative answer to the conjecture described above.

Corollary 1.3. *Brooks' theorem has no analogue for Borel graphs in the following sense: there is no countable family $\{\mathcal{H}_i\}_{i \in I}$ of Borel graphs such that for any Borel graph $\mathcal{G}$ with $\Delta(\mathcal{G}) \leq \Delta$ we have $\chi_B(\mathcal{G}) > \Delta$ if and only if for some $i \in I$ the graph $\mathcal{G}$ contains a Borel homomorphic copy of $\mathcal{H}_i$.*

b) Chromatic number and hyperfiniteness. Recall that a Borel graph $\mathcal{G}$ is called *hyperfinite*, if it is the increasing union of Borel graphs with finite connected components. In [CJM$^+$20] the authors examine the connection between *hyperfiniteness* and notions of Borel combinatorics, such as Borel chromatic number and the Lovász Local Lemma. Roughly speaking, they show that hyperfiniteness has no effect on Borel combinatorics, for example, they establish the following.

Theorem 1.4 ([CJM$^+$20])**.** *There exists a hyperfinite Δ-regular acyclic Borel graph with Borel chromatic number $\Delta + 1$.*

Using homomorphism graphs, we provide a new, short and more streamlined proof of this result. In particular, the conclusion about the chromatic number follows from our general result about $\mathbf{Hom}^e$ (a version of $\mathbf{Hom}^{ac}$), while to get hyperfiniteness we can basically choose any acyclic hyperfinite graph as a target graph. To get both properties at once, we simply pick a variant of the graph $\mathbb{G}_0$ (see [KST99, Section 6]) as our target graph.

c) Graph homomorphism. We also consider a slightly more general context: homomorphisms to finite graphs. Clearly, the Δ-regular examples constructed by Marks do not admit a Borel homomorphism to finite graphs of chromatic number at most Δ, as this would imply that their Borel chromatic number is $\leq \Delta$. No other examples of such graphs were known. We show the following.

Theorem 1.5. *For every $\Delta > 2$ and every $\ell \leq 2\Delta - 2$ there are a finite graph H and a Δ-regular acyclic Borel graph $\mathcal{G}$ such that $\chi(H) = \ell$ and $\mathcal{G}$ does not admit Borel homomorphism to H. The graph $\mathcal{G}$ can be chosen to be hyperfinite.*

This theorem is a step towards the better understanding of Problem 8.12 from [KM20].

Roadmap. The paper is structured as follows. In Section 2 we collect the most important definitions and theorems that are going to be used. Then, in Section 3 we establish the basic properties of homomorphism graphs and their various modifications. Section 4 contains Marks' technique's adaptation to our context, while in Section 5 we prove our main results. We conclude the paper with a couple of remarks Section 6.

2 Preliminaries

For standard facts and notations of descriptive set theory not explained here we refer the reader to [Kec95] (see also [Mos09]).

Given a graph G, we refer to maps $V(G) \to S$ and $G \to S$ as *vertex (S)-labelings* and *edge (S)-labelings*, respectively. An edge labeling is called an *edge coloring*, if incident edges have different labels. Let $\mathcal{F}$ be a family of subsets of $V(G)$, and $n \in \{1, 2, \ldots, \aleph_0\}$. An *$\mathcal{F}$ measurable n-coloring* is an n-coloring c of G such that $c^{-1}(i) \in \mathcal{F}$ for each $i < n$. Using this notion, we define the *$\mathcal{F}$ measurable chromatic number of G*, $\chi_{\mathcal{F}}(G)$ to be the minimal n for which such a coloring exists.

We denote by $[S]^{\mathbb{N}}$ the collection of infinite subsets of the set S, and by $S^{<\mathbb{N}}$ the family of finite sequences of elements of S. Define the *shift-graph (on $[\mathbb{N}]^{\mathbb{N}}$)*, $\mathcal{G}_S$, by letting x and y be adjacent if $y = x \setminus \min x$ or $x = y \setminus \min y$. The shift-graph has a close connection to the notion of so called Ramsey property: for $s \subset \mathbb{N}$ finite and $A \in [\mathbb{N}]^{\mathbb{N}}$ with $\max s < \min A$ let $[s, A] = \{B \in [\mathbb{N}]^{\mathbb{N}} : s \subset B, A \supseteq B \setminus s\}$. A set $S \subseteq [\mathbb{N}]^{\mathbb{N}}$ is called *Ramsey* if for each set of the form $[s, A]$ there exists $B \in [A]^{\mathbb{N}}$ such that $[s, B] \cap S = \emptyset$ or $[s, B] \subseteq S$ (see, e.g., [KST99, Kho12, Tod10] for results on the shift-graph and Ramsey measurability). The following follows from the definition.

Theorem 2.1. *The graph $\mathcal{G}_S$ has no Ramsey measurable finite coloring.*

Note that the Galvin-Prikry theorem asserts that Borel sets are Ramsey measurable. However, adapting Marks' technique to our setting will require the usage of families of sets that are much larger than the collection of Borel sets. By Σ^1_n (Π^1_n) formulas we mean the collection of second-order formulas over the structure $\langle \mathbb{N}, +, \cdot, 0 \rangle$, of the appropriate form (see, e.g., [Jec03, Section 17]). A set $S \subseteq (\mathbb{N}^{\mathbb{N}})^k$ is called *provably $\mathbf{\Delta}^1_2$*, if there are Σ^1_2 and Π^1_2 formulas $\Phi(\cdot, \cdot)$ and $\Psi(\cdot, \cdot)$ and an $a \in \mathbb{N}^{\mathbb{N}}$ such that $x \in S \iff \Psi(x, a) \iff \Phi(x, a)$ and it is provable (from ZFC) that $\forall x \forall a \ (\Psi(x, a) \iff \Phi(x, a))$ (see, [Kan09]).

It will be convenient to consider a slightly more general notion that is sufficiently robust. Assume that X is a Polish space and $A \subseteq X$. We say that A is *weakly provably $\mathbf{\Delta}^1_2$* if there exists a Borel map $f : X \to \mathbb{N}^{\mathbb{N}}$ such that $A = f^{-1}(B)$ where B is provably $\mathbf{\Delta}^1_2$. We will use the following result.

Proposition 2.2. *Let X be a Polish space. Weakly provably $\mathbf{\Delta}^1_2$ subsets of X*

1. *form an algebra,*

2. *have the Baire property,*

3. *are measurable with respect to any Borel probability measure,*

4. *in the case $X = [\mathbb{N}]^{\mathbb{N}}$ have the Ramsey property.*

Proof. The first assertion is easy from the definition. In order to see the second and third assertions note that it has been shown by Solovay (unpublished) and Fenstad-Norman [FN74] that provably $\mathbf{\Delta}^1_2$ subsets of $\mathbb{N}^{\mathbb{N}}$ are universally measurable and are $\aleph_0$-universally Baire. Since these families are closed under taking Borel preimages, we are done. In an upcoming paper we will show that weakly provably $\mathbf{\Delta}^1_2$ sets have the Ramsey property [GKV]; the proof also essentially follows from [IS89, Theorem 3.5]. $\square$

If $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a nonempty pruned tree, and $A \subseteq \mathbb{N}^{\mathbb{N}}$, $G(T, A)$ will denote the two-player infinite game on $\mathbb{N}$ with legal positions in T and payoff set A. We will call the first player Alice and the second Bob. Note that the Borel Determinacy Theorem [Mar75] states that one of the players has a winning strategy in $G(T, A)$, whenever A is Borel. Recall that a subset of A a Polish space X is in the class $\Game\mathbf{\Delta}^1_1$ if there is some Borel set $B \subset X \times \mathbb{N}^{\mathbb{N}}$ such that $A = \{x :$ Alice has a winning strategy in $G(\mathbb{N}^{<\mathbb{N}}, B_x)\}$.

Remark 2.3. *To establish Theorem 1.2, Theorem 1.4 and Theorem 1.5, we only need that the sets in the class $\Game\mathbf{\Delta}^1_1$ have the regularity properties listed in Proposition 2.2, which is a weaker statement than the one above. However, we were unable to locate a reference for this fact that avoids the usage of weakly provably $\mathbf{\Delta}^1_2$-sets. Note also that Proposition 2.2 easily follows from $\mathbf{\Sigma}^1_2$ determinacy (i.e., the assumption that in $G(T, A)$ one of the players has a winning strategy if A is $\mathbf{\Sigma}^1_2$).*

We will need a coding for Borel sets. Let $\mathbf{BC}(X)$ be a set of Borel codes and sets $\mathbf{A}(X)$ and $\mathbf{C}(X)$ with the properties summarized below:

Proposition 2.4. *(see [Mos09, 3.H])*

- $\mathbf{BC}(X) \in \mathbf{\Pi}^1_1(\mathbb{N}^{\mathbb{N}})$, $\mathbf{A}(X) \in \mathbf{\Sigma}^1_1(\mathbb{N}^{\mathbb{N}} \times X)$, $\mathbf{C}(X) \in \mathbf{\Pi}^1_1(\mathbb{N}^{\mathbb{N}} \times X)$,
- *for $c \in \mathbf{BC}(X)$ and $x \in X$ we have $(c, x) \in \mathbf{A}(X) \iff (c, x) \in \mathbf{C}(X)$,*
- *if P is a Polish space and $B \in \mathbf{\Delta}^1_1(P \times X)$ then there exists a Borel map $f : P \to \mathbb{N}^{\mathbb{N}}$ so that $ran(f) \subset \mathbf{BC}(X)$ and for every $p \in P$ we have $\mathbf{A}(X)_{f(p)} = B_p$.*

Moreover, in the case $X = (\mathbb{N}^{\mathbb{N}})^k$ the sets $\mathbf{BC}(X)$, $\mathbf{A}(X)$, and $\mathbf{C}(X)$ can be described by Π^1_1, Σ^1_1, and Π^1_1 formulas, respectively.

Using a recursive encoding we identify trees with elements of $\mathbb{N}^{\mathbb{N}}$. It turns out that sets that arise from the existence of winning strategies in Borel games are weakly provably $\mathbf{\Delta}^1_2$.

Lemma 2.5. *Let X be a Polish space and $B \subseteq X \times \mathbb{N}^{\mathbb{N}}$ be Borel. Then the set*

$$\{(x, T) : x \in X, T \subseteq \mathbb{N}^{<\mathbb{N}} \textit{ is a nonempty pruned tree, } \text{Alice} \textit{ has a winning strategy in } G(T, B_x)\}$$

is weakly provably $\mathbf{\Delta}^1_2$.

Proof. To ease the notation set $\mathbf{BC} := \mathbf{BC}(\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N})$, $\mathbf{C} := \mathbf{C}(\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N})$ and $\mathbf{A} := \mathbf{A}(\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N})$. Note that we have $\mathbf{BC} \in \mathbf{\Pi}^1_1(\mathbb{N}^\mathbb{N})$, $\mathbf{A} \in \mathbf{\Sigma}^1_1(\mathbb{N}^\mathbb{N} \times (\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}))$, $\mathbf{C} \in \mathbf{\Pi}^1_1(\mathbb{N}^\mathbb{N} \times (\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}))$. Consider first the set

$$WS = \{(x, c, T) : x \in \mathbb{N}^\mathbb{N}, c \in \mathbf{BC},$$

$$T \text{ is a nonempty pruned tree on } \mathbb{N}^{<\mathbb{N}} \text{ and Alice has a winning strategy in } G(T, (\mathbf{C}_c)_x)\}.$$

We show that WS is provably $\mathbf{\Delta}^1_2$. Note that S is a winning strategy for Alice in $G(T, (\mathbf{C}_c)_x)$ if and only if $S \subseteq T$ is a strategy for Alice such that $\forall y\ (y \notin [S] \vee y \in (\mathbf{C}_c)_x)$. Using Proposition 2.4 this yields that WS can be described by a Σ^1_2 formula. Moreover, as $\mathbf{A}_c = \mathbf{C}_c$ whenever $c \in \mathbf{BC}$, and $\mathbf{C}_c$ is Borel, by the Borel determinacy theorem Alice has a winning strategy in $G(T, (\mathbf{C}_c)_x)$ if and only if Bob has no winning strategy. That is, for every $S \subseteq T$ strategy for Bob we have $\exists y\ (y \in [S] \wedge y \in (\mathbf{A}_c))$. This yields an equivalent description of WS using a Π^1_2 formula.

To deduce the lemma from this, applying a Borel isomorphism between X and some subset of $\mathbb{N}^\mathbb{N}$ we may assume that $X = \mathbb{N}^\mathbb{N}$. It follows that the set above is Borel isomorphic to a section of WS, which shows our claim. □

3 The homomorphism graph

In this section we define the main objects of our study, *homomorphism graphs*, and establish a couple of their properties.

Let Γ be a countable group and $S \subseteq \Gamma$ be a generating set. Assume that $\Gamma \curvearrowright X$ is an action of Γ on the set X. As there is no danger of confusion we always denote the action with the symbol $\cdot$. The *Schreier graph* $Sch(\Gamma, S, X)$ of such an action is a graph on the set X such that $x \neq x'$ are adjacent iff for some $\gamma \in S \cup S^{-1}$ we have that $\gamma \cdot x = x'$.

Probably the most important example of a Schreier graph is the (right) *Cayley graph,* $\mathrm{Cay}(\Gamma, S)$ that comes from the right multiplication action of Γ on itself. That is, $g, h \in \Gamma$ form an edge in $\mathrm{Cay}(\Gamma, S)$ if there is $\sigma \in S$ such that $g \cdot \sigma = h$. Another example is the graph of the left-shift action of Γ on the space 2^Γ: recall that the left-shift action is defined by

$$\gamma \cdot x(\delta) = x(\gamma^{-1} \cdot \delta)$$

for $\gamma \in \Gamma$ and $x \in 2^\Gamma$. Observe that the Schreier graph of this actions is a Borel graph, when we endow the space 2^Γ with the product topology.

Our examples will come from a generalization of this graph. First note that if we replace 2 by any other standard Borel space X, the space X^Γ still admits a Borel product structure with respect to which the Schreier graph of the left-shift action defined as above, is a Borel graph. The main idea is to start with a Borel graph $\mathcal{H}$ and restrict the corresponding Schreier graph on $V(\mathcal{H})^\Gamma$ to an appropriate subset on which the elements $h \in V(\mathcal{H})^\Gamma$ are homomorphisms from $\mathrm{Cay}(\Gamma, S)$ to $\mathcal{H}$. This allows us to control certain properties (such as chromatic number or hyperfiniteness) of the resulting graph by the properties if $\mathcal{H}$. More precisely:

Definition 3.1. *Let $\mathcal{H}$ be a Borel graph and Γ be a countable group with a generating set S. Let $\mathbf{Hom}(\Gamma, S, \mathcal{H})$ be the restriction of $Sch(\Gamma, S, V(\mathcal{H}))$ to the set*

$$\{h \in V(\mathcal{H})^\Gamma : h \textit{ is a homomorphism from } \mathrm{Cay}(\Gamma, S) \textit{ to } V(\mathcal{H})\}.$$

We will refer to $\mathcal{H}$ as the *target graph*, and we will denote the map $h \mapsto h(1)$ by *Root* (note that the vertices of $\mathrm{Cay}(\Gamma, S)$ are labeled by the elements of Γ). It is clear from the definition that $\mathbf{Hom}(\Gamma, S, \mathcal{H})$ is a Borel graph with degrees at most $|S \cup S^{-1}|$ and that *Root* is a Borel map. We can immediately make the following observation.

Proposition 3.2. *1. The action of $1 \neq \gamma \in S$ on $\mathbf{Hom}(\Gamma, S, \mathcal{H})$ has no fixed-points.*

2. $\mathbf{Hom}(\Gamma, S, \mathcal{H})$ admits a Borel homomorphism to $\mathcal{H}$. Thus, $\chi_B(\mathbf{Hom}(\Gamma, S, \mathcal{H})) \leq \chi_B(\mathcal{H})$.

Proof. Let $h \in \mathbf{Hom}(\Gamma, S, \mathcal{H})$ and $1 \neq \gamma \in S \cup S^{-1}$. Note that as γ^{-1} and 1 are adjacent in $\mathrm{Cay}(\Gamma, S)$ it follows that $Root(\gamma \cdot h) = h(\gamma^{-1})$ and $Root(h) = h(1)$ are adjacent in $\mathcal{H}$ as h is a homomorphism. Consequently, the map $Root$ is a Borel homomorphism from $\mathbf{Hom}(\Gamma, S, \mathcal{H})$ to $\mathcal{H}$ and $h \neq \gamma \cdot h$ as there are no loops in $\mathcal{H}$. $\square$

The Δ-regular tree T_Δ. In this paper we only consider the case of the group

$$\Gamma_\Delta = \langle \alpha_1, \dots, \alpha_\Delta | \alpha_1^2 = \dots = \alpha_\Delta^2 = 1 \rangle$$

together with the generating set $S_\Delta = \{\alpha_1, \dots, \alpha_\Delta\}$. Since $\mathrm{Cay}(\Gamma_\Delta, S_\Delta)$ is isomorphic to the Δ-regular infinite tree, T_Δ, we use $\mathbf{Hom}(T_\Delta, \mathcal{H})$ to denote the graph $\mathbf{Hom}(\Gamma_\Delta, S_\Delta, \mathcal{H})$. Note also that we consider $\mathrm{Cay}(\Gamma_\Delta, S_\Delta)$ and T_Δ equipped with a Δ-edge coloring. As suggested above, an equivalent description of the vertex set of $\mathbf{Hom}(T_\Delta, \mathcal{H})$ is that it is the set of pairs (x, h) where h is a homomorphism from the tree T_Δ to $\mathcal{H}$ and x is a distinguished vertex of T_Δ, a root. Then we have that (x, h) and (y, g) form an (α-)edge if and only if $h = g$ and (x, y) is an α-edge in T_Δ. This is because (a) there is a one-to-one correspondence between a homomorphism from $\mathrm{Cay}(\Gamma_\Delta, S_\Delta)$ to $\mathcal{H}$ and the pairs (x, h), and (b) the shift action $\Gamma_\Delta \curvearrowright \mathbf{Hom}(\Gamma_\Delta, S_\Delta, \mathcal{H})$ corresponds to changing the root for a fixed homomorphism h from T_Δ to $\mathcal{H}$.

Recall that an action $\Gamma \curvearrowright X$ is *free* if for each $1 \neq \gamma \in \Gamma$ and $x \in X$ we have $\gamma \cdot x \neq x$. The *free part*, denoted by $Free(X)$, is the set $\{x : \forall 1 \neq \gamma \in \Gamma \ \gamma \cdot x \neq x\}$. Note that the left-shift action of Γ_Δ on, say, $\mathbb{N}^{\Gamma_\Delta}$ is not free, in particular, the corresponding Schreier-graph has cycles. To remedy this, Marks used a restriction of the graph to the free part, showing that for each $\Delta > 2$ this graph has Borel chromatic number $\Delta + 1$. Analogously, we have the following.

Definition 3.3. *Let $\mathbf{Hom}^{ac}(T_\Delta, \mathcal{H}) = \mathbf{Hom}(T_\Delta, \mathcal{H}) \restriction Free(V(\mathcal{H})^{\Gamma_\Delta})$, that is, the restriction of the graph $\mathbf{Hom}(T_\Delta, \mathcal{H})$ to the free part of the Γ_Δ action.*

In our first application we will use this remedy to get acyclic graphs. Note that for each edge in $\mathbf{Hom}^{ac}(T_\Delta, \mathcal{H})$ there is a unique generator $\alpha \in S_\Delta$ that induces it. In particular, the graph $\mathbf{Hom}^{ac}(T_\Delta, \mathcal{H})$ admits a canonical Borel edge Δ-coloring. The following is straightforward.

Proposition 3.4. *Let $\mathcal{H}$ be a locally countable Borel graph. If $\mathbf{Hom}^{ac}(T_\Delta, \mathcal{H})$ is nonempty, then it is Δ-regular and acylic.*

However, utilizing the homomorphism graph together with an appropriate target graph, we will be able to completely avoid the non-free part, in an automatic manner. This way we will be able to guarantee the hyperfiniteness of the homomorphism graph as well. Recall that $T_\Delta = \mathrm{Cay}(\Gamma_\Delta, S_\Delta)$ comes with a Δ-edge coloring by the elements of S_Δ. Let us consider the a subgraph of the homomorphism graph that arises by requiring h to preserve this information.

Definition 3.5. *Assume that the graph $\mathcal{H}$ is equipped with a Borel edge S_Δ-labeling. Let $\mathbf{Hom}^e(T_\Delta, \mathcal{H})$ be the restriction of $\mathbf{Hom}(T_\Delta, \mathcal{H})$ to the set*

$$\{h \in V(\mathbf{Hom}(T_\Delta, \mathcal{H})) : h \textit{ preserves the edge labels}\}.$$

Clearly, $\mathbf{Hom}^e(T_\Delta, \mathcal{H})$ is also a Borel graph. Note that in the following statement the labeling of the edges of the target graph $\mathcal{H}$ is typically not a coloring.

Proposition 3.6. *Assume that $\mathcal{H}$ is an acyclic graph equipped with a Borel edge S_Δ-labeling and $\mathbf{Hom}^e(T_\Delta, \mathcal{H})$ is nonempty. Then*

1. *$\mathbf{Hom}^e(T_\Delta, \mathcal{H})$ is acyclic,*
2. *If $\mathcal{H}$ is hyperfinite, then so is $\mathbf{Hom}^e(T_\Delta, \mathcal{H})$,*
3. *$\mathbf{Hom}^e(T_\Delta, \mathcal{H})$ is Δ-regular.*

Proof. Observe that if h is a homomorphism from a tree to an acyclic graph that is not injective, then there must be adjacent pairs of vertices x, y and y, z with $x \neq z$ and $h(z) = h(x)$. Thus, if $h \in \mathbf{Hom}^e(T_\Delta, \mathcal{H})$ is edge label preserving then it must be injective, as incident edges have different labels in T_Δ. Therefore, the map *Root* is injective on each connected component of $\mathbf{Hom}^e(T_\Delta, \mathcal{H})$, yielding (1).

To see (2), let $(\mathcal{H}_n)_{n\in\mathbb{N}}$ be a witness to the hyperfiniteness of $\mathcal{H}$. Let $\mathcal{H}'_n$ be the pullback of $\mathcal{H}_n$ by the map *Root*. Since *Root* is injective on every connected component, the graphs $\mathcal{H}'_n$ also have finite components and their union is $\mathbf{Hom}^e(T_\Delta, \mathcal{H})$.

For (3) just notice that using the injectivity of *Root* again, it follows that $\{\gamma \cdot h\}_{\gamma\in\{1\}\cup S_\Delta}$ has cardinality $\Delta + 1$. $\square$

4 Variations on Marks' technique

Now we are ready to adapt Marks' technique [Mar16] to homomorphism graphs. Let us denote by $\chi_{wpr-\mathbf{\Delta}^1_2}(\mathcal{H})$ the weakly provably $\mathbf{\Delta}^1_2$-chromatic number of $\mathcal{H}$ (see Section 2).

Theorem 1.1. *Let $\mathcal{H}$ be a locally countable Borel graph. Then*

$$\chi_{wpr-\mathbf{\Delta}^1_2}(\mathcal{H}) > \Delta \implies \chi_B(\mathbf{Hom}^{ac}(T_\Delta, \mathcal{H})) > \Delta.$$

The games we will define naturally yield elements $h \in \mathbf{Hom}(T_\Delta, \mathcal{H})$ rather than $\mathbf{Hom}^{ac}(T_\Delta, \mathcal{H})$. In order to deal with the cyclic part of the graph, we will show slightly more, using the same strategy as Marks. Let $V \subseteq V(\mathbf{Hom}(T_\Delta, \mathcal{H}))$, an *anti-game labeling of V* is a map $c : V \to \Delta$ such that there are no $i \in \Delta$ and distinct vertices $h, h' \in V$ with $c(h) = c(h') = i$ and $\alpha_i \cdot h = h'$.

Remark 4.1. *One can define analogously anti-game labelings for graphs with with edges labeled by Δ. Note that in the case when the graph is Δ-regular and the labeling is an edge Δ-coloring, the existence of an anti-game coloring is equivalent to solving the well known* edge grabbing problem *(that is, every vertex picks one adjacent edge but no edge can be picked from both sides); see also [BBE+20].*

Lemma 4.2. *There exists a Borel anti-game labeling $c : V(\mathbf{Hom}(T_\Delta, \mathcal{H})) \setminus V(\mathbf{Hom}^{ac}(T_\Delta, \mathcal{H})) \to \Delta$.*

Proof. Let us use the notation $C = V(\mathbf{Hom}(T_\Delta, \mathcal{H})) \setminus V(\mathbf{Hom}^{ac}(T_\Delta, \mathcal{H}))$. By definition, the Γ_Δ action on every connected component $\mathbf{Hom}(T_\Delta, \mathcal{H}) \restriction C$ is not free. Using [KM04, Lemma 7.3] we can find a Borel maximal family $\mathcal{F} \subseteq C^{<\mathbb{N}}$ of pairwise disjoint finite sequences each of length at least 2 such that for each $(h_i)_{i<k} \in \mathcal{F}$ there is a sequence $(\alpha_{n_i})_{i<k} \in S^k_\Delta$ such that $\alpha_{n_i} \neq \alpha_{n_{i+1}}$, $\alpha_{n_i} \cdot h_i = h_{i+1}$ for $i < k-1$, and $\alpha_{n_0} \neq \alpha_{n_{k-1}}$, $\alpha_{n_{k-1}} \cdot h_{k-1} = h_0$. (Note that it is possible that $k = 2$ in which case there are two distinct generators $\alpha_{n_0} \neq \alpha_{n_1}$ such that $\alpha_{n_0} \cdot h_0 = \alpha_{n_1} \cdot h_0 = h_1$.)

Now label an element $h \in C$ by n_i if $h = h_i$ for some $(h_i)_{i<k} \in \mathcal{F}$. Otherwise, let $c(h)$ be the minimal i such that $\alpha_i \cdot h$ has strictly smaller distance to $\mathcal{F}$ than h with respect to the graph distance in $\mathbf{Hom}(T_\Delta, \mathcal{H})$. It is easy to check that c is an anti-game labeling. $\square$

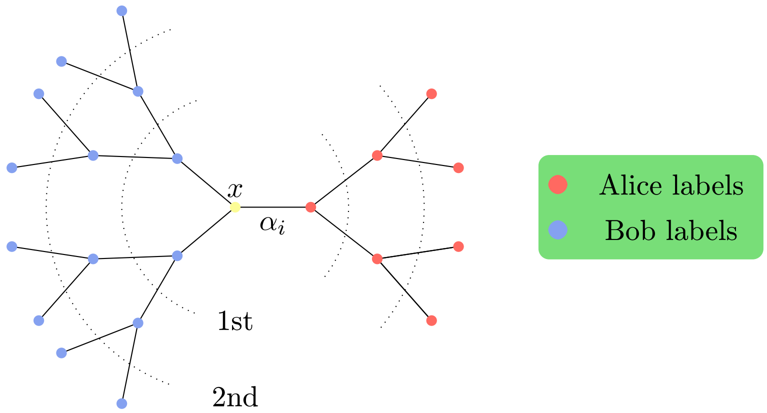

Figure 1: The game $\mathbb{G}(x,i)$

Proof of Theorem 1.1. We will show that there is no Borel anti-game labeling $c : V(\mathbf{Hom}(T_\Delta, \mathcal{H})) \to \Delta$. Note that this yields Theorem 1.1, as if $\mathbf{Hom}^{ac}(T_\Delta, \mathcal{H})$ admitted a Borel Δ-coloring, combining this with the coloring constructed in Lemma 4.2 we would obtain a Borel anti-game labeling on $V(\mathbf{Hom}(T_\Delta, \mathcal{H}))$. So, towards contradiction, assume that c is a Borel anti-game labeling.

Without loss of generality we may assume that $\mathcal{H}$ has no isolated points. This ensures that the games below can be always continued.

We define a family of games $\mathbb{G}(x,i)$ parametrized by elements of $x \in V(\mathcal{H})$, $i \in \Delta$ for two players. In a run of the game $\mathbb{G}(x,i)$ players Alice and Bob alternate and build a homomorphism h from T_Δ to $\mathcal{H}$, i.e., an element of $\mathbf{Hom}(T_\Delta, \mathcal{H}) \subset V(\mathcal{H})^{\Gamma_\Delta}$, with the property that $Root(h) = x$.

In the k-th round, first Alice labels vertices of distance k from the 1 on the side of the α_i edge. After that, Bob labels all remaining vertices of distance k, etc (see Fig. 1). In other words, Alice labels the elements of Γ_Δ corresponding to reduced words of length k starting with α then Bob labels the rest of the reduced words of length k. For the reader familiar with Marks' construction, let us point out that for a fixed x, the games are analogues to the ones he defines, with the following differences: allowed moves are vertices of the target graph $\mathcal{H}$ and restricted by its edge relation.

The winning condition is defined as follows:

$$\text{Alice wins the game } \mathbb{G}(x,i) \text{ iff } c(h) \neq i.$$

Lemma 4.3. 1. *For any $x \in V(\mathcal{H})$ and $i \in \Delta$ one of the players has a winning strategy in the game $\mathbb{G}(x,i)$.*

2. *The set $\{(x,i) : \text{Alice has a winning strategy in } \mathbb{G}(x,i)\}$ is weakly provably $\mathbf{\Delta}^1_2$.*

Proof. Let us denote by $E_\mathcal{H}$ the connected component equivalence relation of $\mathcal{H}$. Observe that as T_Δ is connected, the range of any element $h \in \mathbf{Hom}(T_\Delta, \mathcal{H})$ is contained in a single $E_\mathcal{H}$ class. By the Feldman-Moore theorem, there is a countable collection of Borel functions $f_i : V(\mathcal{H}) \to V(\mathcal{H})$ such that $E_\mathcal{H} = \bigcup_{j \in \mathbb{N}} graph(f_j^{\pm 1})$. Therefore, the games $\mathbb{G}(x,i)$ above can be identified by games played on $\mathbb{N}$, namely, labeling a vertex in T_Δ by a vertex $y \in V(\mathcal{H})$ corresponds to playing the minimal natural number j with $f_j(x) = y$. Since the functions $(f_j)_{j \in \mathbb{N}}$ are Borel, this correspondence is Borel as well. Moreover, the rule that h must be homomorphism determines a pruned subtree of legal positions $T_{x,i} \subset \mathbb{N}^{<\mathbb{N}}$ and the map $(x,i) \mapsto T_{x,i}$ is Borel. This yields that there exists a Borel set $B \subseteq V(\mathcal{H}) \times \Delta \times \mathbb{N}^\mathbb{N}$ such that

$$\text{Alice has a strategy in } \mathbb{G}(x,i) \iff \text{Alice has a winning strategy in } G(T_{x,i}, B_{x,i}).$$

Now, the first claim follows from the Borel determinacy theorem, while the second follows from Lemma 2.5. ☐

Claim 4.4. *For every $x \in X$ there is an $i \in \Delta$ such that* Bob *wins* $\mathbb{G}(x, i)$*.*

Proof. Suppose not. Then we can combine strategies of Alice for each i in the natural way to build a homomorphism that is not in the domain of c (see, e.g., [Mar] or [Mar16]). ☐

Now we can finish the proof of Theorem 1.1. Define $d : V(\mathcal{H}) \to \Delta$ by

$$d(x) = i \iff i \text{ is minimal such that Bob has a winning strategy in } \mathbb{G}(x, i). \tag{1}$$

Since weakly provably $\mathbf{\Delta}^1_2$ sets form an algebra, d is weakly provably $\mathbf{\Delta}^1_2$ measurable and by Claim 4.4 it is everywhere defined. By our assumptions on $\mathcal{H}$ there are $x \neq x'$ adjacent with $d(x) = d(x')$. Now, we can play the two winning strategies corresponding to games $\mathbb{G}(x, i)$ and $\mathbb{G}(x', i)$ of Bob against each other, as if the first move of Alice was x' (resp. x). This yields distinct homomorphisms h, h' with $\alpha_i \cdot h = h'$ and $c(h) = c(h') = i$, contradicting that c is an anti-game labeling. ☐

4.1 Generalizations

Edge labeled graphs. As mentioned above, a novel feature of our approach is that requiring the homomorphisms to be edge label preserving and ensuring that $\mathcal{H}$ is acyclic, we can get rid of the investigation of the cyclic part (see Proposition 3.6). In order to achieve this, we have to assume slightly more about the chromatic properties of the target graph.

Assume that $\mathcal{H}$ is equipped with and edge S-labeling. The *edge-labeled chromatic number,* $el\chi(\mathcal{H})$ of $\mathcal{H}$ is the minimal n, for which there exists a map $c : V(\mathcal{H}) \to n$ so that for each $i \in n$ the set $c^{-1}(i)$ doesn't span edges with every possible label. In other words, $el\chi(\mathcal{H}) > n$ if and only if no matter how we assign n many colors to the vertices of $\mathcal{H}$, there will be a color class containing edges with every label. We define μ measurable, Baire measurable, etc. versions of the edge-labeled chromatic number in the natural way.

Theorem 4.5. *Let $\mathcal{H}$ be a locally countable Borel graph with a Borel S_Δ-edge labeling, such that for every vertex x and every label α there is an α-labeled edge incident to x. Then*

$$el\chi_{wpr-\mathbf{\Delta}^1_2}(\mathcal{H}) > \Delta \implies \chi_B(\mathbf{Hom}^e(T_\Delta, \mathcal{H})) > \Delta.$$

Proof. The proof is similar to the proof of Theorem 1.1, but with taking the edge colors into consideration. Let us indicate the required modifications. We define $\mathbb{G}(x, i)$ as above, with the extra assumption that players must build a homomorphism that respects edge labels, i.e., an element $h \in \mathbf{Hom}^e(T_\Delta, \mathcal{H})$. The condition on the edge-labeling ensures that the players can continue the game respecting the rules at every given finite step.

The analogue of Claim 4.4 clearly holds in this case, and we can define d as in (1). Finally, $el\chi_{wpr-\mathbf{\Delta}^1_2}(\mathcal{H}) > \Delta$ guarantees the existence of $i \in \Delta$ and $x, x' \in V(\mathcal{H})$ such that $d(x) = d(x') = i$ and that the edge between x and x' has label α_i, which in turn allows us to use the winning strategies of Bob in $\mathbb{G}(x, i)$ and $\mathbb{G}(x', i)$ against each other, as above. ☐

Graph homomorphism. In what follows, we will consider a slightly more general context, namely, instead of the question of the existence of Borel colorings, we will investigate the existence of Borel homomorphisms to a given finite graph H. The following notion is going to be our key technical tool.

Definition 4.6 (Property Δ-(*))**.** *Let $\Delta > 2$ and H be a finite graph. We say that H satisfies property Δ-(*) if there are sets $R_0, R_1 \subseteq V(H)$ such that H restricted to $V(H) \setminus R_i$ has chromatic number at most $(\Delta - 1)$ for $i \in \{0, 1\}$, and there is no edge between vertices of R_0 and R_1.*

Note that $\chi(H) \leq \Delta$ implies Δ-(*): indeed, if $A_1, \dots, A_\Delta$ are independent sets that cover $V(H)$, we can set $R_0 = R_1 = A_1$. Later, in Proposition 5.8 we show that there are graphs that satisfy Δ-(*) and have chromatic number $2\Delta - 2$. This is best possible as, if a graph H satisfies Δ-(*) then $\chi(H) \leq 2\Delta - 2$. In order to see this, take $R_0, R_1 \subseteq V(H)$ witnessing Δ-(*). Then, as there is no edge between R_0 and R_1, so in particular, between $R_0 \setminus R_1$ and $R_0 \cap R_1$, it follows that the chromatic number of H's restriction to R_0 is $\leq \Delta - 1$. But then we can construct proper $\Delta - 1$-colorings of R_0 and $V(H) \setminus R_0$, which shows our claim.

Theorem 4.7. *Let $\mathcal{H}$ be a locally countable Borel graph and assume that H is a finite graph with Δ-(*). Assume that $\mathcal{H}$ is equipped with a Borel S_Δ-edge labeling, such that every vertex is incident to some edge with every label. Then*

$$el\chi_{wpr-\mathbf{\Delta}^1_2}(\mathcal{H}) > 2^{\Delta \cdot 2^{|V(H)|}} \implies \mathbf{Hom}^e(T_\Delta, \mathcal{H}) \textit{ has no Borel homomorphism to } H.$$

Proof. Assume for contradiction that such a Borel homomorphism c exists. We will need a further modification of Marks' games. Let $R \subseteq V(H)$. For $x \in V(\mathcal{H})$ define the game $\mathbb{G}(x, i, R)$ as in the proof of Theorem 4.5, with the winning condition modified to

$$\text{Alice wins the game } \mathbb{G}(x, i, R) \text{ iff } c(h) \notin R.$$

Observe that playing the strategies of Alice against each other as in Claim 4.4 we can establish the following.

Claim 4.8. *For every $x \in V(\mathcal{H})$ and every sequence $(R_i)_{i \in \Delta}$ with $\bigcup_i R_i = V(H)$ there is some i such that* Alice *has no winning strategy in $\mathbb{G}(x, i, R_i)$.*

Now let N be the powerset of the set $\{(i, R) : i \in \Delta, R \subseteq V(H)\}$. Of course, $|N| = 2^{\Delta \cdot 2^{|V(H)|}}$. Define a mapping $d : V(\mathcal{H}) \to N$ by

$$(i, R) \in d(x) \iff \text{Alice has a winning strategy in } \mathbb{G}(x, i, R).$$

As in Lemma 4.3, the map d is weakly provably $\mathbf{\Delta}^1_2$-measurable. By our assumption on $\mathcal{H}$, there is a subset C on which d is constant and C spans an edge with each label.

Lemma 4.9. *Let $i \in \Delta$ and $R_0, R_1 \subseteq V(H)$ be sets such that there is no edge between points of R_0 and R_1 in G. Then for every $x \in C$* Bob *has no winning strategy in at least one of $\mathbb{G}(x, i, R_0)$ and $\mathbb{G}(x, i, R_1)$. In particular, if R is independent in G then* Bob *cannot have a winning strategy in $\mathbb{G}(x, i, R)$.*

Proof. If there exists an $x \in C$ for which $\mathbb{G}(x, i, R_0)$ and $\mathbb{G}(x, i, R_1)$ can be won by Bob, then, as d is constant on C, this is the case for every $x \in C$. So we could find $x_0, x_1 \in C$ connected with an α_i labeled edge so that Bob has winning strategies in $\mathbb{G}(x_0, i, R_0)$ and $\mathbb{G}(x_1, i, R_1)$. Then we can play the two winning strategies of Bob against each other as in the proof of Theorem 1.1. This would yield elements h_0, h_1 in $\mathbf{Hom}^e(T_\Delta, \mathcal{H})$ that form an α_i-edge with $c(h_i) \in R_i$, contradicting our assumption on c and R_i. $\square$

To finish the proof of the theorem, fix the sets R_0, R_1 from the definition of the condition Δ-(*), and take an arbitrary $x \in C$. By Lemma 4.9, we get that for one of them, say R_0, Alice has a winning strategy $\mathbb{G}(x, \alpha_0, R_0)$. Let $A_1, \ldots, A_{\Delta-1}$ be independent sets from the definition of Δ-(*), i.e., with the property that $R_0 \cup \bigcup_i A_i = V(G)$. Using Lemma 4.9 again, we obtain that Alice has a winning strategy in $\mathbb{G}(x, \alpha_i, A_i)$ for each $i \in \Delta$. This contradicts Claim 4.8.

□

5 Applications

In this section we apply the theorems proven before to establish our main results. We will choose a target graph using three prominent notions from descriptive set theory: *measure*, *category* and *Ramsey property*.

5.1 Complexity of coloring problem

First we will utilize the shift-graph $\mathcal{G}_S$ on $[\mathbb{N}]^{\mathbb{N}}$ to establish the complexity results. Let us mention that it would be ideal to use the main result of [TV21] (i.e., that deciding the Borel chromatic number of graphs is complicated) directly and apply the $\mathbf{Hom}^{ac}(T_\Delta, \cdot)$ map together with Theorem 1.1 to show that this already holds for acyclic bounded degree graphs. Unfortunately, since the mentioned theorem requires large weakly provably $\mathbf{\Delta}^1_2$-chromatic number, this does not seem to be possible (the graphs constructed in [TV21] only have large Borel chromatic numbers, at least a priori). Instead, we will rely on the uniformization technique from [TV21]. Roughly speaking, the technique enables us to prove that in certain situations deciding the existence of, say, Borel colorings is $\mathbf{\Sigma}^1_2$-hard, whenever we are allowed to put graphs "next to each other".

Let X, Y be uncountable Polish spaces, $\mathbf{\Gamma}$ be a class of Borel sets and $\Phi : \mathbf{\Gamma}(X) \to \mathbf{\Pi}^1_1(Y)$ be a map. Define $\mathcal{F}^\Phi \subset \mathbf{\Gamma}(X)$ by $A \in \mathcal{F}^\Phi \iff \Phi(A) \neq \emptyset$ and let the *uniform family, $\mathcal{U}^\Phi$,* be defined as follows: for $B \in \mathbf{\Gamma}(\mathbb{N}^{\mathbb{N}} \times X)$ let

$$\bar{\Phi}(B) = \{(s, y) \in \mathbb{N}^{\mathbb{N}} \times Y : y \in \Phi(B_s)\},$$

and

$$B \in \mathcal{U}^\Phi \iff \bar{\Phi}(B) \text{ has a full Borel uniformization}$$

(that is, it contains the graph of a Borel function $\mathbb{N}^{\mathbb{N}} \to Y$).

Let $\mathbf{\Delta}$ be a family of subsets of Polish spaces. Recall that a subset A of a Polish space X is $\mathbf{\Delta}$*-hard,* if for every Y Polish and $B \in \mathbf{\Delta}(Y)$ there exists a continuous map $f : Y \to X$ with $f^{-1}(A) = B$. A set is $\mathbf{\Delta}$*-complete* if it is $\mathbf{\Delta}$-hard and in $\mathbf{\Delta}$. A family $\mathcal{F}$ of subsets of a Polish space X is said to be $\mathbf{\Delta}$*-hard on* $\mathbf{\Gamma}$, if there exists a set $B \in \mathbf{\Gamma}(\mathbb{N}^{\mathbb{N}} \times X)$ so that the set $\{s \in \mathbb{N}^{\mathbb{N}} : B_s \in \mathcal{F}\}$ is $\mathbf{\Delta}$-hard. The next definition captures the central technical condition.

Definition 5.1. *The family $\mathcal{F}^\Phi$ is said to be* nicely $\mathbf{\Sigma}^1_1$-hard on $\mathbf{\Gamma}$ *if for every $A \in \mathbf{\Sigma}^1_1(\mathbb{N}^{\mathbb{N}})$ there exist sets $B \in \mathbf{\Gamma}(\mathbb{N}^{\mathbb{N}} \times X)$ and $D \in \mathbf{\Sigma}^1_1(\mathbb{N}^{\mathbb{N}} \times Y)$ so that $D \subset \bar{\Phi}(B)$ and for all $s \in \mathbb{N}^{\mathbb{N}}$ we have*

$$s \in A \iff D_s \neq \emptyset \iff \Phi(B_s) \neq \emptyset \; (\iff B_s \in \mathcal{F}^\Phi).$$

A map $\Phi : \mathbf{\Gamma}(X) \to \mathbf{\Pi}^1_1(Y)$ is called $\mathbf{\Pi}^1_1$ *on* $\mathbf{\Gamma}$ if for every Polish space P and $A \in \mathbf{\Gamma}(P \times X)$ we have $\{(s, y) \in P \times Y : y \in \Phi(A_s)\} \in \mathbf{\Pi}^1_1$. Now we have the following theorem.

Theorem 5.2 ([TV21], Theorem 1.6). *Let X, Y be uncountable Polish spaces, $\mathbf{\Gamma}$ be a class of subsets of Polish spaces which is closed under continuous preimages, finite unions and intersections and $\mathbf{\Pi}^0_1 \cup \mathbf{\Sigma}^0_1 \subset \mathbf{\Gamma}$. Suppose that $\Phi : \mathbf{\Gamma}(X) \to \mathbf{\Pi}^1_1(Y)$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Gamma}$ and that $\mathcal{F}^\Phi$ is nicely $\mathbf{\Sigma}^1_1$-hard on $\mathbf{\Gamma}$. Then the family $\mathcal{U}^\Phi$ is $\mathbf{\Sigma}^1_2$-hard on $\mathbf{\Gamma}$.*

Let us identify infinite subsets of $\mathbb{N}$ with their increasing enumeration. If $x, y \in [\mathbb{N}]^{\mathbb{N}}$ let us use the notation $y \leq^{\infty} x$ in the case the set $\{n : y(n) \leq x(n)\}$ is infinite and $y \leq^* x$ if it is co-finite. Set $\mathcal{D} = \{(x, y) : y \leq^{\infty} x\}$. It follows form the fact that $\mathcal{G}_S$ restricted to sets of the form $\mathcal{D}_x$ has a Borel 3-coloring that the graphs $\mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S \restriction \mathcal{D}_x)$ admit a Borel 3-coloring, uniformly in x:

Lemma 5.3. *There exists a Borel function $f_{dom} : [\mathbb{N}]^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ so that for each $x \in [\mathbb{N}]^{\mathbb{N}}$ we have $f_{dom}(x) = \langle c_0, \dots, c_{\Delta-1}\rangle$ with $c_i \in \mathbf{BC}([\mathbb{N}]^{\mathbb{N}})$, $\mathbf{A}([\mathbb{N}]^{\mathbb{N}})_{c_i}$ are $\mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S)$-independent subsets of $V(\mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S))$ for every $i < \Delta$ and*

$$V(\mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S \restriction \mathcal{D}_x)) = \bigcup_{i=0}^{\Delta-1} \mathbf{A}([\mathbb{N}]^{\mathbb{N}})_{c_i}.$$

Proof. Note that it suffices to construct a Borel map $c : [\mathbb{N}]^{\mathbb{N}} \times \mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S) \to \Delta$ that is a coloring of the graph $\mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S \restriction \mathcal{D}_x)$ for each x: indeed, we can use Proposition 2.4 for $(B_i)_x = \{(x, h) : c(x, h) = i\}$ to obtain Borel maps $f_i : [\mathbb{N}]^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ so that for every $x \in [\mathbb{N}]^{\mathbb{N}}$ we have $\mathbf{A}([\mathbb{N}]^{\mathbb{N}})_{f_i(x)} = B_i$ and let $f_{dom}(x) = \langle f_0(x), \dots, f_{\Delta-1}(x)\rangle$.

It has been established in [TV21, Lemma 4.5] (see also [DPT15]) that there exists a Borel map $c' : \mathcal{D} \to 3$ such that for each x it is a 3-coloring of the graph $\mathcal{G}_S \restriction \mathcal{D}_x$. As the map $Root : \mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S) \to \mathcal{G}_S$ is a Borel homomorphism by Proposition 3.2, it follows that the map $c(x, h) := c'(x, Root(h))$ is the desired Δ-coloring (in fact, 3-coloring). $\square$

Let $\mathcal{H}$ be the graph on $\mathbb{N}^{\mathbb{N}} \times V(\mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S))$ defined by making $(x, h), (x', h')$ adjacent if $x = x'$ and h is adjacent to h' in $\mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S)$. Fixing a Polish topology on $V(\mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S))$ that is compatible with the Borel structure, we might assume that $V(\mathcal{H})$ is a Polish space.

Putting together results proved in the previous sections, we get the following corollary.

Corollary 5.4. *The Borel chromatic number of $\mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S)$ is $\Delta + 1$.*

Proof. This follows from Proposition 2.2, Theorem 2.1, and Theorem 1.1. $\square$

Now we are ready to prove the following.

Proposition 5.5. *There exists a Borel set $C \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times V(\mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S))$ so that the set $\{s : \chi_B(\mathcal{H} \restriction C_s) \leq \Delta\}$ is $\mathbf{\Sigma}^1_2$-hard.*

Proof. We check the applicability of Theorem 5.2, with $X = V(\mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S))$, $Y = \mathbb{N}^{\mathbb{N}}$, $\mathbf{\Gamma} = \mathbf{\Delta}^1_1$ and

$$\Phi(A) = \{c : (\forall x, y \in A)\Big(c = \langle c_0, \dots, c_{\Delta-1}\rangle,\ c_i \in \mathbf{BC}(X), x \in \bigcup_i \mathbf{C}(X)_{c_i}$$

$$\text{and } (x, y) \in \mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S) \Rightarrow (\forall i)\big(\neg(x, y \in \mathbf{A}(X)_{c_i})\big)\Big)\},$$

in other words, $\Phi(A)$ contains the Borel codes of the Borel Δ-colorings of $\mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S) \restriction A$. Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be analytic and take a closed set $F \subset \mathbb{N}^{\mathbb{N}} \times [\mathbb{N}]^{\mathbb{N}}$ so that $proj_0(F) = A$. Let

$$B' = \{(s, y) : (\forall x \leq^* y)(x \notin F_s)\}.$$

The following has been proved in [TV21, Lemma 4.6].

Lemma 5.6. *1. Φ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Delta}^1_1$.*

2. $B' \in \mathbf{\Pi}^0_2$.

3. For any Borel set C we have $C \in \mathcal{U}^\Phi$ if and only if $\chi_B(\mathcal{H} \restriction C) \leq \Delta$.

Now define
$$B = \{(s,h) : h \in \mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S \restriction B'_s)\},$$
and
$$D = \{(s,c) : s \in A \text{ and } (\exists x \in F_s)(f_{dom}(x) = c)\},$$
where f_{dom} is the function from Lemma 5.3.

We will show that B and D witness that $\mathcal{F}^\Phi$ is nicely $\mathbf{\Sigma}^1_1$-hard. The set B is Borel by (2) of the lemma above, while by its definition D is analytic.

Suppose that $s \in A$. Then for each $x' \in F_s$ we have
$$B'_s = \{y : (\forall x \leq^* y)(x \notin F_s)\} \subset \{y : y \leq^\infty x'\} = \mathcal{D}_{x'}.$$
Thus, by Lemma 5.3 $B_s \in \mathcal{F}^\Phi$ and $D_s \neq \emptyset$. Moreover, if $c \in D_s$ then for some $x \in F_s$ we have $f_{dom}(x) = c$ with $c = \langle c_0, \dots, c_{\Delta-1}\rangle$, again by Lemma 5.3 we have $B_s \subseteq \bigcup_{i=0}^{\Delta-1} \mathbf{A}([\mathbb{N}]^\mathbb{N})_{c_i}$ and the sets $\mathbf{A}([\mathbb{N}]^\mathbb{N})_{c_i}$ are $\mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S)$-independent, thus, $D_s \subseteq \Phi(B_s)$. Conversely, if $s \notin A$ then $F_s = D_s = \emptyset$ and $B'_s = [\mathbb{N}]^\mathbb{N}$. Then $B_s = \mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S)$, which set does not admit a Borel Δ-coloring by Corollary 5.4. Consequently, $\Phi(B_s) = \emptyset$.

So, Theorem 5.2 is applicable and it yields a Borel set $C \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \times [\mathbb{N}]^\mathbb{N}$ so that $\{s : C_s \in \mathcal{U}^\Phi\}$ is $\mathbf{\Sigma}^1_2$-hard. This implies the desired conclusion by (3) of the Lemma above. □

We can prove Theorem 1.2. Let us restate the theorem, describing precisely what we mean by "form a $\mathbf{\Sigma}^1_2$-complete set".

Theorem 1.2. *Let X be an uncountable Polish space and $\Delta > 2$. The set*

$S = \{c \in \mathbf{BC}(X^2) : \mathbf{C}(X^2)_c$ *is a Δ-regular acyclic Borel graph with Borel chromatic number* $\leq \Delta\}$

is $\mathbf{\Sigma}^1_2$-complete.

In particular, Brooks' theorem has no analogue for Borel graphs in the following sense: there is no countable family $\{\mathcal{H}_i\}_{i \in I}$ of Borel graphs such that for any Borel graph $\mathcal{G}$ with $\Delta(\mathcal{G}) \leq \Delta$ we have $\chi_B(\mathcal{G}) > \Delta$ if and only if for some $i \in I$ the graph $\mathcal{G}$ contains a Borel homomorphic copy of $\mathcal{H}_i$.

Proof of Theorem 1.2. First, note that using the fact that the codes of Borel functions between Polish spaces form a $\mathbf{\Pi}^1_1$ set, it is straightforward to show that S is a $\mathbf{\Sigma}^1_2$ set (see e.g., [TV21, Proof of Theorem 1.3]). Similarly, one can check that if there was a collection $\{\mathcal{H}_i : i \in I\}$ as above, then this would yield that the set S is $\mathbf{\Pi}^1_2$. Thus, in order to show both parts of the theorem it suffices to prove that S is $\mathbf{\Sigma}^1_2$-hard.

Second, by [Sab12], it follows that if we replace continuous functions with Borel ones in the definition of $\mathbf{\Sigma}^1_2$-hard sets we get the same class. As uncountable Polish spaces are Borel isomorphic, it is enough to show that S is $\mathbf{\Sigma}^1_2$-hard for some X.

Take the graph $\mathcal{H}$ and the set C from Proposition 5.5. Note that the graph $\mathbf{Hom}^{ac}(T_\Delta, \mathcal{G}_S)$ is acyclic and has degrees $\leq \Delta$ by its construction. Therefore, the same holds for $\mathcal{H} \restriction C_s$ for each s. Using that the sets $D_i = \{(s,x,h) : \text{the degree of } (x,h) \text{ in } \mathcal{H} \restriction C_s \text{ is } i\}$ are Borel, it is straightforward to modify $\mathcal{H}$ so that we obtain a Borel graph $\mathcal{H}_\Delta$ on a Polish space of the form $X = \mathbb{N}^\mathbb{N} \times Y$ such that for each s the graph $\mathcal{H}_\Delta \restriction \{s\} \times Y$ is Δ-regular, acyclic and that the set $\{s : \chi_B(\mathcal{H}_\Delta \restriction \{s\} \times Y)) \leq \Delta\} = \{s : \chi_B(\mathcal{H} \restriction C_s) \leq \Delta\}$ (indeed, to vertices in D_i we can attach $\Delta - i$-many disjoint infinite rooted trees that are Δ-regular except for the root, which has degree $\Delta - 1$, in a Borel way). The third part of Proposition 2.4 gives a Borel reduction from the former set to S. Since the latter set is $\mathbf{\Sigma}^1_2$-hard, this yields the desired result by using [Sab12] as above. □

5.2 Hyperfiniteness

In this section we use Baire category arguments to obtain a new proof of Theorem 1.4.

Theorem 1.4. *There exists a hyperfinite Δ-regular acyclic Borel graph with Borel chromatic number $\Delta + 1$.*

We will utilize a version of the graph $\mathbb{G}_0$ constructed in [KST99]. For $s \in 2^{<\mathbb{N}}$ define

$$\mathbb{G}_s = \{(s^\frown(0)^\frown c, s^\frown(1)^\frown c) : c \in 2^{\mathbb{N}}\}$$

on $2^{\mathbb{N}}$. Fix some collection $(s_n)_{n\in\mathbb{N}} \subseteq 2^{<\mathbb{N}}$ such that $|s_n| = n$, i.e., $s_n \in 2^n$, for every $n \in \mathbb{N}$, together with a function $e : \mathbb{N} \to \Delta$ such that $(s_n)_{e(n)=i} \subseteq 2^{<\mathbb{N}}$ is dense in $2^{<\mathbb{N}}$ for every $i \in \Delta$. Set $\mathbb{G}_0 = \bigcup_{n\in\mathbb{N}} \mathbb{G}_{s_n}$. Label an edge α_i if it is in the graph $\bigcup_{e(n)=i} \mathbb{G}_{s_n}$. Finally, write $\mathcal{H}$ for the restriction of $\mathbb{G}_0$ to those vertices x such that every vertex in the connected component of x is adjacent to at least one edge of each label. Standard arguments yield the following claim.

Claim 5.7.

1. *$\mathcal{H}$ is acyclic and locally countable.*
2. *$\mathcal{H}$ is defined on a comeager subset of $2^{\mathbb{N}}$.*
3. *The Baire measurable edge-labeled chromatic number of $\mathcal{H}$ is infinite.*
4. *$\mathcal{H}$ is hyperfinite.*

Proof. $\mathcal{H}$ the fact the $\mathcal{H}$ is locally countable is clear from its definition, while acyclicity follows from the assumption that $|s_n| = n$. To see the second part, note that the set $\{x \in 2^{\mathbb{N}} : \forall i \in \Delta\ \exists n\ (e(n) = i \wedge s_n \sqsubset x)\}$ is open and dense in $2^{\mathbb{N}}$. Now, $\mathcal{H}$ is the restriction of $\mathbb{G}_0$ to a set that is an intersection of the image of this set under countably many homeomorphisms of the form $s_n^\frown(i)^\frown c \mapsto s_n^\frown(1-i)^\frown c$, hence its vertex set is comeager. The proof of the third part is identical to the proof of [KST99, Proposition 6.2]. Finally, the hyperfiniteness of $\mathcal{H}$ follows from the fact that $E_{\mathcal{H}} \subset \mathbb{E}_0$ (see, e.g., [JKL02, Proposition 1.3]). □

Proof of Theorem 1.4. By Claim 5.7 and Proposition 3.6 the graph $\mathbf{Hom}^e(T_\Delta, \mathcal{H})$ is hyperfinite, Δ-regular and acyclic. By Theorem 4.5 its Borel chromatic number is $\Delta + 1$. □

Note that the above theorem also implies Theorem 1.6 in [CJM+20] that asserts that Lovász Local Lemma (LLL) cannot be solved in a Borel way, even on hyperfinite graphs if the probability of a bad event is polynomial in the degree Δ of the dependency graph (for related results see [CGA+16] and [Ber20]). This is because the sinkless orientation problem from [BFH+16] can be thought of as an instance of the LLL as follows: Each edge corresponds to a random binary variable representing its orientation. At each node the bad event has probability $2^{-\Delta}$: all incident edges are oriented towards it. It remains to observe that a Borel solution to the sinkless orientation problem implies easily a Borel solution to the edge grabbing problem which in turn, by Remark 4.1, implies the existence of a Borel anti-game coloring. However, it follows from the proof of Theorem 4.5 that $\mathbf{Hom}^e(T_\Delta, \mathcal{H})$ does not admit a Borel anti-game coloring.

5.3 Graph homomorphisms

In this section we prove Theorem 1.5 that we restate here for the convenience of the reader.

Theorem 1.5. *For every $\Delta > 2$ and every $\ell \leq 2\Delta - 2$ there are a finite graph H and a Δ-regular acyclic Borel graph $\mathcal{G}$ such that $\chi(H) = \ell$ and $\mathcal{G}$ does not admit Borel homomorphism to H. The graph $\mathcal{G}$ can be chosen to be hyperfinite.*

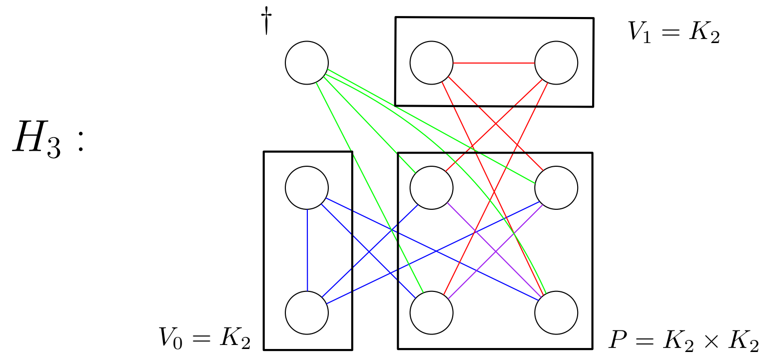

Figure 2: The maximal graph with the property Δ-(*) for $\Delta = 3$.

We remark that the first proof of the theorem (without the conclusion about hyperfiniteness) relied on a construction from the random graph theory, see [BCG+21] for motivation and connection to the LOCAL model. We sketch the construction here for completeness. Fix $k \in \mathbb{N}$, large enough depending on Δ, and consider $k\Delta$ pairings on a set n sampled independently uniformly at random. In another words, we have a $k\Delta$-regular graph and there is a canonical edge Δ-labeling such that each vertex is adjacent to exactly k edges of each color. Now taking a *local-global* limit of such graphs as $n \to \infty$ produces with probability 1 a an acyclic graphing with large edge-labeled chromatic number as needed, see [Bol81, HLS14].

Proof of Theorem Theorem 1.5. Note that it is enough to show the existence of such H and $\mathcal{G}$ with $\chi(H) = 2\Delta - 2$. Indeed, since erasing a vertex decreases the chromatic number by at most 1, we can produce subgraphs of H with chromatic number exactly ℓ for each $\ell \leq 2\Delta - 2$.

In the next paragraph we show that there is a finite graph H that satisfies the condition Δ-(*), see Definition 4.6, and has chromatic number $2\Delta - 2$. Then it is easy to see that taking the target graph $\mathcal{H}$ as in Claim 5.7 gives the conclusion by Proposition 3.6 and Theorem 4.7. $\square$

On condition Δ-(*). As we have seen, in Section 4.1, condition Δ-(*) is (formally) a weaker condition than having chromatic number at most Δ, but still allows us to use a version of Marks' technique. We will show that–similarly to the way the complete graph on Δ-many vertices, K_Δ, is maximal among graphs of chromatic number $\leq \Delta$–there exists a maximal graph (under homomorphisms) with property Δ-(*). It turns out that the chromatic number of the maximal graphs is $2\Delta - 2$.

Let us describe maximal examples of graphs that satisfy the condition Δ-(*). Recall that the (categorical) product $G \times H$ of graphs G, H is the graph on $V(G) \times V(H)$, such that $((g,h),(g',h')) \in E(G \times H)$ if and only if $(g,g') \in E(G)$ and $(h,h') \in E(H)$.

Write P for the product $K_{\Delta-1} \times K_{\Delta-1}$. Let V_0 and V_1 be vertex disjoint copies of $K_{\Delta-1}$. We think of vertices in V_i and P as having labels from $[\Delta-1]$ and $[\Delta-1] \times [\Delta-1]$, respectively. The graph H_Δ is the disjoint union of V_0, V_1, P and an extra vertex † that is connected by an edge to every vertex in P, and additionally, if v is a vertex in V_0 with label $i \in [\Delta-1]$, then we connect it by an edge with $(i',j) \in P$ for every $i' \neq i$ and $j \in [\Delta-1]$, and if v is a vertex in V_1 with label $j \in \Delta - 1$, then we connect it by an edge with $(i,j') \in P$ for every $j' \neq j$ and $i \in [\Delta-1]$. The graph H_3 is depicted in Fig. 2.

Proposition 5.8. *1. H_Δ satisfies Δ-(*).*

2. $\chi(H_\Delta) = 2\Delta - 2$.

3. A graph G satisfies Δ-(*) if and only if it admits a homomorphism to H_Δ.

Proof. (1) Set $R_0 = V(V_0) \cup \{\dagger\}$ and $R_1 = V(V_1) \cup \{\dagger\}$. By the definition there are no edges between R_0 and R_1. Consider now, e.g., $V(H_\Delta) \setminus R_0$. Let A_j consist of all elements in P that have second coordinate equal to j together with the vertex in V_1 that has the label j. By the definition, the set A_i is independent and $\bigcup_{i\in[\Delta-1]} A_i$ covers $H_\Delta \setminus R_0$, and similarly for R_1.

(2) By (1) and the claim after the definition of Δ-(*), it is enough to show that $\chi(H_\Delta) \geq 2\Delta - 2$. Towards a contradiction, assume that c is a proper coloring of H_Δ with $< 2\Delta - 2$-many colors. Note the vertex $\dagger$ guarantees that $|c(V(P))| \leq 2\Delta - 4$, and also $\Delta - 1 \leq |c(V(P))|$.

First we claim that there are no indices $i, j \in [\Delta - 1]$ (even with $i = j$) such that $c(i, r) \neq c(i, s)$ and $c(r, j) \neq c(s, j)$ for every $s \neq r$: indeed, otherwise, by the definition of P we had $c(i, r) \neq c(s, j)$ for every r, s unless $(i, r) = (s, j)$, which would the upper bound on the size of $c(V(P))$.

Therefore, without loss of generality, we may assume that for every $i \in [\Delta - 1]$ there is a color α_i and two indices $j_i \neq j'_i$ such that $c(i, j_i) = c(i, j'_i) = \alpha_i$. It follows form the definition of P and $j_i \neq j'_i$ that $\alpha_i \neq \alpha_{i'}$ whenever $i \neq i'$.

Moreover, note that any vertex in V_1 is connected to at least one of the vertices (i, j_i) and (i, j'_i), hence none of the colors $\{\alpha_i\}_{i\in[\Delta-1]}$ can appear on V_1. Consequently, since V_1 is isomorphic to $K_{\Delta-1}$ we need to use at least $\Delta - 1$ additional colors, a contradiction.

(3) First note that if G admits a homomorphism into H_Δ, then the pullbacks of the sets witnessing Δ-(*) will witness that G has Δ-(*).

Conversely, let G be a graph that satisfies Δ-(*). Fix the corresponding sets R_0, R_1 together with $(\Delta - 1)$-colorings c_0, c_1 of their complements. We construct a homomorphism Θ from G to H_Δ. Let

$$\Theta(v) = \begin{cases} \dagger & \text{if } v \in R_0 \cap R_1, \\ c_0(v) & \text{if } v \in R_1 \setminus R_0, \\ c_1(v) & \text{if } v \in R_0 \setminus R_1, \\ (c_0(v), c_1(v)) & \text{if } v \notin R_0 \cup R_1. \end{cases}$$

Observe that $R = R_0 \cap R_1$ is an independent set such that there is no edge between R and $R_0 \cup R_1$. Using this observation, one easily checks case-by-case that Θ is indeed a homomorphism. $\square$

Remark 5.9. *It can be shown that for $\Delta = 3$ both the Chvátal and Grötsch graphs satisfies the condition* 3-*(*).*

Remark 5.10. *Interestingly, recent results connected to counterexamples to Hedetniemi's conjecture yield Theorem 1.5 asymptotically, as $\Delta \to \infty$. Recall that Hedetniemi's conjecture is the statement that if G, H are finite graphs then $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$. This conjecture has been recently disproven by Shitov [Shi19], and strong counterexamples have been constructed later (see, [TZ19, Zhu21]). We claim that these imply for $\varepsilon > 0$ the existence of finite graphs H with $\chi(H) \geq (2-\varepsilon)\Delta$ to which Δ-regular Borel forests cannot have, in general, a Borel homomorphism, for every large enough Δ. Indeed, if a Δ-regular Borel forest admitted a Borel homomorphism to each finite graph of chromatic number at least $(2 - \varepsilon)\Delta$, it would have such a homomorphism to their product as well. Thus, we would obtain that the chromatic number of the product of any graphs of chromatic number $(2-\varepsilon)\Delta$ is at least $\Delta+1$. This contradicts Zhu's result [Zhu21], which states that the chromatic number of the product of graphs with chromatic number n can drop to $\approx \frac{n}{2}$.*

6 Remarks and further directions

Since the construction of homomorphism graphs is rather flexible, we expect that this method will find further applications. A direction that we do not take in this paper is to investigate homomorphism graphs corresponding to countable groups other than Γ_Δ. Another possible direction could be to understand the connection of our method with hyperfiniteness.

While Theorem 1.2 is optimal in the Borel context, one might hope that there is a positive answer in the case of graphs arising as compact, free subshifts of 2^{Γ_Δ}.

Question 6.1. *Is there a characterization of Borel graphs with Borel chromatic number $\leq \Delta$ that are compact, free subshifts of the left-shift action of Γ_Δ on 2^{Γ_Δ}?*

A way to answer this question on the negative would be to extend the machinery developed in [STD16] or [Ber21], so that the produced equivariant maps preserve the Borel chromatic number, their range is compact, and then apply Theorem 1.2; however, this seems to require a significant amount of new ideas.

Let us point out that Theorem 1.1 has a particularly nice form, if we assume Projective Determinacy or replace the Axiom of Choice with the Axiom of Determinacy (see, e.g., [CK11] for related results).

Theorem 6.2. *Let $\Delta > 2$.*

- *($\mathtt{PD}$) Let $\mathcal{H}$ be a locally countable Borel graph. Then*

$$\chi_{pr}(\mathcal{H}) > \Delta \iff \chi_{pr}(\mathbf{Hom}^{ac}(T_\Delta, \mathcal{H})) > \Delta,$$

  where χ_{pr} stands for the projective chromatic number of $\mathcal{H}$.

- *($\mathtt{AD} + \mathtt{DC}_{\aleph_0}$) Let $\mathcal{H}$ be a locally countable graph on a Polish space. Then*

$$\chi(\mathcal{H}) > \Delta \iff \chi(\mathbf{Hom}^{ac}(T_\Delta, \mathcal{H})) > \Delta.$$

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