Cognitive Hierarchies in Extensive Form Games*

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Abstract

In the cognitive hierarchy (CH) framework, players in a game have heterogeneous levels of strategic sophistication. Each player believes that other players in the game are less sophisticated, and these beliefs correspond to the truncated distribution of a "true" distribution of levels. We develop the dynamic cognitive hierarchy (DCH) solution by extending the CH framework to extensive form games. Initial beliefs are updated as the history of play provides information about players' levels of sophistication. We establish some general properties of DCH and fully characterize the DCH solution for a wide class of centipede games. DCH predicts a "representation effect": there will be earlier taking if the centipede game is played sequentially rather than simultaneously in its reduced normal form. Experimental evidence supports this prediction. In all three centipede games for which the DCH representation effect is predicted, termination occurs earlier when played sequentially (using the direct response method) compared to when the game is played in the reduced normal form (using the strategy method). In a fourth centipede game, where this effect is not predicted, it is not observed.

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"What surprised me was how bad they played."

—Beth Harmon, *The Queen's Gambit*

1 Introduction

In many situations, people interact with one another over time, in a multi-stage environment, such as playing chess or bargaining with alternating offers. The standard approach to studying these situations is to model them as extensive form games where equilibrium theory is applied, usually with refinements such as subgame perfection or other notions of sequential rationality. However, the standard equilibrium concepts, such as sequential equilibrium and its refinements, impose strong assumptions about the strategic sophistication of the players—perhaps implausibly strong from an empirical standpoint, as behavior in many laboratory experiments has suggested (see, for example, Camerer (2003)).

In response to these anomalous findings, researchers have proposed a variety of models that relax the requirement of mutual consistency of beliefs embodied in standard equilibrium concepts. The focus of this paper is the "level-k" family of models, which assume a hierarchical structure of strategic sophistication among the players, where level-k sophisticated players can think k strategic steps and believe everyone else is less sophisticated in the sense that they think fewer than k strategic steps. The *standard* level-k model assumes level-k players believe all other players are level-(k-1) (see Nagel (1995)).

However, applications of the level-k approach have been limited almost exclusively to the analysis of games in strategic form, where all players make their moves simultaneously, and the theory has not been formally developed for the analysis of general games in extensive form.[1] To apply the standard level-k model to extensive form games, one would assume that at each decision node, a level-k player will choose an action that maximizes the continuation value of the game, assuming all other players are level-(k-1) players in the continuation game. As a result, each player's belief about other players' level is *fixed* at the beginning. However, as the game proceeds, this fixed belief can lead to a logical conundrum, as a level-k player can be "surprised" by an opponent's previous move that is not consistent with the strategy of a level-(k-1) player.

If one closely examines this problem, the incompatibility derives from two sources that imply players cannot learn: (1) each level of player's prior belief about the other players' levels is degenerate, i.e., a singleton; and (2) players ignore the information contained in the history of the game. To solve both of these problems at the same time, as an alternative to the standard level-k approach, we use the cognitive hierarchy (CH) version of level-k, as proposed by Camerer et al. (2004), and extend it to games in extensive form. Like the standard level-k model, the CH framework posits that players are heterogeneous with respect to levels of strategic sophistication and believe that other players are less sophisticated. However, their beliefs are not degenerate. A level-k player believes all other players have lower levels distributed anywhere from level 0 to k-1.

[1] Some special cases have been studied, which we discuss below.

Furthermore, the CH framework imposes a partial consistency requirement that ties the players' prior beliefs on the level-type space to the true underlying distribution of levels. Specifically, a level-k player's beliefs are specified as the truncated true distribution of levels, conditional on levels ranging from 0 to k-1, i.e., players have "truncated rational expectations." This specification has the important added feature, relative to the standard level-k model, that more sophisticated players also have beliefs that are closer to the true distribution of levels, and very high level types have approximate rational expectations about the behavior of the other players. Thus, the CH approach blends aspects of purely behavioral models and equilibrium theory.

In our extension of CH to games in extensive form, a player's prior beliefs over lower levels are updated as the history of play in the game unfolds, revealing information about the distribution of other players' levels of sophistication. These learning effects can be quite substantial as we illustrate later in the paper. Hence, the main contribution of this paper is to propose a new CH framework—dynamic cognitive hierarchy (DCH)—for the general analysis of games in extensive form and, in doing so, provide new insights beyond those offered by the original CH model.

Our first result establishes that in games of perfect information every player will update their belief about each of the opponent's levels independently (Proposition 1). Second, we show that when the history of play in the game unfolds, players become more certain about the opponents' levels of sophistication, in a specific way. Formally, the support of their beliefs shrinks as the history gets longer (Proposition 2). Third, we show that the probability of paths with strictly dominated strategies being realized converges to zero as the distribution of levels increases (Proposition 3). Nonetheless, solution concepts based on iterated dominance, such as forward induction, can be violated even at the limit when the average level of sophistication converges to infinity. Relatedly, even though the players fully exploit the information from the history, it is not guaranteed that high-level players will use strategies that are consistent with the subgame perfect equilibrium of the game. In fact, behavior of the most sophisticated players can be inconsistent with backward induction, even at the limit when the level of sophistication of all players is arbitrarily high.

Although backward induction is a cornerstone principle of game theory, laboratory experiments reveal systematic behavioral deviations even in very simple games of perfect information. One prominent class of games where observed behavior is grossly inconsistent with backward induction is the increasing-pie "centipede game." This is an alternating move two-person game, where, in turn, each player can either "take" the larger of two pieces of the current pie, which terminates the game and leaves the other player with the smaller piece of the current pie, or "pass," which increases the size of the pie and allows the other player to take or pass. The game continues for a predetermined maximum number of turns. The subgame perfect equilibrium of this game of perfect information is solved by backward induction. Payoffs are such that it is optimal to take at the last stage, and both players have an optimal strategy to take if they expect the opponent will take at the next stage. Thus, backward induction implies that the game should end immediately.[2]

[2] Rosenthal (1981) introduced the centipede game to demonstrate how backward induction can be challenging and implausible to hold in some environments due to logical issues about updating off-path beliefs. His example is a ten-node game with a linearly increasing pie. Later on a shorter variant with an exponen-

Starting with McKelvey and Palfrey (1992), several laboratory and field experiments have reported experimental data from centipede games in a range of environments, such as different lengths of the game (see McKelvey and Palfrey (1992) and Fey et al. (1996)), different subject pools (see Palacios-Huerta and Volij (2009), Levitt et al. (2011), and Li et al. (2021)), different payoff configurations (see García-Pola et al. (2020b), Fey et al. (1996), Zauner (1999), Kawagoe and Takizawa (2012), and Healy (2017)) and different experimental methods (Nagel and Tang (1998), Bornstein et al. (2004) , García-Pola et al. (2020a), and Rapoport et al. (2003)). Although standard game theory predicts the game should end in the first stage, such behavior is rarely observed. To this end, we study a family of centipede games with a linearly increasing pie where our DCH solution makes clear predictions about the evolution of beliefs as the game unfolds. Our particular interest in the analysis is the finding that DCH implies a *representation effect* that predicts specific violations of strategic invariance. Consider the following two strategically equivalent extensive form representations of the centipede game. In the first (the usual representation used in centipede experiments), the game is played as an alternate move game: that is, first player 1 decides to take or pass. If they take the game ends; if they pass, it is player 2's turn to pass or take, and so forth. In the second representation, first player 1 decides at which node to take if the game gets that far, or always pass; the second player then makes the same decision without observing the first player's choice. We show that DCH implies that centipede games played according to the first extensive form representation of the game will end *earlier, with lower payoffs to the players,* compared to the game played according to the second (strategically equivalent) version, where each player independently chooses a stopping strategy, without observing the choice of their opponent (Theorem 1).

Our theoretical analysis of the representation effect in centipede games shows how DCH can provide new insights related to an unresolved debate in experimental methodology: whether or not the *direct-response method* is behaviorally equivalent to *the strategy method* (see Brandts and Charness (2011)). Under the direct-response method of obtaining behavioral data for a game, the game is played sequentially according to its extensive form representation. Thus, each player can observe the actions taken by others in the previous subgames and update their beliefs about other players' levels. In contrast, under the strategy method, the game is played simultaneously according to the reduced normal form of the game. Thus, players take actions in hypothetical situations without observing any moves of the other players, so each player's choices are guided solely by their prior beliefs. Our DCH solution will generally predict different patterns of behavior and outcomes under two different methods despite the fact that they are strategically equivalent.[3]

While the direct-response method is the most commonly used method to implement centipede game experiments, there are a few exceptions. Nagel and Tang (1998) is the first

tially increasing pie, called "share or quit," is studied by Megiddo (1986) and Aumann (1988). The name centipede was coined by Binmore (1987), and named for a 100-node variant.

[3]We consider the representation effect with respect to the *reduced* normal form, because that is the typical implementation of the strategy method in economics experiments, and has always been the case with centipede game experiments. A representation effect does not arise in DCH if level-0 players uniformly randomize across all non-reduced contingent strategies (i.e., separately including outcome equivalent strategies such as "always take" and "take at the first stage but pass at all later stages"), and all strategic types perfectly best respond. This is discussed in section 7.4.

paper to report the results from a centipede game experiment conducted as a simultaneous move game, the reduced normal form. In their 12-node centipede games, each player has seven available strategies that correspond to an intended "take-node" or always passing, and they make their decisions simultaneously. Pooling the data over many repetitions, they find that only 0.5% of outcomes coincide with the equilibrium prediction, suggesting that the non-equilibrium behavior in the centipede games cannot solely be attributed to the violation of backward induction. However, as the authors remarked, the results may be confounded with the effect of reduced normal form: "...There might be substantial differences in behavior in the extensive form game and in the normal form game..." (Nagel and Tang (1998), p. 357). One of our contributions is to show that DCH provides a theoretical rationale for the existence of this representation effect.[4]

A recent experiment by García-Pola et al. (2020a) specifically studies how the direct-response method and the strategy method would affect the behavior in four different centipede games. Our DCH solution makes a sharp prediction about the representation effect—in three of the four games, the distribution of terminal nodes under the strategy method will first order stochastically dominate the distribution under the direct-response method, but not in the fourth game. We revisit the data from that experiment, and show that the results from all four of their games are consistent with the representation effect predicted by DCH.

The paper is organized as follows. The related literature is discussed in the next section. Section 3 sets up the model. Section 4 establishes properties of the DCH belief-updating process and explores the relationship between our model and subgame perfect equilibrium with several examples. In Section 5, the representation effect is explored in a detailed analysis of centipede games with a linearly increasing pie. Experimental data that provide a test the representation effect hypothesis is presented and discussed in section 6. We discuss several additional features of our model in section 7 and conclude in section 8.

2 Related Literature

The idea of limited depth of reasoning in games of strategy has been proposed and studied by economists and game theorists for at least thirty years (see, for example, Binmore (1987, 1988), Selten (1991, 1998), Aumann (1992), Stahl (1993), and Alaoui and Penta (2016, 2018)). On the empirical side, Nagel (1995) conducts the first laboratory experiment explicitly designed to study hierarchical reasoning in simultaneous move games, using the "beauty contest" game. Each player chooses a number between 0 and 100. The winner is the player whose choice is closest to the average of all the chosen numbers discounted by a parameter $p \in (0, 1)$. To analyze the data, Nagel (1995) assumes level-0 players choose randomly. Level-1 players believe all other players are level-0 and best respond to them by choosing $50p$. Following the same logic, level-k players believe all other players are level-(k-1) and best respond to them with $50p^k$.

This iterative definition of hierarchies has been applied to a range of different environments. For instance, Ho et al. (1998) also analyze the beauty contest game while Costa-Gomes et al. (2001) and Crawford and Iriberri (2007a) consider the strategic levels in a vari-

[4] Such a representation effect is also a feature of QRE. See Goeree et al. (2016), pp. 67-72 and 80-85.

ety of simultaneous move games. Costa-Gomes and Crawford (2006) study the "two-person guessing game," a variant of the beauty contest game. Finally, the level-k approach has also been applied to games of incomplete information. Crawford and Iriberri (2007b) apply this approach to reanalyze auction data, and Cai and Wang (2006) and Wang et al. (2010) use the level-k model to organize empirical patterns in experimental sender-receiver games. All these studies assume level-k players best respond to degenerate beliefs of level-(k-1) players.

This standard level-k model has been extended in a number of ways. One such approach is that each level of player best responds to a mixture of all lower levels. Stahl and Wilson (1995) are the first to construct and estimate a specific mixture model of bounded rationality in games where each level of player best responds to a mixture between lower levels and equilibrium players.[5] Camerer et al. (2004) develop the CH framework, where level-k players best respond to a *mixture* of the behavior of all lower level behavioral types from 0 to k-1. In addition, players have correct beliefs about the relative proportions of these lower levels, so it includes a consistency restriction on beliefs in the form of truncated rational expectations.

A second direction is to endogenize the strategic levels of players, using a cost benefit approach. Alaoui and Penta (2016) develop a model of endogenous depth of reasoning, where each player trades off the benefit of additional levels of sophistication against the cost of doing so. Players can have different benefit and cost functions, depending on their beliefs and strategic abilities, respectively. A model is developed for two-person games with complete information and calibrated against experimental data. Alaoui et al. (2020) provide some additional analysis and a laboratory experiment that further explores the implications of this model.

Third, De Clippel et al. (2019) study the implications of the standard level-k approach to mechanism design. They establish a form of the revelation principle for level-k implementation and obtain conditions on the implementability of social choice functions under a range of assumptions about level-0 behavior.

Fourth, there have been several papers that model how an individual's strategic level evolves when the same game is repeated multiple times.[6] The standard level-k model is ideally suited to understanding how naive individuals behave when they encounter a game for the first time. This is a limitation since, in most laboratory experiments in economics and game theory, subjects play the same game with multiple repetitions, in order to gain experience and to facilitate convergence to equilibrium behavior. It is also a limitation since many games studied by economists and other social scientists are aimed at understanding strategic interactions between highly experienced players (oligopoly, procurement auctions, legislative bargaining, for example), where some convergence to equilibrium would be natural to expect.

In this vein, Ho and Su (2013) and Ho et al. (2021) propose a modification of CH that

[5]In the same spirit of Stahl and Wilson (1995), Levin and Zhang (2022) propose the NLK solution which is an equilibrium model where each player best responds to a mixture of an exogenously determined naive strategy (with probability λ) and the equilibrium strategy (with probability $1-\lambda$). The main difference between NLK and our DCH solution is that NLK requires the belief system to be mutually consistent (as NLK is a solution of a fixed problem) while DCH relaxes this requirement.

[6]Nagel (1995) proposes a model for the evolution of levels based on changing reference points to explain unravelling in guessing games.

allows for learning across repeated plays of the same sequential game, in a different way than in Stahl (1996), but in the same spirit. In their setting, an individual player repeatedly plays the same game (such as the guessing game) and updates his or her beliefs about the distribution of levels *after* observing past outcomes of earlier games, but holding fixed beliefs during each play of the game. In addition to updating beliefs about other players' levels, a player also *endogenously* chooses a new level of strategic sophistication for themselves, in the spirit of Stahl (1996), for the next iteration of the game. This is different from our DCH framework where each player updates their beliefs about the levels of other players after each move within a single game. Moreover, because players are forward-looking in DCH, they are *strategic learners*—i.e., they correctly anticipate the evolution of their posterior beliefs in later stages of the game—which leads to a much different learning dynamic compared with naive adaptive learning models.

All of these extensions add significantly to the literature on level-k behavior for games in strategic form, by allowing for a richer set of heterogeneous beliefs, by incorporating cognitive costs into the model, and by showing how the model can be used to address classic mechanism design problems, but all of them are limited by a restriction to simultaneous move games. In extensive form games, the timing structure is crucial, and beliefs evolve as the game unfolds and players have an opportunity to adjust their beliefs in response to past actions, which is the focus of this paper. Our DCH provides an extension in this direction, under the assumption that players are forward looking about the actions of their opponents in the entire game tree. Rampal (2022) develops an alternative approach for multi-stage games of perfect information. He models levels of sophistication by assuming that players have limited foresight in the sense of a rolling horizon; that is, players only look forward a fixed number of stages. This creates a hierarchy of strategic sophistication that depends on the length of a player's rolling horizon. In addition, players are uncertain about their opponents' foresight. The baseline game of perfect information is then transformed into a game of incomplete information, with specific assumptions about players' beliefs about payoffs at non-terminal nodes that correspond to the current limit of their horizon in the game.

At a more conceptual level, our dynamic generalization of CH is related to other behavioral models in game theory. There is a connection between DCH and misspecified learning models (see, for example, Hauser and Bohren (2021)) in the sense that level-k players wrongly believe all other players are less sophisticated. However, in contrast to categorical types of players in misspecified learning models, DCH provides added structure to the set of types in a systematic way, such that higher-level types have a more accurate belief about opponents' rationality at the aggregate level. In the context of social learning, application of our model to the investment game is related to Eyster and Rabin (2010) and Bohren (2016) who model the updating process when there exist some behavioral types of players in the population.

Our DCH solution is also related to two other behavioral solution concepts of the dynamic games—the Agent Quantal Response Equilibrium (AQRE) by McKelvey and Palfrey (1998) and the Cursed Sequential Equilibrium (CSE) by Fong et al. (2023)—in the sense that each of these solution concepts relaxes different requirements of the standard equilibrium theory. DCH is a non-equilibrium model which allows different levels of players to best respond to different conjectures about other players' strategies while AQRE is an equilibrium model where players make stochastic choices. Both DCH and AQRE depict that players are able to perform Bayesian inferences. In contrast, CSE is an equilibrium model where players are able to make best response but fail to perform Bayesian inferences.

3 The Model

This section formally develops the dynamic cognitive hierarchy (DCH) model for extensive form games. In section 3.1, we introduce the notation for extensive form games by following Osborne and Rubinstein (1994). Next, we define the DCH updating process in section 3.2, specifying how players' beliefs about other players' levels evolve from the history of play. This leads to a definition of the *DCH solution* of a game.

3.1 Extensive Form Games

Let $N_c = \{0, 1, \ldots, n\} \equiv \{c\} \cup N$ be a finite set of *players*, where player c is called "chance." Let H be a finite set of *sequences* that satisfies the following two properties.

1. The empty sequence ϕ (the initial history) is a member of H.
2. If $(a^j)_{j=0,\ldots,J} \in H$ and $L < J$, then $(a^j)_{j=0,\ldots,L} \in H$.

Each $h \in H$ is a *history* and each component of a history is an *action* taken by a player. In addition, for any non-initial history $h = (a^j)_{j=0,\ldots,L}$, we use $\alpha(h) = (a^j)_{j=0,\ldots,L-1}$ to denote the unique precedent history of h. A history $(a^j)_{j=0,\ldots,J} \in H$ is a *terminal history* if there is no a^{J+1} such that $(a^j)_{j=0,\ldots,J+1} \in H$. The set of terminal histories is denoted as Z and $H \backslash Z$ is the set of non-terminal histories. Moreover, for every non-terminal history $h \in H \backslash Z$, $A(h) = \{a : (h, a) \in H\}$ is the set of available actions after the history h, and Z_h is the set of terminal histories after h.

The *player function* $P : H \backslash Z \to N_c$ assigns to each non-terminal history a player of N_c. In other words, $P(h)$ is the player who takes an action after history h. With this, for any player $i \in N_c$, $H_i = \{h \in H \backslash Z : P(h) = i\}$ is the set of histories where player $i \in N_c$ is the active player. Therefore, H_c is the set of non-terminal histories where chance determines the action taken after history h. A function σ_c specifies a probability measure $\sigma_c(\cdot|h)$ on $A(h)$ for every $h \in H_c$. That is, $\sigma_c(a|h)$ is the probability that a occurs after the history h.

For each $i \in N$, a partition $\mathcal{I}_i$ of H_i defines i's information sets. Information set $I_i \in \mathcal{I}_i$ specifies a subset of histories contained in H_i that i cannot distinguish from one other, where for any $h \in H_i$, $I_i(h)$ is the element of $\mathcal{I}_i$ that contains h. Furthermore, i's available actions are the same for all histories in the same information set. Formally, for any history $h' \in I_i(h) \in \mathcal{I}_i$, $A(h') = A(h)$.[7] Therefore, we use $A(I_i)$ to denote the set of available actions at information set I_i. In addition, each player $i \in N$ has a *payoff function* (in von Neumann-Morgenstern utilities) $u_i : Z \to \mathbb{R}$. Finally, an *extensive form game*, Γ, is defined by the tuple $\Gamma = \langle N_c, H, P, \sigma_c, (\mathcal{I}_i)_{i\in N}, (u_i)_{i\in N} \rangle$.

[7]We assume that all players in the game have perfect recall. See Osborne and Rubinstein (1994) chapter 11 for a definition.

In an extensive form game Γ, for each player $i \in N$, a *behavioral strategy* of player i is a collection $(\sigma_i(I_i))_{I_i \in \mathcal{I}_i}$ of independent probability measures where $\sigma_i(I_i)$ is a probability measure over $A(I_i)$ and $\sigma_i(a|I_i)$ is the probability that a is chosen after the information set I_i. In short, a behavioral strategy of player i is a function $\sigma_i : \mathcal{I}_i \to \Sigma_i$ where $\Sigma_i \equiv \times_{I_i \in \mathcal{I}_i} \Delta(A(I_i))$ is the set of behavioral strategies for player i. For notational convenience, we use $\Sigma \equiv \times_{i \in N} \Sigma_i$ to denote the set of all behavioral strategy profiles. Lastly, we use the notation $\Sigma_{-i} = \times_{j \neq i} \Sigma_j$ and write elements of Σ as $\sigma = (\sigma_i, \sigma_{-i})$ when we want to focus on a particular player $i \in N$.

3.2 Cognitive Hierarchies and Belief Updating

Each player i is endowed with a level of sophistication, $\tau_i \in \mathbb{N}_0$, where $\Pr(\tau_i = k) = p_{ik}$ for all $i \in N$ and $k \in \mathbb{N}_0$, and the distribution is *independent* across players.[8] Without loss of generality, we assume $p_{ik} > 0$ for all $i \in N$ and $k \in \mathbb{N}_0$. Let $\tau = (\tau_1, \ldots, \tau_n)$ be the level profile and τ_{-i} be the level profile without player i. Each player i has a prior belief about all other players' levels and these prior beliefs satisfy *truncated rational expectations.* That is, for each i and k, a level-k player i believes all other players in the game are at most level-(k-1). For each i, $j \neq i$ and k, let $\mu_{ij}^k(\tau_j)$ be level-k player i's prior belief about player j's level, and $\mu_i^k(\tau_{-i}) = \left(\mu_{ij}^k(\tau_j)\right)_{j \neq i}$ be level-k player i's prior belief profile. Furthermore, for each i and k, level-k player i believes any other player j's level is independently distributed according to the lower truncated probability distribution function:

$$\mu_{ij}^k(\kappa) = \begin{cases} \frac{p_{j\kappa}}{\sum_{m=0}^{k-1} p_{jm}} & \text{if } \kappa < k \\ 0 & \text{if } \kappa \geq k. \end{cases} \tag{1}$$

The assumption underlying μ_{ij}^k is that level-k types of each player have a correct belief about the relative proportions of players who are less sophisticated than they are, but maintain the incorrect belief that other players of level $\kappa \geq k$ do not exist. The j subscript indicates that different players can have different level distributions.

A strategy profile is now a *level-dependent* profile of behavior strategies for each level of each player. Thus, let σ_i^k be the behavioral strategy adopted by level-k player i, where, σ_i^0 uniformly randomizes at each information set.[9] That is, for all $i \in N$, $I \in \mathcal{I}_i$ and $a \in A(I_i)$,

$$\sigma_i^0(a|I) = \frac{1}{|A(I)|}.$$

[8]For the sake of simplicity, we assume that the distribution of levels is *independent* of the probability distribution of chance's moves σ_c.

[9]Uniform randomization is not the only way to model non-strategic level-0 players, but there are several justifications for doing so. One compelling reason is that it is universally applicable to all games, in exactly the same way, unlike almost any other specification since the cardinality of the set of available actions will typically vary across players and information sets, Probably for this reason, it is the most commonly used approach in applications of CH, including the original specification in Camerer et al. (2004). By taking the same approach in our generalization of CH to games in extensive form, it allows for clear comparisons to the original CH framework. In principle the number of specifications of level-0 behavior is enormous, especially for games with many strategies or information sets. Some alternatives to uniform randomization are noted in section 7.4.

In the following we may interchangeably call level-0 players *non-strategic players* and level $k \geq 1$ players *strategic players*. At every information set I, each strategic player i with level $k > 1$ forms a *joint* belief about other players' levels of sophistication and the distribution of histories $h \in I$. Their posterior beliefs at I depend on the level-dependent strategy profile of the other players, and their prior belief about the distribution of levels, μ_i^k.[10] This updating process is formalized with some additional notation. Let $\sigma_j^{-k} = \left(\sigma_j^0, ..., \sigma_j^{k-1}\right)$ be the profile of strategies adopted by the levels below k of player j. In addition, let $\sigma_{-i}^{-k} = \left(\sigma_1^{-k}, ..., \sigma_{i-1}^{-k}, \sigma_{i+1}^{-k}, ..., \sigma_n^{-k}\right)$ denote the strategy profile of the levels below k of all players other than player i.

All strategic players believe every *history* (and hence information set) is reached with positive probability because $\mu_{ij}^k(0) > 0$ for all $i, j, k > 0$ and $\sigma_j^0\left(a|I\right) > 0$ for all j, $I \in \mathcal{I}_j$ and $a \in A(I)$. The fact that every information set in the game is reached with positive probability implies that, for any i, $k > 0$, $I \in \bigcup_{j\in N} \mathcal{I}_j$, and given any level-dependent strategy profile, σ, and prior distribution of levels, μ_i^k, Bayes rule can be applied to derive level-k player i's posterior belief about the joint distribution of other players' levels, τ_{-i} (lower than k) and histories $h \in I$. This posterior belief is denoted by $\nu_i^k\left(\tau_{-i}, h \mid I, \sigma_{-i}^{-k}\right)$ and call $\left\{\nu_i^k\left(\tau_{-i}, h \mid I, \sigma_{-i}^{-k}\right)\right\}_{I\in\bigcup_{j\in N}\mathcal{I}_j}$ level-k player i's *contingent posterior belief system* induced by σ. Finally, we denote level-k player i's marginal posterior belief about other players' levels at information set I as

$$\nu_i^k\left(\tau_{-i} \mid I, \sigma_{-i}^{-k}\right) \equiv \sum_{h\in I} \nu_i^k\left(\tau_{-i}, h \mid I, \sigma_{-i}^{-k}\right)$$

and the marginal posterior belief about player j's level at information set I as $\nu_{ij}^k\left(\tau_j \mid I, \sigma_{-i}^{-k}\right)$.

In the DCH model, players correctly anticipate how they will update their posterior beliefs about other players' levels at all future histories of the game. Thus, level-k player i *believes* others are using the (normalized) strategy profile, $\tilde{\sigma}_{-i}^{-k} = (\tilde{\sigma}_1^{-k}, \ldots, \tilde{\sigma}_{i-1}^{-k}, \tilde{\sigma}_{i+1}^{-k}, \ldots, \tilde{\sigma}_n^{-k})$, where for any $j \neq i$, any $I_j \in \mathcal{I}_j$ and any $a \in A(I_j)$:

$$\tilde{\sigma}_j^{-k}(a \mid I_j) = \sum_{\kappa=0}^{k-1} \nu_{ij}^k\left(\kappa \mid I_j, \sigma_{-i}^{-k}\right) \cdot \sigma_j^\kappa\left(a \mid I_j\right).$$

In general, the posterior distribution of levels of other players will be different for different levels of the same player at the same information set, since the supports of those distributions will generally differ.[11] This, in turn, induces different levels of the same player to have different beliefs about the probability distribution over the terminal payoffs that can be reached from that information set. For each $i \in N$, $k > 0$, σ, and τ_{-i} such that $\tau_j < k$ for

[10] Strategic players whose level is $k = 1$ do not update, since they have a degenerate prior belief that all other players are level $k = 0$. Also, note that player i updates their beliefs at every information set, not only at information sets in $\mathcal{I}_i$.

[11] However, the support of the beliefs of all levels of all players will always include the type profile τ_{-i}^0, in which all other players are level-0. That is, $\nu_i^k\left(\tau_{-i}^0, h \mid I, \sigma_{-i}^{-k}\right) > 0$ for all $i \in N$, $k \in \mathbb{N}$, information set I and $h \in I$.

all $j \neq i$, let $\tilde{\rho}_i^k(z|h, \tau_{-i}, \sigma_{-i}^{-k}, \sigma_i^k)$ be level-k player i's belief about the conditional realization probability of $z \in Z_h$ at history $h \in H \backslash Z$, if the profile of levels of the other players is τ_{-i} and i is using strategy σ_i^k. Finally, level-k of player i's conditional expected payoff at information set I is given by:

$$\mathbb{E}u_i^k(\sigma|I) = \sum_{h' \in I} \sum_{\{\tau_{-i}:\tau_j<k \ \forall j \neq i\}} \sum_{z \in Z_h} \nu_i^k \left(\tau_{-i}, h' \mid I, \sigma_{-i}^{-k}\right) \tilde{\rho}_i^k(z|\tau_{-i}, h', \sigma_{-i}^{-k}, \sigma_i^k) u_i(z). \tag{2}$$

The *DCH solution* of the game is defined as the level-dependent strategy profile, σ^*, such that $\sigma_i^{k*}(\cdot|I_i)$ maximizes $\mathbb{E}u_i^k(\sigma^*|I_i)$ for all i, k, I_i[12] and the belief system induced by σ^* is called the *DCH belief system*.

Remark 1. *For games represented in the normal form, the DCH solution is defined by considering the strategically equivalent extensive form where players each move once, in sequence, without observing other players' actions (see section 4.3 for an example). In this case, the DCH solution reduces to the standard CH solution. In the remainder of the paper, we will refer it as* ***the DCH solution in normal form****.*

Remark 2. *Posterior belief systems are defined on the joint distribution of levels and histories at each information set. An alternative formulation of posterior belief systems is to define level-k player i's updating process on τ_{-i} history-by-history, i.e., $\nu_i^k(\tau_{-i}|h, \sigma_{-i}^{-k})$, and then derive the conditional posterior beliefs, $\pi_i^k(h)$ over the histories in every information set in the game, for all i, k,and I, $\sum_{h' \in I} \pi_i^k(h') = 1$. See Lin and Palfrey (2022). For games with perfect information, these two formulations are identical since every information set is singleton. Therefore, in games with perfect information, for every level-k player i and any non-terminal history $h \in H \backslash Z$, we will simply use $\nu_i^k(\tau_{-i}|h, \sigma_{-i}^{-k})$ to represent the DCH posterior belief system.*

To simplify notation and exposition, most of the remainder of the paper studies DCH in games of perfect information. Some the properties established in the next section for games of perfect information apply more generally, and these cases are noted. We discuss additional extensions to games with imperfect information in sections 7.2 and 7.3.

4 Properties of the DCH Solution

Section 4.1 first establishes the general properties of the belief-updating process. Section 4.2 explores the relationship between the DCH solution and subgame perfect equilibrium. Finally, we illustrate the *representation effect* predicted by DCH in section 4.3. Specifically, we show that the DCH solution might be dramatically different in extensive form and its corresponding reduced normal form.

[12] We assume (as is typical in level-k models) that players randomize uniformly over optimal actions when indifferent. This assumption is convenient because it ensures a unique DCH solution to every game, so we assume it here. Note that while the DCH solution is defined as a fixed point, it can be solved for recursively, starting with the lowest level and iteratively working up to higher levels.

4.1 Properties of the Belief-Updating Process

The first result shows that for games of perfect information, the updating process satisfies an independence property.[13] Specifically, the following proposition establishes that all levels of all players will update their posterior beliefs about other players' levels independently.

Proposition 1. *For any game of perfect information* Γ*, any* $h \in H \backslash Z$*, any* $i \in N$*, and for any* $k \in \mathbb{N}$*, level-*k *player* i*'s posterior belief about other players' levels at history* h *is independent across players. That is,* $\nu_i^k\left(\tau_{-i} \mid h, \sigma_{-i}^{-k}\right) = \prod_{j \neq i} \nu_{ij}^k\left(\tau_j \mid h, \sigma_{-i}^{-k}\right)$.

Proof: We prove this proposition by induction on the length of the sequence of h, which we denote by $|h|$. Let σ be any level-dependent strategy profile and p be any prior distribution over types. First we can notice that at the initial history, i.e., $|h| = 0$, for any $i \in N$ and level $k > 0$, $\nu_i^k\left(\tau_{-i} \mid \phi, \sigma_{-i}^{-k}\right) = \prod_{j \neq i} \mu_{ij}^k(\tau_j)$ as players' levels are independently determined.

In the following, for any $h', h'' \in H$ where h' is a subsequence of h'', we slightly abuse the notation to use h'' to denote the unique action at h' that leads to h''. To establish the base case, suppose $|h| = 1$ with $j = P(\alpha(h))$, and consider any $i \neq j$ who is some level $k > 0$. Because player j has made the only move in the game so far, and the prior distribution of types is assumed to be independent across players, we have, for any τ_{-i} such that $\tau_{i'} < k$ $\forall i' \neq i, j$:

$$\begin{aligned}
\nu_i^k\left(\tau_{-i} \mid h, \sigma_{-i}^{-k}\right) &= \frac{\sigma_j^{\tau_j}(h \mid \alpha(h))\mu_{ij}^k(\tau_j)}{\sum_{l=0}^{k-1} \sigma_j^l(h \mid \alpha(h))\mu_{ij}^k(l)} \prod_{i' \neq i,j} \mu_{ii'}^k(\tau_{i'}) \\
\nu_{ij}^k\left(\tau_j \mid h, \sigma_{-i}^{-k}\right) &= \frac{\sigma_j^{\tau_j}(h \mid \alpha(h))\mu_{ij}^k(\tau_j)}{\sum_{l=0}^{k-1} \sigma_j^l(h \mid \alpha(h))\mu_{ij}^k(l)} \\
\nu_{ii'}^k\left(\tau_{i'} \mid h, \sigma_{-i}^{-k}\right) &= \mu_{ii'}^k(\tau_{i'}) \\
&\Longrightarrow \\
\nu_i^k\left(\tau_{-i} \mid h, \sigma_{-i}^{-k}\right) &= \prod_{j \neq i} \nu_{ij}^k\left(\tau_j \mid h, \sigma_{-i}^{-k}\right)
\end{aligned}$$

where, we know $\sum_{l=0}^{k-1} \sigma_j^l(h \mid \alpha(h)) > 0$ because $\sigma_j^0(h \mid \alpha(h)) = \frac{1}{|A(\alpha(h))|} > 0$. Hence, the result is true for $|h| = 1$. Next, consider any h such that $|h| = t > 1$ and $h \in H \backslash Z$ and suppose that $\nu_i^k\left(\tau_{-i} \mid h, \sigma_{-i}^{-k}\right) = \prod_{j \neq i} \nu_{ij}^k\left(\tau_j \mid h, \sigma_{-i}^{-k}\right)$ for all h such that $|h| = 1, 2, ..., t-1$. Let $j = P(\alpha(h))$ and consider any $i \neq j$ who is some level $k > 0$. Because only player j has

[13] This property of independence of the history-conditional belief updating process on levels holds in general for all extensive form games with perfect recall, including games of imperfect information. See Proposition 1 in Lin and Palfrey (2022).

moved, going from $\alpha(h)$ to h, we have for any τ_{-i} such that $\tau_{i'} < k \ \forall i' \neq i, j$:

$$\begin{aligned}
\nu_i^k\left(\tau_{-i} \mid h, \sigma_{-i}^{-k}\right) &= \frac{\sigma_j^{\tau_j}(h \mid \alpha(h))\nu_{ij}^k\left(\tau_j \mid \alpha(h), \sigma_{-i}^{-k}\right)}{\sum_{l=0}^{k-1}\sigma_j^l(h \mid \alpha(h))\nu_{ij}^k\left(l \mid \alpha(h), \sigma_{-i}^{-k}\right)} \prod_{i' \neq i,j} \nu_{ii'}^k\left(\tau_{i'} \mid \alpha(h), \sigma_{-i}^{-k}\right) \\
\nu_{ij}^k\left(\tau_j \mid h, \sigma_{-i}^{-k}\right) &= \frac{\sigma_j^{\tau_j}(h \mid \alpha(h))\nu_{ij}^k\left(\tau_j \mid \alpha(h), \sigma_{-i}^{-k}\right)}{\sum_{l=0}^{k-1}\sigma_j^l(h \mid \alpha(h))\nu_{ij}^k\left(l \mid \alpha(h), \sigma_{-i}^{-k}\right)} \\
\nu_{ii'}^k\left(\tau_{i'} \mid h, \sigma_{-i}^{-k}\right) &= \nu_{ii'}^k\left(\tau_{i'} \mid \alpha(h), \sigma_{-i}^{-k}\right) \\
&\Longrightarrow \\
\nu_i^k\left(\tau_{-i} \mid h, \sigma_{-i}^{-k}\right) &= \prod_{j \neq i} \nu_{ij}^k\left(\tau_j \mid h, \sigma_{-i}^{-k}\right)
\end{aligned}$$

as desired. ■

What drives this result is that in games of perfect information, when player j moves, all players perfectly observe this history. As a result, all players other than j only update their beliefs about the level of player j, and do not update their beliefs about any of the other players. In addition, the assumption of independence of the distribution of player levels is used. If levels are correlated across players then it's possible that player i can update their beliefs about the level of player j based on actions taken by player l. From Proposition 1, we can see that the marginal posterior belief of level-k player i to player j's belief only depends on player j's moves along the history. Therefore, we can obtain that $\nu_{ij}^k\left(\kappa \mid h, \sigma_{-i}^{-k}\right) = \nu_{ij}^k\left(\kappa \mid h, \sigma_j^{-k}\right)$. Specifically,

$$\nu_{ij}^k\left(\kappa \mid h, \sigma_j^{-k}\right) = \begin{cases} \frac{\mu_{ij}^k(\kappa) f_j(h|\sigma_j^\kappa)}{\sum_{m=0}^{k-1}\mu_{ij}^k(m) f_j(h|\sigma_j^m)} & \text{if } \kappa < k \\ 0 & \text{if } \kappa \geq k, \end{cases}$$

where $f_j(h \mid \sigma_j^\kappa)$ is the probability that player j moves along the path to reach h given player j is using the strategy σ_j^κ.[14]

The second property of the DCH model is that in the later histories, the support of the posterior beliefs is (weakly) shrinking. In this sense, the players would have a more precise posterior belief when the history gets longer. For any player $i, j \in N$ such that $i \neq j$, for any $h \in H \backslash (Z \cup \{\phi\})$, and for any $k \in \mathbb{N}$, we denote the support of level-k player i's belief about player j's level as

$$supp_{ij}^k(h) \equiv \{\tau_j \in \{0, 1, ..., k-1\} \mid \nu_{ij}^k\left(\tau_j \mid h, \sigma_j^{-k}\right) > 0\}.$$

This property is formally stated in the following proposition.

[14] For any history h' and h'', we define a partial order $\prec$ on H such that $h' \prec h''$ if and only if h' is a subsequence of h''. With this notation, the probability $f_j(h \mid \sigma_j^\kappa)$ can be computed by

$$f_j(h \mid \sigma_j^\kappa) = \begin{cases} \Pi_{h' \in H_j \cap \{\tilde{h}:\tilde{h} \prec h\}}\ \sigma_j^\kappa(h \mid h') & \text{if } H_j \cap \{\tilde{h} : \tilde{h} \prec h\} \neq \emptyset \\ 1 & \text{otherwise.} \end{cases}$$

Proposition 2. *In any finite extensive form game of perfect information* Γ*, for all* $i, j \in N$*,* $k \in \mathbb{N}$*, and* $h \in H \backslash (Z \cup \{\phi\})$ $supp_{ij}^k(h) \subseteq supp_{ij}^k(\alpha(h))$ *.*

Proof: To prove the statement, it suffices to show that $\kappa \notin supp_{ij}^k(\alpha(h)) \Rightarrow \kappa \notin supp_{ij}^k(h)$ for all $\kappa = 0, 1, ..., k-1$. There are two possibilities. Either $j = P(\alpha(h))$ or $j \neq P(\alpha(h))$. If $j \neq P(\alpha(h))$, then some player other than j moved at $\alpha(h)$, so $\nu_{ij}^k\left(\tau_j | h, \sigma_j^{-k}\right) = \nu_{ij}^k\left(\tau_j | \alpha(h), \sigma_j^{-k}\right)$ for all $\tau_j = 0, 1, ..., k-1$. Hence $\nu_{ij}^k\left(\kappa | \alpha(h), \sigma_j^{-k}\right) = 0 \Rightarrow \nu_{ij}^k\left(\kappa | h, \sigma_j^{-k}\right) = 0$, so $supp_{ij}^k(h) \subseteq supp_{ij}^k(\alpha(h))$. If $j = P(\alpha(h))$, then j moved at $\alpha(h)$, in which case, by Bayes' rule:

$$\nu_{ij}^k\left(\tau_j \mid h, \sigma_{-i}^{-k}\right) = \frac{\sigma_j^{\tau_j}(h \mid \alpha(h))\nu_{ij}^k\left(\tau_j \mid \alpha(h), \sigma_j^{-k}\right)}{\sum_{l=0}^{k-1} \sigma_j^l(h \mid \alpha(h))\nu_{ij}^k\left(l \mid \alpha(h), \sigma_j^{-k}\right)}$$

for all $\tau_j = 0, 1, ..., k-1$. Hence, $\nu_{ij}^k\left(\kappa | \alpha(h), \sigma_j^{-k}\right) = 0 \Rightarrow \nu_{ij}^k\left(\kappa | h, \sigma_j^{-k}\right) = 0$, so $supp_{ij}^k(h) \subseteq supp_{ij}^k(\alpha(h))$. ■

Notice that although players are unable to perfectly observe all previous actions in games of imperfect information, Proposition 2 still holds—the support of marginal beliefs about others' levels in later information sets is (weakly) shrinking. See Lin and Palfrey (2022). Besides, the assumption of independence of the distribution of levels is not used in the proof. In other words, the shrinkage of the support is irreversible, regardless of how the levels are distributed.

In addition, there are a few additional remarks about the properties of the updating process in the DCH model that are worth highlighting. First, there is a second source of learning, besides the shrinking support property, which is that after each move by an opponent, each strategic player with level $k \geq 2$ updates the probability that the opponent is level-0.[15] This in turn leads to updating of the relative likelihood of the higher strategic types of the opponent, since the probabilities have to sum to 1. Second, as the game unfolds, the beliefs of higher level players about their opponents can be updated in either direction, in the sense of believing an opponent is either more or less sophisticated. Examples in the next section will illustrate this. Third, while players' belief-updating process is adaptive, nonetheless all players are *strategically forward-looking* (rather than myopic) in the sense that players take into account and correctly anticipate how all players in the game will update beliefs at each future history.

Since the players are forward-looking and have truncated rational expectations, it is natural to ask if there is any connection between our model and perfect or sequential equilibrium. We explore this relationship in the next section.

4.2 DCH and Subgame Perfect Equilibrium

In this section, we study the relationship between the DCH solution and subgame perfect equilibrium through three simple examples. One question we address is whether sufficiently high-level players always behave consistently with rational backward induction. As it turns

[15] The updating by strategic players' beliefs about level-0 opponents can be either increasing or decreasing.

out, this is *not* generally true. In the following series of simple two-person extensive form games, we demonstrate how high-level players could violate backward induction either on or off the equilibrium path, suggesting the DCH solution is fundamentally different from subgame perfection. For the sake of simplicity, in this section and for the rest of the paper (except for section 4.2.2) we assume every player's level distribution is identical.

4.2.1 Illustrative Example: Violating Backward Induction at Some Subgame

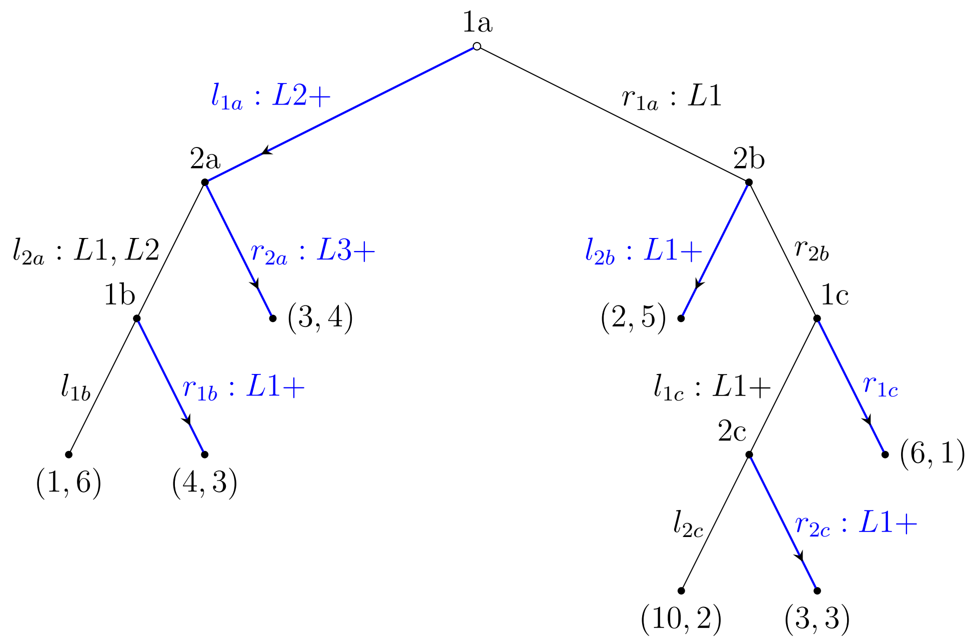

Figure 1: Game Tree of Example 4.2.1. A "+" sign indicates a move is chosen by the specified level and all higher levels. The subgame perfect equilibrium moves are marked with arrows.

Example 4.2.1 demonstrates how backward induction could be violated by every level of player at some subgame. The game tree for this two-person game of perfect information is shown in Figure 1. Suppose every player's level is independently drawn from Poisson(1.5), which has been suggested by Camerer et al. (2004) as an empirically plausible distribution. Every level of players' move choices are labelled in the figure, with a "+" sign indicating a move is chosen by the specified level type and all higher levels. For instance, level-1 player 1 chooses r_{1a} at the beginning while level-2 and above choose l_{1a}. Calculations can be found in Appendix A.

To illustrate the mechanics of the DCH solution in this example, it is useful to begin by focusing on subgame $2a$. In this subgame, level-2 and higher-level of player 2 would update from the information that player 1 is not a level-1 player, leading a level-2 player 2 to choose l_{2a} because the updated belief puts all weight on player 1 being level-0. However, a level-3 player 2 places positive posterior probability on player 1 being level-2, and as long as this posterior probability is high enough, it is optimal for level-3 player 2 to choose r_{2a}—as if player 2 were engaged in the same backward induction reasoning used to justify the subgame perfect equilibrium. Following a similar logic, all high-level players would behave this way in the left branch of the game, where player 1 chooses l_{1a} at the beginning.

However, this is not the case for the right branch of the game after player 1 chooses r_{1a} at the beginning. At subgame $1c$, the move predicted by the subgame perfect equilibrium is *never* chosen by any strategic player 1. Hence, in the DCH solution for this example, high-level player 1's behavior is consistent with subgame perfect equilibrium on the left branch but not on the right branch. □

4.2.2 Dominated Actions

As we examine this example carefully, we can find the key of this phenomenon is that player 1 knows the subgame $h = 1c$ can be reached only if player 2 chooses a strictly dominated action[16] in the previous stage. One can think of player 2's decision at subgame $h = 2b$ as a rationality check in the following sense. Whenever player 2 chooses r_{2b}, the support of strategic player 1's posterior belief will shrink to a singleton—he will believe player 2 is level-0. This extreme posterior belief would lead a strategic player 1 to deviate from subgame perfect strategy.

Generally speaking, if a history contains some player's strictly dominated action, then all other players will immediately believe this player is non-strategic and best respond to such strategy. As a result, it is possible that the strategy profile will not be the subgame perfect equilibrium for every strategic level. This argument holds as long as level-0 player's strategy is the beginning of the hierarchical reasoning process—no matter how small the proportion of level-0 players is. However, since paths with strictly dominated actions can be realized only if some player is level-0, paths containing strictly dominated actions occur with vanishing probability as the proportion of level-0 players converges to 0. Proposition 3 formally shows this conclusion.

Proposition 3. *Consider any finite extensive form game of perfect information where each player i's level is drawn from the distribution $p_i = (p_{ik})_{k=0}^{\infty}$. If some history h can occur only if some player chooses a strictly dominated action, then the probability for such history being realized converges to 0 as $p_0 = (p_{i0})_{i\in N} \to (0, \dots, 0)$.*

Proof: Consider any h that can occur only if some player chooses a strictly dominated action. That is, there is h' that is a subsequence of h with $i = P(h')$ such that there is a strictly dominated action $a' \in A(h')$ and (h', a') is a subsequence of h. Since this is a strictly dominated action, it can only be chosen by a level-0 player. Therefore, the ex ante probability for player i to choose a' at h' is

$$\Pr(a' \mid h') = \sum_{j=0}^{\infty} \sigma_i^j(a' \mid h')p_{ij} = \sigma_i^0(a' \mid h')p_{i0} = \frac{1}{|A(h')|}p_{i0}.$$

[16]Formally speaking, at any history $h \in H\backslash Z$ with $i = P(h)$, an action $a' \in A(h)$ is strictly dominated if there is another action $a'' \in A(h)$ such that

$$\min_{z\in Z_{h''}} u_i(z) > \max_{z\in Z_{h'}} u_i(z).$$

where $h' = (h, a')$ and $h'' = (h, a'')$.

Lastly, the ex ante probability for h to be realized, $\Pr(h)$, is smaller than $\Pr(a'|h')$ and hence

$$\lim_{p_0 \to (0,\dots,0)} \Pr(h) \le \lim_{p_0 \to (0,\dots,0)} \Pr(a' \mid h') = \lim_{p_0 \to (0,\dots,0)} \frac{1}{|A(h')|} p_{i0} = 0.$$

This completes the proof. ■

It is worth noticing that the independence of the distribution of the levels is not required in this proposition as strictly dominated actions will only be chosen by level 0 players. Moreover, one can see this principle in play in Example 4.2.1 where player 1's anomalous behavior only happens when player 2 chooses a strictly dominated action, which is only chosen by level-0. For other parts of the game, if both players are at least level 3, the model predicts the game will follow the subgame perfect equilibrium path.

Since the subgame perfect equilibrium path never contains strictly dominated actions, one might be tempted to conjecture that the equilibrium path is always followed by sufficiently sophisticated players. The next example demonstrates that this is not true. In fact, it is possible that the subgame perfect equilibrium path is *never* chosen by strategic players, so high-level players in our model do not necessarily converge to the subgame perfect equilibrium.

4.2.3 Violating Backward Induction on the Equilibrium Path

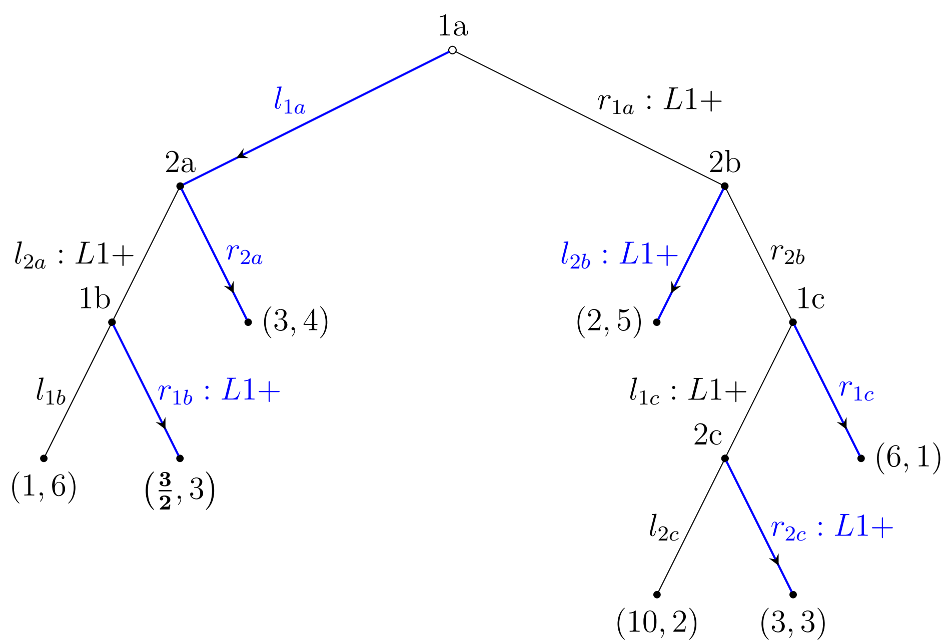

Figure 2: Game Tree of Example 4.2.3. A "+" sign indicates a move is chosen by the specified level and all higher levels. The subgame perfect equilibrium moves are marked with arrows.

Example 4.2.3 is modified from the previous example by changing player 1's payoff from 4 to $\frac{3}{2}$ as he chooses r_{1b} at history $1b$. Decreasing the payoff does not affect the subgame perfect equilibrium. However, this change makes low-level players think the subgame perfect equilibrium actions are not profitable, causing a domino effect that high-level players think the equilibrium actions are not optimal as well. Here we consider an arbitrary prior distribution $p = (p_k)_{k=0}^{\infty}$. The game tree is shown in Figure 2 with every level of players' decisions. The calculations can be found in Appendix A.

Level-1 players will behave the same as in the previous example. However, the change of payoffs makes l_{1a} not profitable for level-2 player 1 at the initial history. Hence, player 2 would believe player 1 is certainly level-0 whenever the game proceeds to the left branch. Moreover, every level of players would behave the same by the same logic. As a result, the subgame perfect equilibrium path is *never* chosen by strategic players. If p_0 is close to 0, the subgame perfect equilibrium outcome will almost never be reached.

Instead, there is an imperfect Nash equilibrium that can be supported by the strategy profile of every strategic level of both players. Loosely speaking, the belief updating process gets "stuck" at this equilibrium, causing all higher-level players behave in the same way.[17]
□

4.3 The Representation Effect: Non-Equivalence of DCH in the Reduced Normal Form

An interesting feature of the DCH model is *the representation effect*: the DCH solution in extensive form differs from the DCH solution in its corresponding reduced normal form in the sense that two solutions produce different outcome distributions. We illustrate this with a toy example in this section, and provide a detailed analysis of the representation effect for a class of increasing-sum centipede games in the subsequent section.

In the following, we first revisit Example 4.2.3 to show how DCH in reduced normal form is obtained compared with the DCH solution in extensive form. Next, we slightly modify the game and show that the DCH solution is not reduced normal form invariant.

4.3.1 Revisit Example 4.2.3

Table 1: Reduced Normal Form of Example 4.2.3

		Player 2					
Player 1		$l_{2a}l_{2b}$	$r_{2a}l_{2b}$	$l_{2a}r_{2b}l_{2c}$	$l_{2a}r_{2b}r_{2c}$	$r_{2a}r_{2b}l_{2c}$	$r_{2a}r_{2b}r_{2c}$
		L1+					
$l_{1a}l_{1b}$	L1+	*1,6*	3,4	1,6	1,6	3,4	3,4
$l_{1a}r_{1b}$		*3/2,3*	3,4	3/2,3	3/2,3	3,4	3,4
$r_{1a}l_{1c}$		***2,5***	*2,5*	*10,2*	*3,3*	*10,2*	*3,3*
$r_{1a}r_{1c}$		*2,5*	2,5	6,1	6,1	6,1	6,1

[17]The following strategy profile defines this imperfect equilibrium: player 1 chooses r_{1a} at the beginning, r_{1b} at subgame $h = 1b$, and chooses l_{1c} at subgame $h = 1c$; player 2 chooses l_{2a} at subgame $h = 2a$, l_{2b} at subgame $h = 2b$, and chooses r_{2c} at subgame $h = 2c$. Therefore, $(2, 5)$ is an equilibrium outcome.

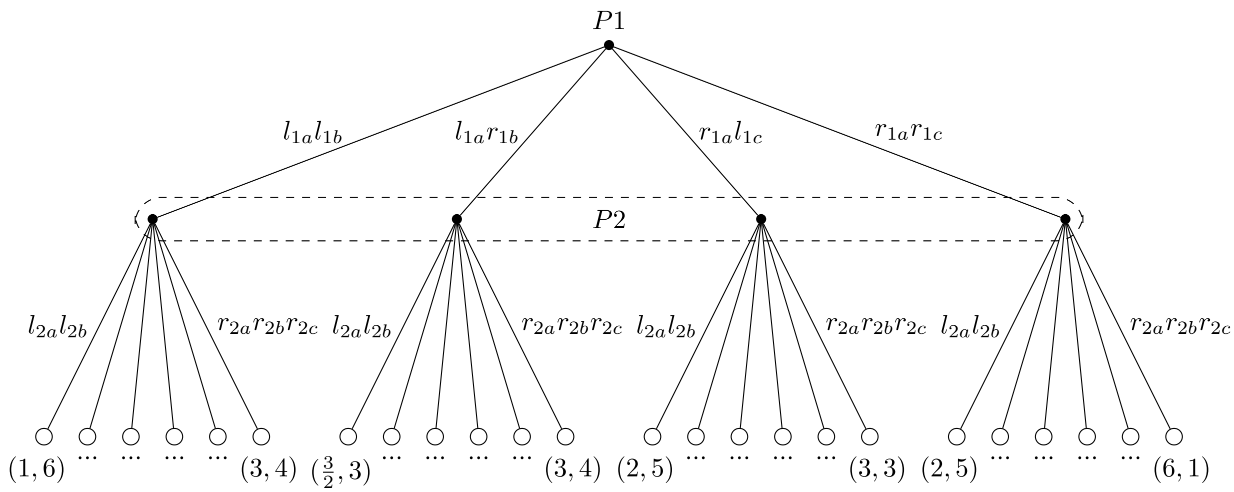

Figure 3: Strategically Equivalent Game of Example 4.2.3

Table 1 displays the 4×6 matrix game that is reduced normal form representation of the extensive form game of Example 4.2.3. Notice that the reduced normal form can also be represented by the extensive form game of Figure 3. It is easy to see that level-1 and higher-level of player 1 will choose the strategy $r_{1a}l_{1c}$ and level-1 and higher-level of player 2 will choose the strategy $l_{2a}l_{2b}$, as indicated in the table. Thus, for this example it turns out that both models lead to a solution where behavior corresponds to an equilibrium outcome that differs from the subgame perfect equilibrium outcome. □

However, it is not true that high-level players in DCH always lead to the same equilibrium outcome in both representations. In the next example, we demonstrate that the DCH solution could be sharply different in extensive form and in reduced normal form.

4.3.2 Illustrative Example: The Representation Effect

Example 4.3, whose game tree is shown in Figure 4, demonstrates how this can happen. This example is almost exactly the same as Example 4.2.3, with the single exception being that player 1's payoff changes from 3 to 8 after choosing r_{2a} at subgame $2a$. This change does not affect the subgame perfect equilibrium, but makes choosing l_{1a} profitable again for high-level player 1. (Here we again assume the prior distribution follows Poisson(1.5).) Consequently, under the extensive form representation, higher levels of DCH players in this game will choose actions that lead to the subgame perfect equilibrium outcome, $(8, 4)$.

This switch to the subgame perfect outcome is a direct consequence of the belief-updating process of the model. Although the payoff 10 is really attractive to player 1, when the game is played under the extensive form, player 1 will realize he can get it only if player 2 is level-0. Therefore, if there is a high enough probability of higher levels of player 2, player 1 will realize he is likely to get the lower payoff of 3 at node $2c$. Hence, a high-level player 1 will choose l_{1a} at the beginning (as if conducting backward induction). As long as there are enough strategic types of player 1 choosing l_{1a}, higher levels of player 2 will update accordingly and choose the subgame perfect equilibrium action r_{2a}. The calculations can be

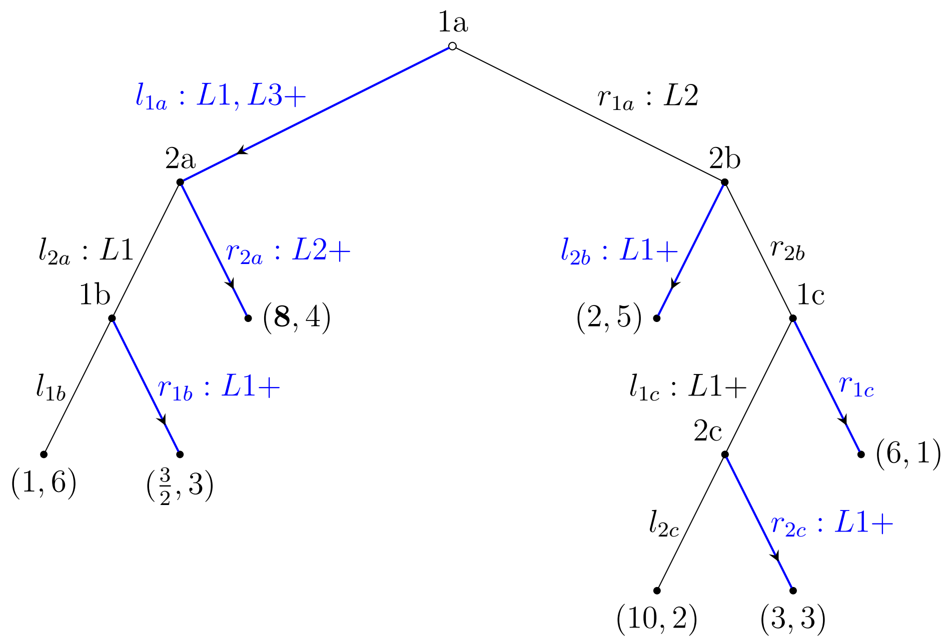

Figure 4: Game Tree of Example 4.3. A "+" sign indicates a move is chosen by the specified level and all higher levels. The subgame perfect equilibrium moves are marked with arrows.

found in Appendix A.

In contrast, under the reduced normal form representation, DCH makes exactly the same prediction as in Example 4.2.3. That is, even though player 1's subgame perfect equilibrium payoff has increased from 3 to 8, *all* strategic types of players 1 and 2 in the reduced normal form will still choose $r_{1a}l_{1c}$ and $l_{2a}l_{2b}$, respectively, again producing the outcome $(2, 5)$ that is not subgame perfect. This difference demonstrates that the DCH solution could be different under different strategically equivalent representations.

In addition, one property of the standard CH model identified by Camerer et al. (2004) is that if a level-k player plays a (pure) equilibrium strategy, then all higher levels of that player will play that strategy too. One may wonder if an analogous property holds in the DCH model. That is, if some level of a player chooses on the equilibrium path, do all higher-levels of that player choose that action too? Example 4.3 provides a counterexample for this conjecture. At the initial history, level-1 player 1 chooses the equilibrium path l_{1a}. However, level-2 player 1 switches to r_{1a}, and level-3 (and above) player 1 switches back to l_{1a}.

The underlying reason is that even if a level-k player chooses the equilibrium path, a higher-level player could still deviate from the equilibrium path if other players do not move along the equilibrium path in later subgames. In this example, level-1 player 1 chooses l_{1a} at the beginning to best respond to level-0 player 2. Yet, level-1 player 2 does not choose the equilibrium path at the subgame $h = 2a$, causing level-2 player 1 to choose r_{1a} at the beginning. Level-2 (and above) player 2 switches to the equilibrium path at the subgame $h = 2a$, and this information can only be updated by level-3 (and above) player 1. Finally, as long as there are enough level-2 (and above) players, high-level player 1 would switch back to the equilibrium path, creating a non-monotonicity. □

5 An Application: Centipede Games

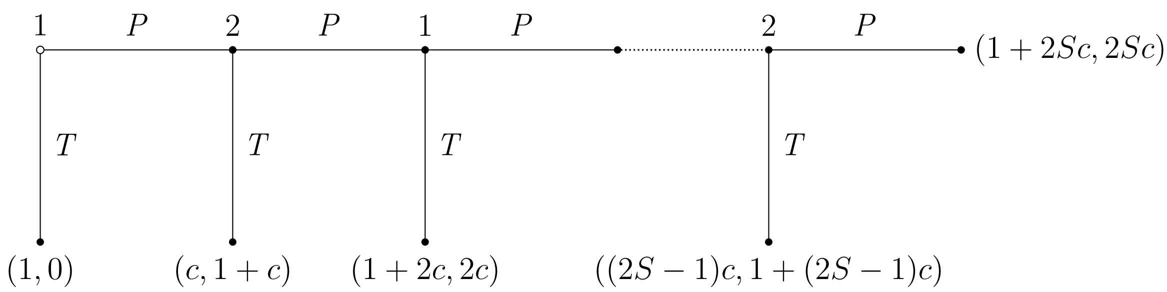

Figure 5: $2S$-node Centipede Game

In this section, we explore the representation effect in much more detail, focusing on the class of "linear centipede games," which is illustrated in Figure 5. The games in this class are described in the following way. Player 1, the first-mover, and player 2, the second-mover, alternate over a sequence of moves. At each move, the player whose turn it is can either end the game ("take") and receive the larger of two payoffs or allow the game to continue ("pass"), in which case both the large and the small payoff are incremented by an amount $c > 0$. The difference between the large and the small payoff equals 1 and does not change. The game continues for at most $2S$ decision nodes (stages) where $S \geq 2$, and we label the decision nodes by $\{1, 2, \ldots, 2S\}$. Player 1 moves at odd nodes and player 2 moves at even nodes. If the game is ended by a player at stage $j \leq 2S$, the payoffs are $(1+(j-1)c,\ (j-1)c)$ if j is odd and $((j-1)c,\ 1+(j-1)c)$ if j is even. If no player ever takes, the payoffs are $(1+2Sc,\ 2Sc)$. Thus, a linear centipede game has two parameters: (S, c). To avoid trivial cases, we assume $\frac{1}{3} < c < 1$.[18]

Specifically, we will compare each level of players' behavior in two different representations of the game given the same prior distribution. This comparison corresponds to a methodological question in experimental economics—whether the direct-response method and the strategy method are behaviorally equivalent. In terms of experimental implementation, the direct-response method and the strategy method correspond, to the extensive form of the sequential and the reduced normal form,[19] respectively. From the perspective of the standard equilibrium theory, these two representations would not induce different behavior because they are strategically equivalent. In contrast, the DCH solution is different in these two representations, suggesting that DCH predicts the direct-response method and the strategy method will induce different behavior.

The key difference between the extensive form and the reduced normal form is that players can observe the other player's previous actions in the extensive form. As the game continues, the only information that can be observed is *how many times the opponent has passed.*[20] This

[18] If $c > 1$, then the unique equilibrium is for every player to pass at every node. If $c < \frac{1}{3}$, then all players with level $k > 0$ will always take, so CH behavior is the same as subgame perfect Nash equilibrium behavior.

[19] In the reduced normal form, a player's strategy is the node at which they will stop the game (or never). Therefore, each player has $S + 1$ available strategies.

[20] Players can also update their beliefs about the opponent's level if the other player surprisingly takes at some stage. However, the game is over at that point.

history seems uninformative at the first glance. However, players can still update their beliefs about the other player's level from this history, and hence behave differently.

5.1 DCH Solution for the Centipede Games in Extensive Form

In the extensive form centipede game, since each player can move at S stages, then a (behavioral) strategy for player i is an S-tuple where each element is the probability to take at the corresponding decision node. That is, $\sigma_1 = (\sigma_{1,1}, \ldots, \sigma_{1,S})$ and $\sigma_2 = (\sigma_{2,1}, \ldots, \sigma_{2,S})$ are player 1 and 2's strategies, respectively. For every $1 \leq j \leq S$, $\sigma_{1,j}$ is the probability that player 1 would take at stage $2j - 1$ and $\sigma_{2,j}$ is the probability that player 2 would take at stage $2j$.

Following the notation introduced earlier, we use σ_1^k and σ_2^k to denote level-k player's strategy. Level-0 players uniformly randomize at each stage. That is, $\sigma_1^0 = \sigma_2^0 = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$. Finally, to simplify the notation, for every stage $1 \leq j \leq 2S$, we let $\nu_j^k(\cdot) : \mathbb{N}_0 \to \Delta(\mathbb{N}_0)$ be level-k stage j-mover's belief about the opponent's level at stage j where $\nu_j^k(\tau_{-P(j)}) \equiv \nu^k_{P(j),-P(j)}(\tau_{-P(j)} \mid j, \sigma^{-k}_{-P(j)})$.

To fully characterize every level of players' strategies, we need to compute every level of players' best responses at every subgame. In principle, we have to solve the behavior of each level recursively. However, since each level of players' strategy is *monotonic*—when the player decides to take at some stage, he will take in all of his later subgames—we can alternatively characterize the solution by identifying the lowest level of player to take at every subgame.

In Lemma 1, we characterize level-1 players' behavior and establish the monotonicity result. These results are straightforward and follow from the assumption that $\frac{1}{3} < c < 1$.

Lemma 1. *In the extensive form linear centipede game, as $\frac{1}{3} < c < 1$,*

1. $\sigma^k_{2,S} = 1$ *for all* $k \geq 1$.
2. $\sigma_1^1 = (0, \ldots, 0)$ *and* $\sigma_2^1 = (0, \ldots, 0, 1)$.
3. *For every* $k \geq 2$ *and every* $1 \leq j \leq S - 1$,
   - *(i)* $\sigma^k_{1,j} = 0$ *if* $\sigma^m_{2,j} = 0$ *for every* $1 \leq m \leq k - 1$;
   - *(ii)* $\sigma^k_{2,j} = 0$ *if* $\sigma^m_{1,j+1} = 0$ *for every* $1 \leq m \leq k - 1$.

Proof: See Appendix B. ■

Lemma 1 has three parts: (1) every strategic player 2 takes at the last stage; (2) completely characterizes level-1 strategies—player 1 passes at every stage and player 2 passes at every stage except for the last stage; (3) provides necessary conditions for higher levels to take at some stage. For any level $k \geq 2$ and any stage $1 \leq j \leq 2S - 1$, a level-k player would take at stage j only if there is some lower level player that would take at the next stage. Otherwise, it is optimal for level-k player to pass at stage j.

The general characterization of level-k optimal strategies is in terms of the following **cutoffs**, specifying, for each stage, the lowest level type to take at that stage.

Definition 1. *For every stage j where $1 \leq j \leq 2S$, define the* ***cutoff****, K_j^* be the lowest level of player that would take at this stage. In other words,*

$$K_j^* = \begin{cases} \arg\min_k \left\{ \sigma^k_{1,\frac{j+1}{2}} = 1 \right\}, & \text{if } j \text{ is odd} \\ \arg\min_k \left\{ \sigma^k_{2,\frac{j}{2}} = 1 \right\}, & \text{if } j \text{ is even} \\ \infty, & \text{if } \nexists k \text{ s.t. } \sigma^k_{1,\frac{j+1}{2}} = 1 \text{ or } \sigma^k_{2,\frac{j}{2}} = 1. \end{cases}$$

Based on Definition 1, the monotonicity obtained in part (3) of Lemma 1 implies the following two results about cutoffs and strategies. Together they show that for any stage, a player's strategy will be to take at that stage if and only if his level is greater or equal to the cutoff.

Proposition 4. *For every $1 \leq j \leq 2S - 1$,*

1. $K_j^* \geq K_{j+1}^* + 1$ *if* $K_{j+1}^* < \infty$;
2. $K_j^* = \infty$ *if* $K_{j+1}^* = \infty$.

Proof: See Appendix B. ■

Proposition 5. *For every $1 \leq j \leq 2S - 1$,*

1. $\sigma^k_{1,\frac{j+1}{2}} = 1$ *for all* $k \geq K_j^*$ *if* j *is odd and* $K_j^* < \infty$;
2. $\sigma^k_{2,\frac{j}{2}} = 1$ *for all* $k \geq K_j^*$ *if* j *is even and* $K_j^* < \infty$.

Proof: See Appendix B. ■

Hence, cutoffs characterize optimal strategies of each level of each player, with a cutoff defining the lowest level that would take at each stage and all higher levels of that player would also take at that stage. The next two propositions establish recursive necessary and sufficient conditions for the existence of some level of some player to take at each stage. The proofs of these propositions provide a recipe for computing cutoffs.

Proposition 6. $K^*_{2S-1} < \infty \iff p_0 < \frac{2^S}{2^S + \left(\frac{3c-1}{1-c}\right)}$

First, we note that the proofs are simplified somewhat by observing the following identity:

$$p_0 < \frac{2^S}{2^S + \left(\frac{3c-1}{1-c}\right)} \iff \frac{p_0 \left(\frac{1}{2}\right)^S}{p_0 \left(\frac{1}{2}\right)^{S-1} + (1 - p_0)} < \frac{1-c}{1+c}.$$

Proof: **Only if:** Suppose $K^*_{2S-1} < \infty$. By Proposition 4, $K_j^* \geq K^*_{2S-1}$ for all $j < 2S - 1$. Hence, the belief of level K^*_{2S-1} of player 1 that player 2 is level-0 at stage $2S - 1$ equals to

$$\nu^{K^*_{2S-1}}_{2S-1}(0) = \frac{p_0 \left(\frac{1}{2}\right)^{S-1}}{p_0 \left(\frac{1}{2}\right)^{S-1} + \sum_{l=1}^{K^*_{2S-1}-1} p_l},$$

since it is optimal for $K^*_{2S-1} < \infty$ to take at $2S-1$. This implies $\frac{p_0\left(\frac{1}{2}\right)^S}{p_0\left(\frac{1}{2}\right)^{S-1}+\sum_{l=1}^{K^*_{2S-1}-1} p_l} < \frac{1-c}{1+c}$ and hence:

$$\frac{p_0\left(\frac{1}{2}\right)^S}{p_0\left(\frac{1}{2}\right)^{S-1}+(1-p_0)} < \frac{1-c}{1+c} \iff p_0 < \frac{2^S}{2^S+\left(\frac{3c-1}{1-c}\right)}.$$

If: Suppose $K^*_{2S-1}=\infty$. Then from Proposition 4, $K^*_j=\infty$ for all $j<2S-1$. That is, all levels of both players pass at every stage up to and including $2S-1$. Hence, the belief of level $k\geq 1$ of player 1 that player 2 is level-0 at stage $2S-1$ equals to

$$\nu^k_{2S-1}(0) = \frac{p_0\left(\frac{1}{2}\right)^{S-1}}{p_0\left(\frac{1}{2}\right)^{S-1}+\sum_{l=1}^{k-1} p_l} > \frac{p_0\left(\frac{1}{2}\right)^{S-1}}{p_0\left(\frac{1}{2}\right)^{S-1}+(1-p_0)}.$$

Since $K^*_{2S-1}=\infty$, it is optimal to pass at $2S-1$ for all levels $k\geq 1$ of player 1, impling

$$\frac{p_0\left(\frac{1}{2}\right)^S}{p_0\left(\frac{1}{2}\right)^{S-1}+(1-p_0)} \geq \frac{1-c}{1+c}.$$

This completes the proof. ■

Thus, p_0 must be sufficiently small, and the condition is easier to satisfy the smaller c is (the potential gains to passing) and the larger is S (the horizon). If this condition holds, there exists some strategic player 1 that takes at stage $2S-1$. The proof also provides an insight for how the cutoffs can be computed. Specifically, the K^*_{2S-1} cutoff is computed as:

$$K^*_{2S-1} = \arg\min_k \left\{\frac{p_0\left(\frac{1}{2}\right)^S}{p_0\left(\frac{1}{2}\right)^{S-1}+\sum_{l=1}^{k-1} p_l} < \frac{1-c}{1+c}\right\}.$$

Cutoffs for earlier stages can be derived recursively as the following proposition establishes.

Proposition 7. *For every* $1\leq j\leq 2S-2$,

$$K^*_j<\infty \iff \frac{p_0\left(\frac{1}{2}\right)^{\lfloor\frac{j}{2}\rfloor+1}+\sum_{l=1}^{K^*_{j+1}-1} p_l}{p_0\left(\frac{1}{2}\right)^{\lfloor\frac{j}{2}\rfloor}+(1-p_0)} < \frac{1-c}{1+c}. \tag{3}$$

Proof: The logic of the proof is similar to Proposition 6. See Appendix B for details. ■

A simple economic interpretation of the conditions obtained in Proposition 6 and 7 is as follows. At any stage s, if the other player will take at the next stage, the net gain to taking at s is $[1+(s-1)c]-[sc]=1-c$. On the other hand, if the other player passes at the next stage, the net gain to taking at stage $s+2$ is $[1+(s+1)c]-[sc]=1+c$. Hence, the right-hand side is simply the ratio of payoffs to the current player depending on the opponent taking or passing at the next stage, assuming the current player will take in the subsequent stage. Thus, a player will take in the current stage if and only if the posterior probability the opponent will take in the next stage is less than this ratio.

The information contained in the history is that *if the game proceeds to later stages, the opponent is less likely to be a level-0 player.* If the game reaches stage j, the player would know the opponent has passed $\lfloor \frac{j}{2} \rfloor$ times, which would occur with probability (conditional on the opponent being level-0) $1/2^{\lfloor \frac{j}{2} \rfloor}$ which rapidly approaches 0.

5.2 DCH Solution for the Centipede Game in Reduced Normal Form

In contrast, the $2S$-leg centipede game in reduced normal form is analyzed as a simultaneous move game where $A_1 = A_2 = \{1, , \ldots, S+1\}$ is the set of actions for each player. Action $s \le S$ represents a plan to pass at the first $s-1$ opportunities and take at the s-th opportunity. Strategy $S+1$ is the plan to always pass. Player 1 and 2's strategies are denoted by a_1 and a_2, respectively. If $a_1 \le a_2$, then the payoffs are $(1+(2a_1-2)c,\ (2a_1-2)c)$; if $a_1 > a_2$, then the payoffs are $((2a_2-1)c,\ 1+(2a_2-1)c)$.

The DCH solution in reduced normal form coincides with the standard CH solution. To characterize the solution, we let a_i^k denote level-k player i's strategy. A level-0 player uniformly randomizes across all available strategies. With a minor abuse of notation, denote $a_i^0 = \frac{1}{S+1}$ for $i \in \{1, 2\}$. Lemma 2 establishes level-1 players' behavior and the monotonicity, similarly to Lemma 1.

Lemma 2. *In the reduced normal form linear centipede game, as* $\frac{1}{3} < c < 1$,

1. $a_1^1 = S+1$ *and* $a_2^1 = S$.
2. *For every* $k \ge 2$,
   - *(i)* $a_1^k \ge \min\{a_2^m : 1 \le m \le k-1\}$;
   - *(ii)* $a_2^k \ge \min\{a_1^m : 1 \le m \le k-1\} - 1$.
3. $a_i^{k+1} \le a_i^k$ *for all* $k \ge 1$ *and for all* $i \in \{1, 2\}$.

Proof: See Appendix B. ■

Lemma 2 has essentially the same three parts as Lemma 1, but stated in terms of the stopping point strategies rather than behavioral strategies. Therefore, as in the extensive form, optimal strategies are given by **cutoffs**, defined analogously to Definition 1.

Definition 2. *For every stage s where $1 \le j \le 2S$, define the* ***cutoff*** $\tilde{K}_j^*$ *to be the lowest level of player that would take no later than this stage. In other words,*

$$\tilde{K}_j^* = \begin{cases} \arg\min_k \left\{ a_1^k \le \frac{j+1}{2} \right\}, & \text{if } j \text{ is odd} \\ \arg\min_k \left\{ a_2^k \le \frac{j}{2} \right\}, & \text{if } j \text{ is even} \\ \infty, & \text{if } \nexists k \text{ s.t. } a_1^k \le \frac{j+1}{2} \text{ or } a_2^k \le \frac{j}{2}. \end{cases}$$

By Lemma 2, we know $a_2^1 = S$. Therefore, $\tilde{K}^*_{2S} = 1$. Proposition 8 and 9 are parallel to Proposition 6 and 7, providing necessary and sufficient conditions for the existence of some strategic players to take before a particular stage.

Proposition 8. $\tilde{K}^*_{2S-1} < \infty \iff p_0 < \frac{S+1}{(S+1)+\left(\frac{3c-1}{1-c}\right)}$.

Proof: See Appendix B. ■

Proposition 9. *For every* $1 \leq j \leq 2S-2$*,*

$$\tilde{K}^*_j < \infty \iff p_0 \left(\frac{S}{S+1} - \frac{2\lfloor \frac{j}{2} \rfloor c}{(S+1)(1+c)} \right) + \sum_{k=1}^{\tilde{K}^*_{j+1}-1} p_k < \frac{1-c}{1+c}. \tag{4}$$

Proof: The logic of the proof is similar to Proposition 8. See Appendix B for details. ■

Propositions 6 and 8 identify conditions on p such that there is some level $k > 0$ of player 1 who would take at some stage in the extensive form DCH solution while in the DCH for the reduced normal form, every strategic level of player 1 would pass at every stage.

Corollary 1. *If* $\frac{S+1}{(S+1)+\left(\frac{3c-1}{1-c}\right)} \leq p_0 < \frac{2^S}{2^S+\left(\frac{3c-1}{1-c}\right)}$*, then* $K^*_{2S-1} < \infty$ *and* $\tilde{K}^*_{2S-1} = \infty$*.*

Proof: Since $2^S > S+1$ for all $S \geq 2$, this follows directly from Propositions 6 and 8. ■

An implication of propositions 7 and 9 is that if p_0 is small, then the difference in behavior under the two different representations of the game will also be small, since the left hand side of inequalities (3) and (4) both converge to $\sum_{k=1}^{\tilde{K}^*_{j+1}-1} p_k$. This result is intuitive. If p_0 is very small, then there is essentially no learning from observing previous actions so behavior will be almost the same in extensive form and in reduced normal form.

However, regardless of how small p_0 is (as long as it is positive), DCH predicts the extensive form and the reduced form representations lead to systematically different behavioral predictions. These differences lead to the main result of this section, Theorem 1, which establishes that players are more likely to take at every stage in the extensive form.

Theorem 1. *For every stage* $1 \leq j \leq 2S$*,*

$$K^*_j \leq \tilde{K}^*_j.$$

Proof: See Appendix B. ■

This result provides a testable prediction that these centipede games will end earlier if played in the extensive form rather than in the reduced normal form. Moreover, this result is robust to the prior distributions of levels. Therefore, DCH predicts players will exhibit more sophisticated behavior in the extensive form since the information from the history is that the opponent is less likely to be a level-0 player.

5.3 Results for the Poisson-DCH Model

In previous applications of the CH model, it has been useful to assume the distribution of levels is given by a Poisson distribution (Camerer et al., 2004). We obtain some additional results here for this one-parameter family of distributions that allow us to further pin down the differences between the extensive form and the reduced normal form of the centipede game. The Poisson-DCH model assumes:

$$p_k \equiv \frac{e^{-\lambda}\lambda^k}{k!}, \text{ for all } k = 0, 1, 2, ...$$

where $\lambda > 0$ is the mean of the Poisson distribution.

Finally, we write the cutoffs as functions of λ. In the extensive form, the cutoff function for stage j is $K_j^*(\lambda)$. In the reduced normal form, the cutoff function for stage j is $\tilde{K}_j^*(\lambda)$.

As we have discussed before, in the extensive form representation, players become more sure that the opponent is not level-0 when the game moves to later stages. Therefore, player 1 has the best information in stage $2S-1$. Proposition 10 quantitatively demonstrates the difference between two models at stage $2S-1$.

Proposition 10. *As the prior distribution follows Poisson(λ), then*

(i) $K_{2S-1}^*(\lambda) < \infty \iff \lambda > ln\left[1 + \left(\frac{1}{2}\right)^S \left(\frac{3c-1}{1-c}\right)\right]$;

(ii) $\tilde{K}_{2S-1}^*(\lambda) < \infty \iff \lambda > ln\left[1 + \left(\frac{1}{S+1}\right) \left(\frac{3c-1}{1-c}\right)\right]$.

Proof: The result is obtained by substituting $p_0 = e^{\lambda}$ in the formulas given by Propositions 6 and 8, and with some algebra. See Appendix B for details. ■

Proposition 10 provides a closed form solution for the minimum λ to support some level of player 1 to take at stage $2S-1$ in both the extensive form and the reduced normal form. The left panel of Figure 6 plots the lowest λ. From the figure, we can notice that at stage $2S-1$, the minimum value of λ to start unraveling is much smaller in the extensive form than in the reduced normal form. Moreover, the minimum λ converges to 0 much faster in the extensive form than in the reduced normal form as S gets higher, which is derived from the belief updating of DCH.

On the other hand, in the right panel of Figure 6, we focus on the four-node centipede game ($S = 2$) and plot the CDF of terminal nodes under both representations predicted by DCH. First of all, we can observe the distribution of terminal nodes under the reduced normal form first order stochastically dominates the distribution under the extensive form. In fact, the FOSD relationship holds for any S, c, λ. This leads to a second interpretation of Theorem 1—since the cutoffs in the extensive form representation are uniformly smaller than the cutoffs in the reduced normal form representation, there are more levels of players that would take at every stage, generating the FOSD relationship.

When λ gets larger, the distribution of levels will shift to the right and players tend to be more sophisticated at the aggregate level. Proposition 11 shows that for sufficiently large λ, highly sophisticated players would take at every stage in both the extensive form and reduced normal form of the centipede game.

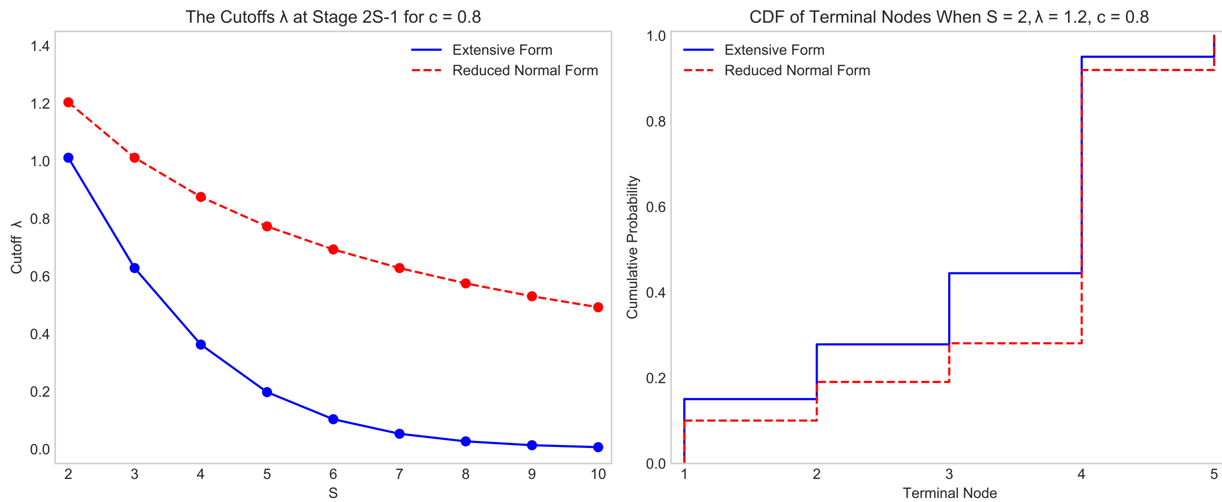

Figure 6: (Left) The minimum value of λ to support taking at stage $2S-1$ under both the extensive form (solid) and the reduced normal form representation (dash) when $c = 0.8$ with S on the horizontal axis and λ on the vertical axis. (Right) The CDFs of terminal nodes in four-node centipede games under different representations predicted by DCH.

Proposition 11. *For both the dynamic and static models, there exists sufficiently high* λ *such that unraveling occurs. That is, for each* S :

(i) $\exists \lambda^* < \infty$ *such that* $K_1^*(\lambda) < \infty$ *for all* $\lambda > \lambda^*$;

(ii) $\exists \tilde{\lambda}^* < \infty$ *such that* $\tilde{K}_1^*(\lambda) < \infty$ *for all* $\lambda > \tilde{\lambda}^*$.

Proof: See Appendix B. ■

This result shows that DCH predicts unravelling occurs if λ is sufficiently high, in both representations. However, it leaves open questions about how this unravelling differs between the two representations. To this end, Proposition 12 provides some insight on this issue, in particular that the reduced normal form requires strictly more "density shift" (higher λ) in order to completely unravel for high-level players.

Proposition 12. *For any* j *where* $1 \leq j \leq 2S-1$, *let* λ_{2S-j}^{**} *be the lowest* λ *such that* $K_{2S-j}^*(\lambda) = j+1$ *for all* $\lambda > \lambda_{2S-j}^{**}$, *and let* $\tilde{\lambda}_{2S-j}^{**}$ *be the lowest* λ *such that* $\tilde{K}_{2S-j}^*(\lambda) = j+1$ *for all* $\lambda > \tilde{\lambda}_{2S-j}^{**}$. *Then* $\lambda_{2S-j}^{**} < \tilde{\lambda}_{2S-j}^{**}$ *for all* $1 \leq j \leq 2S-1$.

Proof: See Appendix B. ■

In other words, we can view the difference of density shifts between two models (so that every level of players completely unravels) as a measure of the effect of belief updating. As shown in Proposition 12, we can always find a non-trivial set of λ such that players have already unravelled in the extensive form but not in the reduced normal form.

Finally, in the Poisson family, we can obtain an unambiguous comparative static result on the change of λ. Proposition 13 shows that when λ increases, the cutoff level of each stage is weakly decreasing. That is, when the average sophistication of the players increases, play is closer to the fully rational model—i,e., there is more taking.

Proposition 13. *For every $1 \leq j \leq 2S$, $K_j^*(\lambda)$ and $\tilde{K}_j^*(\lambda)$ are weakly decreasing in $\lambda > 0$.*

Proof: See Appendix B. ■

5.4 Non-Linear Centipede Games

The results of this section about the exact characterization of behavior in extensive form and reduced normal form centipede games only consider games with a linearly increasing pie. A natural robustness question is whether the qualitative findings apply more generally to other families of centipede games. The key assumption in our analysis is that the increment of pie is not too fast or too slow. If the increment is too fast (i.e., $c > 1$), then it is optimal to pass everywhere. On the other hand, if the increment is too slow (i.e., $c < \frac{1}{3}$), even the lowest level of players would take at the first stage. In all cases within this range, learning occurs in the extensive form but not in the reduced normal form. This would seem to be a general property of increasing-pie centipede games. That is, unless the pie sizes grow so fast that all positive levels of players will always pass, or so slowly that positive levels will always take, then there will be some opportunity for learning, which will lead to different behavior under different representations of the game. Moreover, the main effect of learning in the extensive form will be to update the prior probability of level-0 players in a downward direction, which in turn will lead to earlier taking.

For example, the analysis can be extended to the class of centipede games with an exponentially increasing pie, as studied in the McKelvey and Palfrey (1992) experiment. Similar to the previous analysis, two players alternate over a sequence of moves in an exponential centipede game with $2S$ legs. At each move, if a player passes, both the large and small (positive) payoffs would be *multiplied* by $c > 1$. In addition, the ratio between the large and the small payoff is equal to $\pi > 1$ and does not change as the game progresses. Therefore, an exponential centipede game is parameterized by (S, π, c): if the game is terminated by a player at stage $j \leq 2S$, the payoffs are $(c^{j-1}\pi, c^{j-1})$ if j is odd and $(c^{j-1}, c^{j-1}\pi)$ if j is even. If no one ever takes, then the payoffs will be $(c^{2S}\pi, c^{2S})$. In this class of centipede games, the multiplier c governs the growth rate of pie, and the logic of the proofs of propositions for the linear games is similar for exponential games as long as:

$$\frac{-1 + \sqrt{1 + 8\pi^2}}{2\pi} < c < \pi,$$

which rules out trivial cases, in the same way as the assumption of $\frac{1}{3} < c < 1$ rules out trivial cases in linear centipede games.

All of the qualitative results for linearly increasing centipede games also hold for exponential centipede games, with the only difference being the analytical expression of the cutoffs. In particular, Theorem 1, the representation effect continues to hold.

6 Experimental Evidence

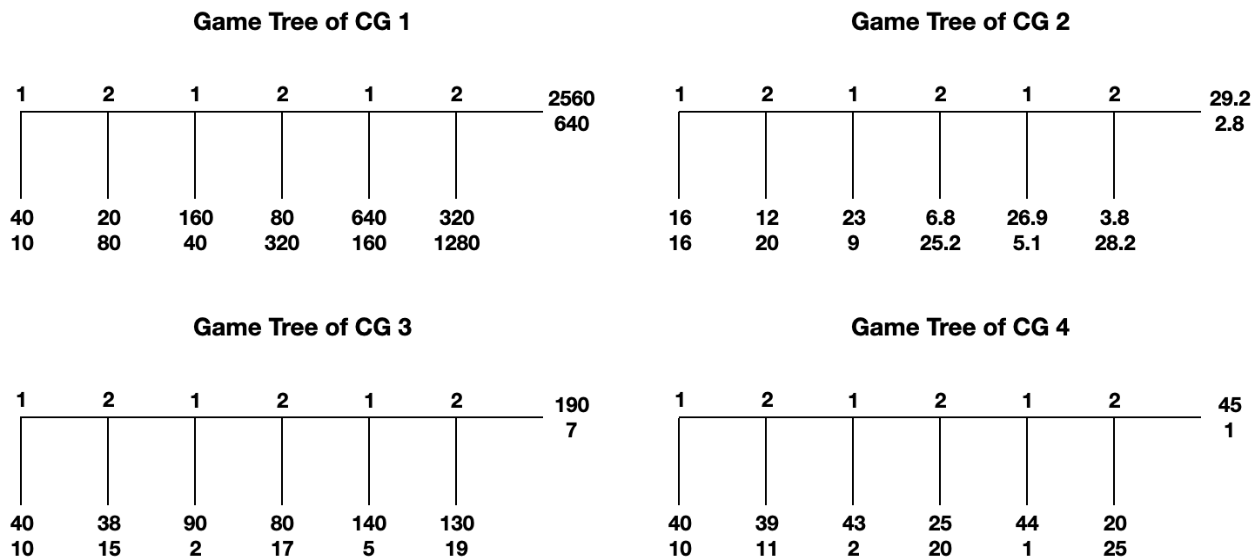

Figure 7: The game trees of CG 1 to CG 4 studied in García-Pola et al. (2020a)

Since DCH is a solution concept developed for games in extensive form, the timing of actions can affect the nature of learning, causing the solution look dramatically different under strategically equivalent *representations* of the game. Specifically, DCH predicts players would behave differently if centipede games of a certain class are implemented under the direct response method (extensive form) and the strategy method (reduced normal form). To empirically test the representation effect predicted by DCH, we revisit a recent experiment conducted by García-Pola et al. (2020a) which compared the behavior in four centipede games under the direct response method and the strategy method.

The game trees of the four centipede games (CG 1 to CG 4) studied in the experiment are plotted in Figure 7. CG 1 is a centipede game with an exponentially-increasing pie while CG 2 is a centipede game with a constant-sum pie. By contrast, the change of the pie size in CG 3 and CG 4 is not monotonic and player 1's payoff is always greater than player 2's payoff. Among these four centipede games, DCH predicts that in CG 1, CG 2 and CG 4, the distribution of terminal nodes under the strategy method will *first order stochastically dominate* the distribution of terminal nodes under the direct response method. Yet, DCH predicts the FOSD relationship does not necessarily hold in CG 3.[21]

DCH Prediction: *In CG 1, CG 2 and CG 4, the distribution of terminal nodes under the strategy method will first order stochastically dominate the distribution of terminal nodes under the direct response method. However, the FOSD relationship is violated in CG 3.*

[21] Since CG 1 is an exponential centipede game, as discussed in section 5.4, we can use the same argument as Theorem 1 to show the FOSD relationship. In the Online Appendix, we show that DCH also predicts the FOSD relationship in CG 2 and CG 4. We also prove, in sharp contrast, that the FOSD relationship is *violated* in CG 3 for empirically plausible prior distributions of levels, for example if levels are distributed Poisson with mean equal to 1.

This experiment consists of two treatments—the direct response method (the hot treatment) and the strategy method (the cold treatment)—with between-subject design. That is, each subject only participates in one of the two treatments. There are 151 subjects in the cold treatment, 76 in the role of first-mover and 75 in the role of second-mover. Each subject chose a stopping point for 16 different centipede games (including the four games in the figure) without feedback between games, and was subsequently matched with a random player of the other role to determine payment. There were 352 subjects in the hot treatment, and each subject only played only one of the four centipede games in the figure.[22]

The two treatments share the following features. The subjects are given identical instructions in both treatments, except for the specific way subject decisions are elicited. In particular, the "frame" of the game is explained and presented on subject computer interfaces in game-tree form to reflect the timing of the centipede game in both treatments. In the cold (strategy-method) treatment, each subject was instructed to click on the first node at which they wanted to stop, if the game got that far. Thus, the task for a subject in the cold treatment was to choose one of four pure strategies of the reduced normal form of the game: take at the first opportunity (T); pass at the first opportunity and take at the second opportunity (PT); pass at the first two opportunities and take at the third opportunity (PPT); or never take (PPP). The instructions provide subjects with explanations of the decision screens, the matching protocols, the payment method, etc. Thus, there is no possibility of a "framing" effect as could happen, for example if the cold treatment were presented as a 4×4 matrix game.[23]

Figure 8 plots the empirical CDFs of the terminal nodes under two different methods in all four games.[24] In the hot treatment, since players are randomly paired at the beginning of the game, we plot the observed distribution of terminal nodes. On the other hand, in the cold (reduced normal form) treatment, players are not paired into groups before each game, and hence a distribution of terminal nodes is not directly observed. However, the empirical conditional take probabilities of each stage is directly observed, and from this one can easily the implied distribution of terminal nodes.[25]

Focusing on CG 1, CG 2 and CG 4, we can see that the distribution of terminal nodes under the strategy method indeed first order stochastically dominates the distribution under the direct response method although the difference is not statistically significant in CG 4 (CG 1: KS Test p-value = 0.046; CG 2: KS Test p-value = 0.003; CG 4: KS Test p-value = 0.653). In stark contrast, the *opposite* FOSD relationship is observed in CG 3—earlier *taking* in the

[22] In the hot treatment, there were 90 subjects (45 in each role) participating in each of CG 1, CG 2 and CG 4, while there were 82 subjects (41 in each role) participating in CG 3. Every subject in a session played 10 repetitions of the same game in the experiment, with feedback, using a matching protocol that was designed to minimize reputation effects. To avoid the analysis from being confounded with learning effects from repetition, our analysis only uses data from the first match of each game.

[23] See García-Pola et al. (2020a) and García-Pola et al. (2020b) for copies of the instructions and exact details of the experimental procedures.

[24] Appendix C provides a table containing the values that are plotted in the figure.

[25] For instance, the probability that the game ends at the first stage is equal to the fraction of subjects in the first-mover role who chose the "T" strategy. The probability that the game ends at the second stage is equal to one minus the fraction of subjects in the first-mover role who chose the "T" strategy times the fraction of subjects in the second-mover role who chose the "T" strategy. The probabilities the game ends at later stages are computed in the similar way.

cold treatment (KS Test p-value = 0.940).[26] This experimental evidence supports DCH at the aggregate level.

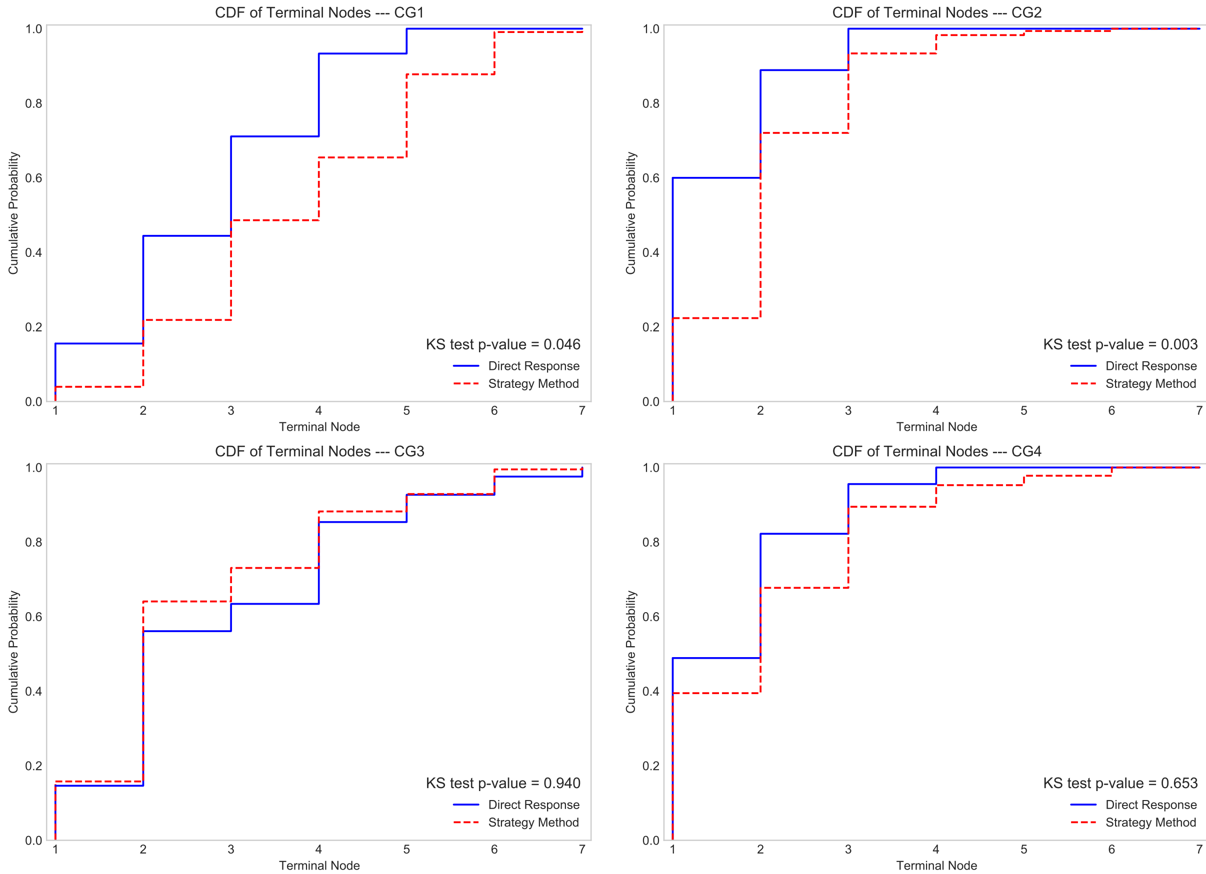

Figure 8: The empirical CDF of CG 1 to CG 4 in García-Pola et al. (2020a) under the direct response method (blue) and the strategy method (red).

7 Discussion

In this section, we briefly discuss several additional features and potential applications of DCH. Section 7.1 illustrates how reputation effects can arise with DCH, and in fact are a built-in feature of the solution concept. This follows from the fact that strategic players are not myopic, but are *forward looking* and take into account how their current actions will affect other players' beliefs and actions. Thus, in DCH higher-level players can mimic

[26]To compute the p-values, we follow the approach of García-Pola et al. (2020a) and generate 100,000 random sub-samples of subjects from the cold treatment to match the number of subjects in each role in each game in the corresponding hot treatment. Next, in each random sample, we randomly match subjects into pairs to obtain the distribution of terminal nodes. This process yields 100,000 simulated CDFs of the terminal nodes from the cold treatment. Then, we perform the KS test on each simulated CDF of the terminal nodes against the CDF from the hot treatment and report the median p-value. The KS p-values computed using this process are almost identical to the KS p-values reported by García-Pola et al. (2020a).

lower-level types in order to affect lower levels of other players' beliefs and hence their future play. In sections 7.2 and 7.3, we highlight some complications of the DCH belief system that arise in games of imperfect or incomplete information. Finally, section 7.4 discusses general issues related to the equivalence or non-equivalence of DCH when analyzed in the *non-reduced* normal form.

7.1 Reputation Formation

In addition to the representation effect, another interesting phenomenon that can arise in DCH is *reputation building* by higher-level players. Since in DCH, players will update their beliefs about others' levels as the history unfolds, it is possible for higher-level players to *mimic* some lower-level players' strategies in order to maintain the reputation of being some lower levels and benefit from this reputation.

It is worth noticing that the reputation concerns predicted by DCH do not only appear in games of incomplete information, but also in games of *complete information*. In other words, the reputation formation in DCH is driven by manipulating the beliefs about levels of sophistication rather than the beliefs about exogenous types. We illustrate this point with the "chain-store game" introduced by Selten (1978).

Illustrative Example

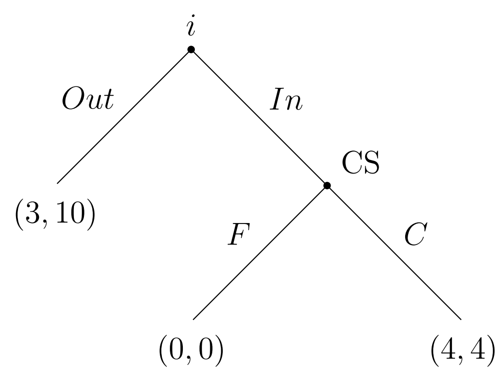

Figure 9: Game tree of period i in the chain-store game. The first number of each pair is competitor i's payoff and the second number is CS's payoff.

In this game, there are $N+1$ players: one chain-store (CS) and N competitors, numbered $1, \dots, N$. In each period, one of the potential competitors decides whether to compete with CS or not ("In" or "Out"); in period i, it is competitor i's turn to decide. If competitor i chooses "In," then CS decides either to fight ("F") or cooperate ("C"). CS responds to competitor i before competitor $(i+1)$'s turn. Hence, in each period, there are three possible outcomes $\{Out, (In, F), (In, C)\}$. The game tree of each period i is plotted in Figure 9. In addition, at every point of the game, all players know all actions taken previously, which makes this game an extensive form game with perfect information. Finally, the payoff to the chain store in this game is the sum of its payoffs in N periods.

There are multiple Nash equilibria where the outcome in any period is either *Out* or (In, C). Specifically, in any equilibrium where competitor i chooses "Out," CS's strategy is to fight if the competitor chooses "In." However, these equilibria are *imperfect*. The unique subgame perfect equilibrium is that all competitors will choose "In" and CS will always choose "C," which fails to capture the reality that CS may choose "F" to deter entrance.

Unlike Kreps and Wilson (1982)'s approach to rationalize the deterrence by introducing payoff-relevant private types, DCH predicts that higher-level CS might purposely choose to fight in early periods to make potential competitors think they are facing a level-0 CS. We demonstrate this by considering the following distribution of levels

$$(p_0,\ p_1,\ p_2,\ p_3) = (0.10,\ 0.15,\ 0.60,\ 0.15).$$

Under this distribution, it suffices to characterize the DCH solution by analyzing level-1 to level-3 players' behavior.

Level-1 players believe all other players are level-0 and best respond to this belief. Therefore, level-1 competitors will choose "Out" and level-1 CS will choose "C." Next, from level-2 CS's perspective, all competitors are either level-0 or level-1 and the only strategic level of competitors will choose "Out," the most preferred outcome of CS. Therefore, level-2 CS does not have incentive to "F," and will behave like level-1 to always choose "C."

On the other hand, level-2 competitors will update their beliefs about CS's level based on the history. If level-2 competitors have observed that CS has cooperated for T times and never fought, the belief about CS being level-0 is

$$\nu \equiv \frac{p_0\left(\frac{1}{2}\right)^T}{p_0\left(\frac{1}{2}\right)^T + p_1} = \frac{2\left(\frac{1}{2}\right)^T}{2\left(\frac{1}{2}\right)^T + 3}.$$

The expected payoff of choosing "In" is $2\nu + 4(1-\nu)$ and therefore, it's optimal for a level-2 competitor to choose "In" if and only if $2\nu + 4(1-\nu) < 3 \iff \nu < \frac{1}{2}$. In this case, if CS has never fought, level-2 competitors will always choose "In." However, if CS has ever fought,[27] level-2 competitors will believe CS is level-0 and always choose "Out." Besides, because level-2 CS behaves the same as level-1, level-3 competitors will therefore behave the same as level-2—they will choose "In" if CS has never chosen "F" but choose "Out" if CS has ever chosen "F."

Finally, in the early periods,[28] if the relative proportion of level-2 players is sufficiently high, it would be profitable for level-3 CS to purposely choose "F" to make all competitors believe the chain store is level-0 and choose "Out" in later periods. Specifically, level-3 CS

[27] By Proposition 2, we know if CS has fought once, then all competitors will always believe CS is level-0, regardless of how many time CS has cooperated.

[28] In the last period, there is no reputation concern and therefore, level-3 CS will choose "C" if he has a chance to move.

would choose to fight if

$$
\begin{aligned}
4+&\left[\frac{p_0}{\sum_{k=0}^{2} p_k}\times 7+\frac{p_1}{\sum_{k=0}^{2} p_k}\times 10+\frac{p_2}{\sum_{k=0}^{2} p_k}\times 4\right]< \\
0+&\left[\frac{p_0}{\sum_{k=0}^{2} p_k}\times 7+\frac{p_1}{\sum_{k=0}^{2} p_k}\times 10+\frac{p_2}{\sum_{k=0}^{2} p_k}\times 10\right] \Longleftrightarrow \frac{p_2}{p_0+p_1+p_2}>\frac{2}{3}. \qquad \square
\end{aligned}
$$

7.2 Correlated Beliefs in Games of Imperfect Information

There is a wide range of applications of extensive form games in economics and political science where players have private information, either due to privately known preferences and beliefs about other players, or from imperfect observability of the histories of play in the game. These applications would include many workhorse models, such as signaling, information transmission, information design, social learning, entry deterrence, reputation building, crisis bargaining, and so forth. Hence the natural next step is to investigate more deeply our approach to dynamic games with incomplete information. In such environments, one complication is that players not only learn about the opponents' levels of sophistication but at the same time also learn about more basic elements of the game structure, such as the opponents' private information, payoff types, and prior moves.

One observation is that allowing for imperfect information in the DCH approach does not introduce any problems of off-path beliefs. The reason is that at every information set of the game, all types of all players have posterior beliefs over the opponents' types that include a positive probability they are facing level-0 players. Hence, there is no issue of specifying off-path beliefs in an ad hoc fashion and therefore we avoid the complications of belief-based refinements.

In games of imperfect information, the DCH belief system is a level-dependent function that assigns to every information set a *joint distribution* of other players' levels and the histories in the information set. When information sets are not singleton, at some information set, the marginal beliefs about other players' levels can be correlated across players. In other words, Proposition 1 might break down in games of imperfect information. We illustrate this by using the following three-person game where each player moves once. The game tree is shown in Figure 10.

Illustrative Example

In the first stage, Player 1 chooses whether to go left or right. In the second stage, player 2 chooses to go left or right. If players 1 and 2 make the same decision, the game ends. Otherwise, the game moves to stage 3 and player 3 makes the final decision. However, in that stage, player 3 only knows that one of the previous players chose l and the other chose r, but does not know which one chose l.

Level-1 players believe all other players are level-0. As we compute the expected payoff of each action, level-1 player 1 will choose r at the initial node. Level-1 player 2 will choose

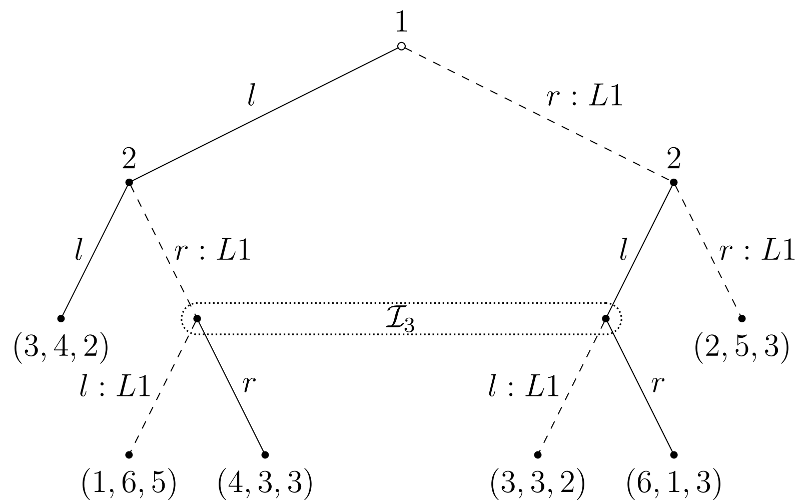

Figure 10: Game Tree of Example 7.2. Dashed lines are the paths selected by level-1 players.

r at subgame $h = l$ and $h = r$. At player 3's information set, since level-1 player 3 thinks both players are level-0, he would believe both histories are equally likely, and choose l.

Conditional on the game reaching player 3's information set $\mathcal{I}_3$, level-2 player 3's DCH belief $\nu_3^2(\tau_{-3}, h|\mathcal{I}_3)$ is joint distribution of the histories $h \in \{lr, rl\}$ and other players' level profile $\tau_{-3} = (\tau_1, \tau_2) \in \{(0,0),(0,1),(1,0),(1,1)\}$. Let $(p_k)_{k\in\mathbb{N}_0}$ be the true distribution of levels of all players, and level-2 player 3's DCH belief can be summarized in Table 2.

Table 2: Level 2 player 3's DCH belief at information set $\mathcal{I}_3$

$\nu_3^2(\tau_{-3}, h\|\mathcal{I}_3)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$h = lr$	$\frac{0.25p_0}{0.5p_0+p_1}$	$\frac{0.5p_1}{0.5p_0+p_1}$	0	0
$h = rl$	$\frac{0.25p_0}{0.5p_0+p_1}$	0	$\frac{0.5p_1}{0.5p_0+p_1}$	0

First observe that given level-0 and level-1 players' strategies, at player 3's information set, level-2 player 3 would think player 1 and player 2 cannot both be level-1 players; otherwise, the game will not reach this information set. In other words, level-2 player 3's marginal belief $\nu_3^2\left(\tau_{-3} = (1,1)|\mathcal{I}_3\right) = 0$. Nonetheless, the marginal belief of *each* player is $\nu_3^2\left(\tau_1 = 1|\mathcal{I}_3\right) = \nu_3^2\left(\tau_2 = 1|\mathcal{I}_3\right) = \frac{0.5p_1}{0.5p_0+p_1}$, implying that the marginal beliefs about levels are correlated across players since:

$$\nu_3^2\left(\tau_1 = 1|\mathcal{I}_3\right) \times \nu_3^2\left(\tau_2 = 1|\mathcal{I}_3\right) \neq 0 = \nu_3^2\left(\tau_{-3} = (1,1)|\mathcal{I}_3\right). \qquad \square$$

7.3 DCH in Multi-Stage Games with Observed Actions

As illustrated in the previous example, for games with imperfect information, DCH belief about other players' levels might be correlated across players at an information set. In other words, the DCH belief system in this case need not be a product measure.

However, Lin (2022) shows that the DCH belief system is indeed a product measure across players in the framework of *multi-stage games with observed actions* introduced by Fudenberg and Tirole (1991). This framework models the situation where every player observes the action of every other player—the only uncertainty is about other players' payoff-relevant private information which is determined by an initial chance move.

In a multi-stage game with observed actions, each player $i \in N$ has a payoff-relevant *type* θ_i drawn from a finite type set Θ_i according to the distribution $\mathcal{F}_i$.[29] After the types are assigned, each player learns their own type but not the other players' types. The game is played in stages $t = 1, \ldots, T$. In each stage, players simultaneously choose actions and the action profile is revealed to all players at the end of the stage. Therefore, in a multi-stage game with observed actions, each player i's information set can be specified as (θ_i, h) where h is a non-terminal public history.[30]

For any $k \in \mathbb{N}$ and $i \in N$, level-k player i's DCH belief system at information set (θ_i, h) is a joint distribution about other players' types and levels, denoted as $\nu_i^k(\tau_{-i}, \theta_{-i}|\theta_i, h)$. Proposition 2 of Lin (2022) shows that $\nu_i^k(\tau_{-i}, \theta_{-i}|\theta_i, h)$ is indeed a product measure across players. That is,

$$\nu_i^k(\tau_{-i}, \theta_{-i}|\theta_i, h) = \prod_{j \neq i} \nu_{ij}^k(\tau_j, \theta_j|\theta_i, h).$$

From the comparison between Example 7.2 and multi-stage games with observed actions, we can find that the observability of actions plays a crucial role in the independence of the DCH belief system.[31]

7.4 (Non-)Equivalence on the Non-Reduced Normal Form

The analysis in section 4.3 and section 5 shows that the DCH solution is *not reduced-normal-form invariant*—the DCH solution in the extensive form and in the corresponding reduced normal form can be dramatically different. To this end, one may naturally wonder whether DCH is *non-reduced-normal-form* invariant.[32]

The intuition behind such invariance, which is explained in more detail in Battigalli (2022), is as follows. First, observe that a level-0 player's *behavioral strategy* (uniform randomization at every subgame) in extensive form and a level-0 player's *mixed strategy* (uniform randomization over all contingent strategies) in non-reduced normal form are *outcome equivalent*.[33] Second, either in extensive form or in non-reduced normal form, all strategic

[29]Without loss of generality, we assume that $\mathcal{F}_i(\theta_i) > 0$ for all $\theta_i \in \Theta_i$ and $\mathcal{F}_i$ is independent of $\mathcal{F}_j$ for any $i, j \in N$ and $i \neq j$.

[30]See Lin (2022) for a detailed description of the framework of multi-stage games with observed actions and the DCH solution for this framework.

[31]See Battigalli (2022) for additional discussion about the role of observability.

[32]We are grateful to Pierpaolo Battigali for observing that DCH applied to games in extensive form is outcome equivalent to the non-reduced normal form. The formal analysis for games with perfect information can be found in Battigalli (2022).

[33]Two (mixed or behavioral) strategies are outcome-equivalent if for every collection of pure strategies of

players (with level $k \geq 1$) *best respond* to totally mixed strategies due to the ever-presence of level-0 players. Since expected payoff maximization is dynamically consistent, the *ex ante* best response (the optimal contingent strategy in non-reduced normal form) must be outcome equivalent to the *sequential* best response (the optimal behavioral strategy in extensive form). Hence, the DCH solution in extensive form will be outcome equivalent to the DCH solution in non-reduced normal form.

The equivalence of DCH solution in extensive form and non-reduced normal form relies heavily on two assumptions: (1) *uniform randomization* by level-0 players and (2) the dynamic consistency of best response.

In standard level-k and CH models, level-0 is assumed to uniformly randomize across all available actions[34], and this approach has carried over to most applications. The specification of uniform randomization by level-0 players has several advantages: it is well-defined (and applied equally and in the same way) for all games; it is nondegenerate, so all paths of play can be rationalized by all strategic players; and it is simple and parsimonious. Some non-uniform specifications of level-0 behavior have been tailored to specific games of interest in particular applications. For example, alternative approaches include modeling level-0 players as choosing (or avoiding) a *salient* action (e.g. Crawford and Iriberri (2007a)), an *instinctive* action (e.g. Rubinstein (2007)) or a *minimum-payoff averse* action (e.g. Chong et al. (2016)) from the action set.

A second approach that is often used relaxes the perfect best response assumption of strategic types. In this alternative approach, strategic levels of players are typically assumed to make *better responses*, whereby players choose actions at each information set stochastically (with full support), and the choice probabilities are increasing in the continuation values, usually specified by a quantal response function such as the logit choice rule (e.g. Camerer et al. (2016), Stahl and Wilson (1995)). In this case, for any strategic level of player, the quantal response behavioral strategy in extensive form will generally not be outcome equivalent to the quantal response mixed strategy in non-reduced normal form. This relaxation of perfect best responses could be useful for estimating DCH in experimental data sets. It results in a smoother updating process, and implies full support of beliefs of about other players' (lower) levels at every information set of the game.

8 Conclusions

We conclude by emphasizing the key motivation for this paper: to provide a theoretical framework that characterizes hierarchical reasoning in sequential games. As documented in the literature, sequential equilibrium based on fully rational backward induction is not only mathematically fragile but also empirically implausible to hold. To narrow the gap between the theory and empirical patterns in sequential games, it is natural to extend the level-k approach to such games, as it has already demonstrated considerable success in narrowing the gap for games played simultaneously. However, the conundrum for directly applying the

the other players the two strategies induce the same distribution of outcomes (see Osborne and Rubinstein (1994) chapter 11).

[34] Uniform randomization is specifically assumed in the original formulation of CH in Camerer et al. (2004)

standard level-k approach is that players may observe actions that are incompatible with their beliefs, which leads to the widely known problem of specifying off-path beliefs. The DCH model avoids this issue with a simple structure that allows players with heterogeneous levels of sophistication to update their beliefs everywhere as history unfolds, using Bayes' rule.

We characterize properties of the belief-updating process and explore how it can affect players' strategic behavior. The key of our framework is that the history of play contains substantial information about other players' levels of sophistication, and therefore as play unfolds, players learn about their opponents' strategic sophistication and update their beliefs about the continuation play in the game accordingly. In this way our DCH model departs from the standard level-k approach and generates new insights, including experimentally testable implications.

We obtain two main results that apply generally to all finite extensive form games. Proposition 1 establishes that a player's updating process is independent across the other players. That is, for every player and every non-terminal history, the joint distribution of the beliefs of the levels of the other players is the product of the individual posterior distribution of the levels of each of those other players. In games of imperfect information, the information sets are non-singleton and the beliefs could be correlated across the histories at some information set.

In addition, Proposition 2 establishes that the updating process filters out possible level types of opponents as the game proceeds, and it is irreversible. That is, over the course of play, it is possible that a player eliminates some levels of another player from the support of his beliefs, and as the game continues, these levels can never be added back to the support. Hence, in addition to updating posterior beliefs over the support of level types, the support also shrinks over time. However, the level-0 players always remain in the support of beliefs, and hence every player believes every future information set can be reached with positive probability.

The second half of the paper provides a rigorous analysis of a class of increasing-pie centipede games and generates testable predictions about how play depends on whether the game is represented in extensive form or in its reduced normal form. One direct implication is the representation effect given by Theorem 1. The theorem states that playing a centipede game in its extensive form representation, i.e., as a sequential move game, would lead to more taking than the reduced normal form representation, where the two players simultaneously announce the stage at which they will take.

This result provides a prediction that may be useful for experimental testing, since the claim is independent of the length of the centipede and the increment of the pie. Moreover, the statement is true for any prior belief about the strategic levels. García-Pola et al. (2020a) recently reported the results of an experiment that is ideally suited to test the representation effect implied by DCH. That experiment explored whether there were differences in behavior in four different centipede games, depending on whether they were played sequentially or (simultaneously) in the reduced normal form. DCH predicts a representation effect with earlier taking in the sequential treatment, in three of the four games, and this is exactly what they find, and the effect is statistically significant in two of the three games. DCH does not predict this representation effect in the fourth game, which is also what their experiment finds. This provides empirical support for the representation effect we identify, and suggests additional experimental studies would be valuable to establish robustness of these effects, and to see if the findings extend the linear centipede games that we focused on in section 5.

Another direction worth pursuing would be to incorporate some salient features of alternative behavioral models of learning in extensive form games into our approach. In the approach taken here, the learning process is "extreme" in the sense that players will completely rule out some levels from their beliefs whenever they observe incompatible actions. For example, players will believe the opponent is level-0 with certainty if a strictly dominated action is taken. Yet, it is possible that the player is strategic and the action is taken by mistake. In this sense, one could incorporate some elements of the extensive form QRE, where players choose actions at each information set stochastically, and the choice probabilities are increasing in the continuation values. In fact, this approach has been used with some success in simultaneous move games (Crawford and Iriberri, 2007a). As shown in Proposition 2, in the present model of DCH, there is no way to *expand* the support of a player's belief about the other players' types. However, if players choose stochastically, then no level type is ever ruled out from the support, which smoothes out the updating process. Because players' beliefs maintain full support on lower types throughout the game, a natural conjecture is that arbitrarily high-level players will approach backward induction when the error is sufficiently small.

As a final remark, while the main point of the present paper is to develop a general theoretical foundation for applying CH to extensive form games, the ultimate hope is that this framework can be usefully applied to gain insight into specific economic models. There are a number of possible such applications one might imagine, where some or all agents in the model have opportunities to learn about the strategic sophistication of the other agents in ways that could significantly affect their choices in the game. One such application is reputation building, which we briefly examined in section 7.1 and deserves more extensive study. As another possible application, Chamley and Gale (1994) analyze a dynamic investment game with social learning, where investments are valuable only if enough other agents are able to invest, and learning occurs as investment decisions are observed over time. The DCH model, which combines learning and updating, but without common knowledge of rationality or fully rational expectations, might be a useful alternative approach to this problem. Models of sequential voting on agendas (McKelvey and Niemi (1978) and Banks (1985)), limit pricing and entry deterrence (Selten (1978) and Milgrom and Roberts (1982)), and dynamic public good provision (Marx and Matthews (2000), Duffy et al. (2007), and Choi et al. (2008)) are some additional areas of applied interest where the DCH approach could be useful.

Appendix A: Proofs of Results in Section 4

Let τ_i be player i's level. Following previous notations, we use $\sigma_i(h)$ to denote player i's (pure) action at h. In addition, $\mu_i^k(\tau_{-i})$ is level-k player i's prior belief about the opponent's level, and $\nu_i^k(\tau_{-i} \mid h)$ is level-k player i's posterior belief about the opponent's level at history h. Finally, level-0 players would uniformly randomize at every node. The analysis of the examples is summarized in the following claims.

Example 4.2.1

Claim 1. *In Example 4.2.1, each level of players' strategies are:*

1. *for any $k \in \mathbb{N}$, $\sigma_1^k(1b) = r_{1b}$, $\sigma_1^k(1c) = l_{1c}$, $\sigma_2^k(2b) = l_{2b}$, and $\sigma_2^k(2c) = r_{2c}$;*
2. *$\sigma_1^1(1a) = r_{1a}$ and $\sigma_1^k(1a) = l_{2a}$ for $k \geq 2$; $\sigma_2^1(2a) = \sigma_2^2(2a) = l_{2a}$ and $\sigma_2^k(2a) = r_{2a}$ for $k \geq 3$.*

Proof: The calculation consists of two parts.

1. First, all strategic levels of players would choose the action with a higher payoff at the last node. Hence, $\sigma_1^k(1b) = r_{1b}$ and $\sigma_2^k(2c) = r_{2c}$ for all $k \geq 1$. Player 2 has a dominant action at history $h = 2b$, so $\sigma_2^k(2b) = l_{2b}$ for all $k \geq 1$. Notice that whenever a dominant action is not chosen, players would believe the opponent is level-0 with certainty. At history $h = 1c$, every level of player 1 thinks player 2 is level-0 and hence for all $k \geq 1$, $\sigma_1^k(1c) = l_{1c}$ since the expected payoff is $13/2 > 6$.

2. Level-1 players believe the other player would randomize at every node. On the one hand, $\sigma_1^1(1a) = r_{1a}$ and $\sigma_2^1(2a) = l_{2a}$ so that they can maximize the expected payoff. On the other hand, level-2 players' initial beliefs are $\mu_i^2(0) = e^{-1.5}/(e^{-1.5} + 1.5e^{-1.5}) = 2/5$ and $\mu_i^2(1) = 3/5$. Thus, $\sigma_1^2(1a) = l_{1a}$ since the expected payoff for l_{1a} is $19/5 > 29/10$. On the other hand, when history $h = 2a$ is realized, level-2 player 2 would believe the opponent is definitely level-0 and hence $\sigma_2^2(2a) = \sigma_2^1(2a) = l_{2a}$.

The behavior of higher-level players can be solved by induction. Level-3 players' prior beliefs are $\mu_i^3(0) = 8/29$, $\mu_i^3(1) = 12/29$, and $\mu_i^3(2) = 9/29$. In this case, $\sigma_1^3(1a) = l_{1a}$ since the expected payoff for l_{1a} is $112/29 > 76/29$. In addition, when history $h = 2a$ is realized, level-3 player 2's posterior belief becomes $\nu_2^3(0 \mid 2a) = 0.5e^{-1.5}/(0.5e^{-1.5} + 1.125e^{-1.5}) = 4/13$ and $\nu_2^3(2 \mid 2a) = 9/13$, and hence $\sigma_2^3(2a) = r_{2a}$ since $4 > 45/13$. Suppose for some $k > 3$, $\sigma_1^\kappa(1a) = l_{1a}$ for all $2 \leq \kappa \leq k$ and $\sigma_2^\kappa(2a) = r_{2a}$ for all $3 \leq \kappa \leq k$. We want to show that $\sigma_1^{k+1}(1a) = l_{1a}$ and $\sigma_2^{k+1}(2a) = r_{2a}$. Level-($k$+1) players' prior beliefs are $\mu_i^{k+1}(\kappa) = p_\kappa/(\sum_{\kappa=0}^k p_\kappa)$ for $0 \leq \kappa \leq k$. By the induction hypothesis, $\sigma_1^{k+1}(1a) = l_{1a}$ if and only if

$$\frac{7}{2}\left(\frac{p_0}{\sum_{\kappa=0}^k p_\kappa}\right) + 4\left(\frac{p_1 + p_2}{\sum_{\kappa=0}^k p_\kappa}\right) + 3\left(\frac{\sum_{\kappa=3}^k p_\kappa}{\sum_{\kappa=0}^k p_\kappa}\right) > \frac{17}{4}\left(\frac{p_0}{\sum_{\kappa=0}^k p_\kappa}\right) + 2\left(1 - \frac{p_0}{\sum_{\kappa=0}^k p_\kappa}\right),$$

which is equivalent to $(7/4)p_0 - p_1 - p_2 < \sum_{\kappa=0}^k p_\kappa$. This holds because $(7/4)p_0 - p_1 - p_2 = -(7/8)e^{-1.5} < 0$. Finally, by the induction hypothesis, level-(k+1) player 2's posterior belief at $h = 2a$ is $\nu_2^{k+1}(0 \mid 2a) = 0.5p_0/(0.5p_0 + \sum_{\kappa=2}^{k} p_\kappa)$ and $\nu_2^{k+1}(j \mid 2a) = p_j/(0.5p_0 + \sum_{\kappa=2}^{k} p_\kappa)$ where $2 \leq j \leq k$. Thus, $\sigma_2^{k+1}(2a) = r_{2a}$ if and only if

$$\frac{9}{2}\nu_2^{k+1}(0 \mid 2a) + 3\left(1 - \nu_2^{k+1}(0 \mid 2a)\right) < 4 \iff \nu_2^{k+1}(0 \mid 2a) < \frac{2}{3}.$$

Moreover, the induction hypothesis suggests that

$$\nu_2^{k+1}(0 \mid 2a) = \frac{\frac{1}{2}p_0}{\frac{1}{2}p_0 + \sum_{\kappa=2}^{k} p_\kappa} < \frac{\frac{1}{2}p_0}{\frac{1}{2}p_0 + \sum_{\kappa=2}^{k-1} p_\kappa} = \nu_2^{k}(0 \mid 2a) < \frac{2}{3},$$

implying the optimal choice for level-(k+1) player 2 is r_{2a}. This completes the proof. ■

Example 4.2.3

Claim 2. *Suppose τ_i's are independently drawn from $p = (p_k)_{k=0}^{\infty}$, then in Example 4.2.3,*

1. *for any $k \in \mathbb{N}$, $\sigma_1^k(1a) = r_{1a}$, $\sigma_1^k(1b) = r_{1b}$, $\sigma_1^k(1c) = l_{1c}$, $\sigma_2^k(2a) = l_{2a}$, $\sigma_2^k(2b) = l_{2b}$, and $\sigma_2^k(2c) = r_{2c}$;*
2. *the ex ante probability of the subgame perfect equilibrium path being realized converges to 0 as $p_0 \to 0^+$.*

Proof: The calculation consists of two parts.

1. By the analysis of Example 4.2.1, we only need to check player 1's action at the initial node and player 2's action at history $h = 2a$. We can prove the statement by induction on k. For $k = 1$, players would think the opponent is level-0. In this case, $\sigma_1^1(1a) = r_{1a}$ since the expected payoff is $17/4 > 9/4$ and $\sigma_2^1(2a) = l_{2a}$ with the expected payoff being $9/2 > 4$. Suppose there is some K such that $\sigma_1^k(1a) = r_{1a}$ and $\sigma_2^k(2a) = l_{2a}$ for all $1 \leq k \leq K$. For level-(K+1) player 1, the prior belief is $\mu_1^{K+1}(0) = p_0/(\sum_{\kappa=0}^{K} p_\kappa)$ and $\sigma_1^{K+1}(1a) = r_{1a}$ if and only if

$$\frac{17}{4}\mu_1^{K+1}(0) + 2\left(1 - \mu_1^{K+1}(0)\right) > \frac{9}{4}\mu_1^{K+1}(0) + \frac{3}{2}\left(1 - \mu_1^{K+1}(0)\right),$$

which holds as $\mu_1^{K+1}(0) > 0$. On the other hand, by the induction hypothesis, player 2 would believe player 1 is level-0 with certainty when history $h = 2a$ is realized, so $\sigma_2^{K+1}(2a) = \sigma_2^1(2a) = l_{2a}$.

2. Statement **1** implies the probability of the subgame perfect equilibrium path r_{2a} being realized is

$$\Pr(r_{2a}) = \Pr((1a, 2a) \mid 1a)\Pr(r_{2a} \mid 2a) = \left[\sigma_1^0(1a, 2a)p_0\right]\left[\sigma_{2,2a}^0(r_{2a})p_0\right] = \frac{1}{4}p_0^2.$$

Therefore, we can find the limit of the probability is

$$\lim_{p_0 \to 0^+} \Pr(r_{2a}) = \lim_{p_0 \to 0^+} \frac{1}{4}p_0^2 = 0.$$

This completes the proof. ■

Example 4.3

Claim 3. *In Example 4.3, each level of players' strategies are:*

1. *for any* $k \in \mathbb{N}$, $\sigma_1^k(1b) = r_{1b}$, $\sigma_1^k(1c) = l_{1c}$, $\sigma_2^k(2b) = l_{2b}$, *and* $\sigma_2^k(2c) = r_{2c}$;
2. $\sigma_1^k(1a) = l_{1a}$ *for all* $k \neq 2$, *and* $\sigma_1^2(1a) = r_{1a}$; $\sigma_2^1(2a) = l_{2a}$, *and* $\sigma_2^k(2a) = r_{2a}$ *for all* $k \geq 2$.

Proof: The proof consists of two parts.

1. The proof is the same as the proof of Claim 1.

2. First, level-1 players believe the other player randomizes everywhere, so $\sigma_1^1(1a) = l_{1a}$ and $\sigma_2^1(2a) = l_{2a}$ in order to maximize their expected payoffs. Level-2 players' prior beliefs are $\mu_i^2(0) = 2/5$ and $\mu_i^2(1) = 3/5$. Therefore, $\sigma_1^2(1a) = r_{1a}$ since the expected payoff is $29/10 > 28/10$. Level-2 player 2's posterior belief at history $h = 2a$ is $\nu_2^2(0 \mid 2a) = 0.5e^{-1.5}/(0.5e^{-1.5} + 1.5e^{-1.5}) = 1/4$ and $\nu_2^2(1 \mid 2a) = 3/4$. In this case, $\sigma_2^2(2a) = r_{2a}$ because $4 > 27/8$.

Finally, we can solve higher-level players' behavior by induction. Level-3 players' prior beliefs are $\mu_i^3(0) = 8/29$, $\mu_i^3(1) = 12/29$, and $\mu_i^3(2) = 9/29$, and hence $\sigma_1^3(1a) = l_{1a}$ since the expected payoff is $128/29 > 76/29$. At history $h = 2a$, level-3 player 2's posterior belief is the same as level-2, and so $\sigma_2^3(2a) = \sigma_2^2(2a) = r_{2a}$. Suppose there is some $K > 3$ such that $\sigma_1^k(1a) = l_{1a}$ for all $3 \leq k \leq K$ and $\sigma_2^k(2a) = r_{2a}$ for all $2 \leq k \leq K$. Level-$(K+1)$ players' prior beliefs are $\mu_i^{K+1}(j) = p_j / \sum_{i=0}^K p_i$ for $0 \leq j \leq K$. By the induction hypothesis, $\sigma_1^{K+1}(1a) = l_{1a}$ if and only if

$$\frac{19}{4}\left(\frac{p_0}{\sum_{i=0}^K p_i}\right) + \frac{3}{2}\left(\frac{p_1}{\sum_{i=0}^K p_i}\right) + 8\left(\frac{\sum_{i=2}^K p_i}{\sum_{i=0}^K p_i}\right) > \frac{17}{4}\left(\frac{p_0}{\sum_{i=0}^K p_i}\right) + 2\left(1 - \frac{p_0}{\sum_{i=0}^K p_i}\right),$$

which is equivalent to $5.5p_0 + 6.5p_1 < 6\sum_{i=0}^K p_i$. This holds when the distribution of levels follows Poisson(1.5). On the other hand, by the induction hypothesis, level-$(K+1)$ player 2's posterior belief at history $h = 2a$ is $\nu_2^{K+1}(0 \mid 2a) = 0.5p_0/(0.5p_0 + p_1 + \sum_{i=3}^K p_i)$ and $\nu_2^{K+1}(j \mid 2a) = p_j/(0.5p_0 + p_1 + \sum_{i=3}^K p_i)$ where $j \neq 0$ or 2, and hence $\sigma_2^{K+1}(2a) = r_{2a}$ if and only if

$$\frac{9}{2}\nu_2^{K+1}(0 \mid 2a) + 3\left(1 - \nu_2^{K+1}(0 \mid 2a)\right) < 4 \iff \nu_2^{K+1}(0 \mid l) < \frac{2}{3}.$$

Moreover, the induction hypothesis implies:

$$\nu_2^{K+1}(0 \mid 2a) = \frac{\frac{1}{2}p_0}{\frac{1}{2}p_0 + p_1 + \sum_{i=3}^K p_i} < \frac{\frac{1}{2}p_0}{\frac{1}{2}p_0 + p_1 + \sum_{i=3}^{K-1} p_i} = \nu_2^K(0 \mid 2a) < \frac{2}{3},$$

as desired. ■

Appendix B: Proofs of Results in Section 5

Proof of Lemma 1

1. Since stage $2S$ is the last stage of the game, for any $k \geq 1$, player 2 would take at this stage if and only if

$$1 + (2S-1)c > 2Sc \iff 1 > c,$$

which holds by assumption. Therefore, $\sigma^k_{2,S} = 1$ for all $k \geq 1$.

2. Consider a level-1 type of player 1 and any of player 1's decision nodes $j \in \{1, ..., S\}$. The payoff from Take is $1 + (2j-2)c$ and the expected payoff from Pass is greater than or equal to $\frac{1}{2}(2j-1)c + \frac{1}{2}(1+(2j)c)$. Thus, $\sigma^k_{1,j} = 0$ is strictly optimal if and only if:

$$\begin{aligned} 1 + (2j-2)c &< \frac{1}{2}(2j-1)c + \frac{1}{2}(1+(2j)c) \\ &\iff \frac{1}{3} < c. \end{aligned}$$

Hence, $\sigma^1_1 = (0, \ldots, 0)$. A similar argument shows that $\sigma^1_2 = (0, \ldots, 0, 1)$.

3. The argument is similar to the proof of the first statement. Consider a level-k type of player 1 and any of player 1's decision nodes $j \in \{1, ..., S-1\}$, and suppose $\sigma^m_{2,j} = 0$ for every $1 \leq m \leq k-1$. Then the payoff from Take is $1 + (2j-2)c$ and the expected payoff from Pass is greater than or equal to $\frac{1}{2}\nu^k_{2j-1}(0)(2j-1)c + (1 - \frac{1}{2}\nu^k_{2j-1}(0))(1+(2j)c)$, which in turn is greater than or equal to $\frac{1}{2}(2j-1)c + \frac{1}{2}(1+(2j)c)$ because $\nu^k_{2j-1}(0) \leq 1$. Thus $\sigma^k_{1,j} = 0$ is strictly optimal if and only if:

$$\begin{aligned} 1 + (2j-2)c &< \frac{1}{2}(2j-1)c + \frac{1}{2}(1+(2j)c) \\ &\iff \frac{1}{3} < c. \end{aligned}$$

Hence, $\sigma^k_{1,j} = 0$. A similar argument shows that $\sigma^k_{2,j} = 0$ if $\sigma^m_{1,j+1} = 0$ for every $1 \leq m \leq k-1$. This completes the proof. ■

Proof of Proposition 4

1. The statement can be proved by induction. Consider stage $2S-1$. By Lemma 1, we know $K^*_{2S} = 1$ and $\sigma^1_{1,S} = 0$, suggesting $K^*_{2S-1} \geq 2 = K^*_{2S} + 1$. Now, fix any $2 \leq m \leq 2S-1$ and suppose the statement holds for all stages $m \leq j \leq 2S-1$. Without loss of generality, we consider an even m. We want to show that if $K^*_m < \infty$, then $K^*_{m-1} \geq K^*_m + 1$. By construction, we know $\sigma^k_{2,\frac{m}{2}} = 0$ for all $1 \leq k \leq K^*_m - 1$. Therefore, Lemma 1 implies $\sigma_{1,\frac{m}{2}} = 0$ for all $1 \leq k \leq K^*_m$, and $K^*_{m-1} \geq K^*_m + 1$.

2. Consider any j such that $K^*_{j+1} = \infty$. Without loss of generality, we consider an odd j. Hence, $\sigma^k_{2,\frac{j+1}{2}} = 0$ for all $k \geq 1$ and we want to show $\sigma^k_{1,\frac{j+1}{2}} = 0$ for all $k \geq 1$ by induction.

Lemma 1 implies $\sigma^1_{1,\frac{j+1}{2}} = 0$. Suppose there is $\overline{K} \geq 2$ such that $\sigma^k_{1,\frac{j+1}{2}} = 0$ for all $1 \leq k \leq \overline{K}$. Since $\sigma^k_{2,\frac{j+1}{2}} = 0$ for all $k \geq 1$, Lemma 1 implies $\sigma^{\overline{K}+1}_{1,\frac{j+1}{2}} = 0$, as desired. ■

Proof of Proposition 5

We prove this by induction. Consider stage $2S-1$. Level K^*_{2S-1} player 1 believes that only level-0 player 2 will pass at stage $2S$, so:

$$
\begin{aligned}
1+(2S-2)c &> \left(1-\frac{1}{2}\nu^{K^*_{2S-1}}_{2S-1}(0)\right)[(2S-1)c] + \frac{1}{2}\nu^{K^*_{2S-1}}_{2S-1}(0)[1+2Sc] \\
&> \left(1-\frac{1}{2}\nu^{k}_{2S-1}(0)\right)[(2S-1)c] + \frac{1}{2}\nu^{k}_{2S-1}(0)[1+2Sc] \text{ for all } k > K^*_{2S-1}
\end{aligned}
$$

since $\frac{1}{2}\nu^k_{2S-1}(0) < \frac{1}{2}\nu^{K^*_{2S-1}}_{2S-1}(0)$ and therefore $\sigma^k_{1,S} = 1$ for all $k \geq K^*_{2S-1}$.

Next, suppose for any m where $2 \leq m \leq 2S-1$, the statement holds for all j such that $m \leq j \leq 2S-1$. Suppose m is odd and $K^*_{m-1} < \infty$. (A similar argument applies if m is even.) By construction, $\sigma^k_{2,\frac{m-1}{2}} = 0$ for all $1 \leq k \leq K^*_{m-1}-1$. By Lemma 1, we have $\sigma^k_{1,s} = 0$ for all $1 \leq s \leq \frac{m-1}{2}$ and for all $1 \leq k \leq K^*_{m-1}$. Level K^*_{m-1} player 2's belief at stage $m-1$ that the other player would pass at stage m is

$$
\begin{aligned}
\frac{1}{2}\nu^{K^*_{m-1}}_{m-1}(0) + \sum_{\kappa=1}^{K^*_m-1} \nu^{K^*_{m-1}}_{m-1}(\kappa) &= \frac{1}{2}\frac{p_0\left(\frac{1}{2}\right)^{\frac{m-1}{2}}}{p_0\left(\frac{1}{2}\right)^{\frac{m-1}{2}} + \sum_{\kappa=1}^{K^*_{m-1}-1} p_\kappa} + \frac{\sum_{\kappa=1}^{K^*_m-1} p_\kappa}{p_0\left(\frac{1}{2}\right)^{\frac{m-1}{2}} + \sum_{\kappa=1}^{K^*_{m-1}-1} p_\kappa} \\
&= \frac{p_0\left(\frac{1}{2}\right)^{\frac{m+1}{2}} + \sum_{\kappa=1}^{K^*_m-1} p_\kappa}{p_0\left(\frac{1}{2}\right)^{\frac{m-1}{2}} + \sum_{\kappa=1}^{K^*_{m-1}-1} p_\kappa}.
\end{aligned}
$$

Since $\sigma^{K^*_{m-1}}_{1,s} = 0$ for all $1 \leq s \leq \frac{m-1}{2}$, then for any $k > K^*_{m-1}$, at stage $m-1$ level-k player 2's belief about the probability that the other player would pass at stage m is

$$
\begin{aligned}
\frac{1}{2}\nu^{k}_{m-1}(0) + \sum_{\kappa=1}^{K^*_m-1} \nu^{k}_{m-1}(\kappa) &\leq \frac{1}{2}\nu^{K^*_{m-1}+1}_{m-1}(0) + \sum_{\kappa=1}^{K^*_m-1} \nu^{K^*_{m-1}+1}_{m-1}(\kappa) \\
&= \frac{1}{2}\frac{p_0\left(\frac{1}{2}\right)^{\frac{m-1}{2}}}{p_0\left(\frac{1}{2}\right)^{\frac{m-1}{2}} + \sum_{\kappa=1}^{K^*_{m-1}} p_\kappa} + \frac{\sum_{\kappa=1}^{K^*_m-1} p_\kappa}{p_0\left(\frac{1}{2}\right)^{\frac{m-1}{2}} + \sum_{\kappa=1}^{K^*_{m-1}} p_\kappa} \\
&< \frac{p_0\left(\frac{1}{2}\right)^{\frac{m+1}{2}} + \sum_{\kappa=1}^{K^*_m-1} p_\kappa}{p_0\left(\frac{1}{2}\right)^{\frac{m-1}{2}} + \sum_{\kappa=1}^{K^*_{m-1}-1} p_\kappa}.
\end{aligned}
$$

since $\sum_{\kappa=1}^{K^*_{m-1}} p_\kappa > \sum_{\kappa=1}^{K^*_{m-1}-1} p_\kappa$. This implies that for any level $k > K^*_{m-1}$, higher level of player 2 at stage $m-1$ would think the other player is less likely to pass at stage m. Since it is already profitable for level K^*_{m-1} player 2 to take at stage $m-1$, we can conclude that $\sigma^k_{2,\frac{m-1}{2}} = 1$ for all $k \geq K^*_{m-1}$. ■

Proof of Proposition 7

Without loss of generality, we can consider an even j, so $\lfloor \frac{j}{2} \rfloor = \frac{j}{2}$ and it is player 2's turn at stage j.

Only if: Suppose $K_j^* < \infty$. Then from Proposition 4, $K_{j'}^* \geq K_j^* + 1$ for all $j' < j$. Hence, the belief of level K_j^* of player 2 that player 1 is level-0 at stage j equals to

$$\nu_j^{K_j^*}(0) = \frac{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}}}{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + \sum_{\kappa=1}^{K_j^*-1} p_\kappa} > \frac{p_0 \left(\frac{1}{2}\right)^{S-1}}{p_0 \left(\frac{1}{2}\right)^{S-1} + (1-p_0)}.$$

Level K_j^* player 2's belief at stage j that the player 1 would pass at stage $j+1$ is

$$\begin{aligned}\frac{1}{2}\nu_j^{K_j^*}(0) + \sum_{\kappa=1}^{K_{J+1}^*-1} \nu_j^{K_j^*}(\kappa) &= \frac{1}{2}\frac{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}}}{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + \sum_{\kappa=1}^{K_j^*-1} p_\kappa} + \frac{\sum_{\kappa=1}^{K_{j+1}^*-1} p_\kappa}{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + \sum_{\kappa=1}^{K_j^*-1} p_\kappa} \\ &= \frac{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}+1} + \sum_{\kappa=1}^{K_{j+1}^*-1} p_\kappa}{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + \sum_{\kappa=1}^{K_j^*-1} p_\kappa}\end{aligned}$$

where we know that $K_{j+1}^* \leq K_j^* - 1$ Since it is optimal for level $K_j^* < \infty$ to take at j this implies $\frac{p_0\left(\frac{1}{2}\right)^{\frac{j}{2}+1} + \sum_{\kappa=1}^{K_{j+1}^*-1} p_\kappa}{p_0\left(\frac{1}{2}\right)^{\frac{j}{2}} + \sum_{\kappa=1}^{K_j^*-1} p_\kappa} < \frac{1-c}{1+c}$ and therefore

$$\frac{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}+1} + \sum_{\kappa=1}^{K_{j+1}^*-1} p_\kappa}{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + (1-p_0)} < \frac{1-c}{1+c}.$$

If: Suppose $K_j^* = \infty$. Then from Proposition 4, $K_{j'}^* = \infty$ for all $j' < j$. That is, all levels of both players pass at every stage up to and including j. Hence the belief of level $k \geq 1$ of player 2 that player 1 is level-0 at stage j equals to

$$\nu_j^k(0) = \frac{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}}}{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + \sum_{\kappa=1}^{k-1} p_\kappa} > \frac{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}}}{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + (1-p_0)}.$$

Since $K_j^* = \infty$ it is optimal to pass at j for all levels $k \geq 1$ of player 2, which implies $\frac{\frac{1}{2}p_0\left(\frac{1}{2}\right)^{\frac{j}{2}} + \sum_{\kappa=1}^{K_{j+1}^*-1} p_\kappa}{p_0\left(\frac{1}{2}\right)^{\frac{j}{2}} + \sum_{\kappa=1}^{k-1} p_\kappa} \geq \frac{1-c}{1+c}$, for all k, where possibly $K_{j+1}^* = \infty$, so:

$$\frac{\frac{1}{2}p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + \sum_{\kappa=1}^{K_{j+1}^*-1} p_\kappa}{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + (1-p_0)} \geq \frac{1-c}{1+c},$$

as desired. ■

Proof of Lemma 2

1. To prove the statement, we can discuss player 1 and 2 separately.

Player 2:

(i) $a_2^1 = S$ strictly dominates $a_2^1 = S+1$: $\mathbb{E}\left[u_2(a_1^0, S)\right] - \mathbb{E}\left[u_2(a_1^0, S+1)\right] = \frac{1-c}{S+1} > 0$ since $c < 1$.

(ii) $a_2^1 = j+1$ strictly dominates $a_2^1 = j$ for all $1 \le j \le S-1$: For $1 \le j \le S-1$, since $c > \frac{1}{3}$,

$$\mathbb{E}\left[u_2(a_1^0, j+1)\right] - \mathbb{E}\left[u_2(a_1^0, j)\right] = \frac{1}{S+1}\left[-1 + (2S - 2j + 1)c\right] \ge \frac{1}{S+1}(-1+3c) > 0.$$

Hence, we can obtain that $a_2^1 = S$.

Player 1: By the same logic as (ii) above, $a_1^1 = j+1$ strictly dominates $a_1^1 = j$ for all $1 \le j \le S$: For $1 \le j \le S$, since $c > \frac{1}{3}$,

$$\mathbb{E}\left[u_1(j+1, a_2^0)\right] - \mathbb{E}\left[u_1(j, a_2^0)\right] = \frac{1}{S+1}\left[-1 + (2S - 2j + 3)c\right] \ge \frac{1}{S+1}(-1+3c) > 0.$$

Hence, we can obtain that $a_1^1 = S+1$.

2. *(i)* Notice that for any a_2, $u_1(a_1, a_2)$ is maximized at $a_1 = a_2$. Fix level $k \ge 2$. If level-k player 1 chooses s, then the expected payoff is:

$$\begin{aligned} V_1^k(s) &\equiv \sum_{\kappa=0}^{k-1} \tilde{p}_\kappa^k \mathbb{E}\left[u_1(s, a_2^\kappa)\right] \\ &= \tilde{p}_0^k \mathbb{E}\left[u_1(s, a_2^0)\right] + \sum_{\kappa=1}^{k-1} \tilde{p}_\kappa^k u_1(s, a_2^\kappa). \end{aligned}$$

Suppose $\min\{a_2^m : 1 \le m \le k-1\} = 1$, then *(i)* holds trivially. If $\min\{a_2^m : 1 \le m \le k-1\} \ge 2$, then we can prove the statement by contradiction. Suppose $a_1^k < \min\{a_2^m : 1 \le m \le k-1\}$, then

$$\begin{aligned} V_1^k(a_1^k) &= \tilde{p}_0^k \underbrace{\mathbb{E}\left[u_1(a_1^k, a_2^0)\right]}_{<\mathbb{E}\left[u_1(a_1^k+1, a_2^0)\right]} + \sum_{\kappa=1}^{k-1} \tilde{p}_\kappa^k \underbrace{u_1(a_1^k, a_2^\kappa)}_{\le u_1(a_1^k+1, a_2^l)} \\ &< V_1^k(a_1^k + 1). \end{aligned}$$

$\mathbb{E}\left[u_1(a_1^k, a_2^0)\right] < \mathbb{E}\left[u_1(a_1^k+1, a_2^0)\right]$ follows from the first statement. Furthermore, $a_1^k < \min\{a_2^m : 1 \le m \le k-1\}$ implies $u_1(a_1^k, a_2^\kappa) \le u_1(a_1^k + 1, a_2^\kappa)$ for all $1 \le \kappa \le k-1$. Hence, $a_1^k < \min\{a_2^m : 1 \le m \le k-1\}$ is not optimal for level-k player 1, a contradiction.

(ii) The logic is similar for player 2.

3. We prove this statement by induction on k. First, it holds for $k = 1$, by the first statement. Next, we suppose it holds for any k where $1 \leq k \leq K-1$ and prove it holds for $k = K$. For level $K+1$ player 1, the expected payoff for choosing s is:

$$\begin{aligned} V_1^{K+1}(s) &= \tilde{p}_0^{K+1}\mathbb{E}\left[u_1(s, a_2^0)\right] + \sum_{\kappa=1}^{K} \tilde{p}_\kappa^{K+1} u_1(s, a_2^\kappa) \\ &= \left(\frac{\sum_{\kappa=0}^{K-1} p_\kappa}{\sum_{\kappa=0}^{K} p_\kappa}\right) V_1^K(s) + \tilde{p}_K^{K+1} u_1(s, a_2^K). \end{aligned}$$

Suppose, by way of contradiction, that $a_1^{K+1} > a_1^K$. Then $V_1^K(a_1^{K+1}) < V_1^K(a_1^K)$. From the induction hypothesis, $a_2^K \leq a_2^{K-1}$, and from the second statement, $a_1^K \geq a_2^{K-1}$ and hence $a_1^{K+1} > a_1^K \geq a_2^{K-1} \geq a_1^K$.This implies $u_1(a_1^{K+1}, a_2^K) \leq u_1(a_1^K, a_2^K)$, so $V_1^{K+1}(a_1^{K+1}) < V_1^{K+1}(a_1^K)$, which contradicts that a_1^{K+1} is the optimal strategy for level $K+1$ player 1. Hence $a_1^{K+1} \leq a_1^K$, so the result is proved for $i = 1$. A similar argument proves the result for $i = 2$. ■

Proof of Proposition 8

With slight abuse of notation, denote a level-k player's prior belief that the opponent is level-κ by $\mu_\kappa^k \equiv \frac{p_\kappa}{\sum_{j=0}^{k-1} p_j}$, $\kappa = 1, ..., k-1$.

Only if: Suppose $p_0 \geq \frac{S+1}{(S+1)+\left(\frac{3c-1}{1-c}\right)}$, then we want to show that $a_1^k = S+1$ for all $k \geq 1$. We can prove this statement by induction on k. By Lemma 2, we know $a_1^1 = S+1$. Now, suppose this statement holds for all $1 \leq k \leq K$ for some $K \in \mathbb{N}$, then we want to show this holds for level $K+1$ player 1. First, by Lemma 2, we have $a_2^k = S$ for all $1 \leq k \leq K$. Level $K+1$ player 1 would choose S if and only if

$$\begin{aligned} &\mu_0^{K+1}\left[\frac{1}{S+1}\left[1 + 2Sc + \sum_{i=2}^{S+1}(2i-3)c\right]\right] + \left(1 - \mu_0^{K+1}\right)(2S-1)c \\ &< \ \mu_0^{K+1}\left[\frac{1}{S+1}\left[2(1+(2S-2)c) + \sum_{i=2}^{S}(2i-3)c\right]\right] + \left(1-\mu_0^{K+1}\right)(1+(2S-2)c) \\ &\iff \mu_0^{K+1}\left[\frac{1}{S+1}(1-3c)\right] + \left(1-\mu_0^{K+1}\right)(1-c) > 0 \\ &\iff \mu_0^{K+1} < \frac{S+1}{(S+1)+\left(\frac{3c-1}{1-c}\right)}. \end{aligned}$$

However, we know $\mu_0^K > p_0$ and we have assumed $p_0 \geq \frac{S+1}{(S+1)+\left(\frac{-1+3c}{1-c}\right)}$, so $\mu_0^{K+1} > p_0 \geq \frac{S+1}{(S+1)+\left(\frac{3c-1}{1-c}\right)}$, implying that $a_1^{K+1} = S+1$.

If: Suppose $p_0 < \frac{S+1}{(S+1)+\left(\frac{3c-1}{1-c}\right)}$, then there exists $N^* < \infty$ such that $\mu_0^{N^*} < \frac{S+1}{(S+1)+\left(\frac{3c-1}{1-c}\right)}$. Therefore, by a previous calculation we have that

$$\tilde{K}_{2S-1}^* = \arg\min_{N^*}\left\{\mu_0^{N^*} < \frac{S+1}{(S+1)+\left(\frac{3c-1}{1-c}\right)}\right\} < \infty,$$

which is the lowest level of player 1 who would take at no later than stage $2S-1$. ■

Proof of Proposition 9

First, an immediate implication of Lemma 2 is that for all level $k \geq 1$, the optimal choice for level-$(k+1)$ is either the same as level-k or to take at one stage earlier. Given this observation, the logic of the proof is similar to Proposition 7.

Only if: For any $1 \leq j \leq 2S-2$, suppose $p_0\left(\frac{S}{S+1} - \frac{2\lfloor \frac{j}{2} \rfloor c}{(S+1)(1+c)}\right) + \sum_{\kappa=1}^{\tilde{K}^*_{j+1}-1} p_\kappa \geq \frac{1-c}{1+c}$, then we want to show $\tilde{K}^*_j = \infty$. Without loss of generality, we consider an odd j. If $\tilde{K}^*_{j+1} = \infty$, then the statement holds immediately. Otherwise, we can prove $a_1^k > \frac{j+1}{2}$ for all $k \geq 1$ by induction. By construction, we know $a_2^m > \frac{j+1}{2}$ for all $1 \leq m \leq \tilde{K}^*_{j+1} - 1$ and $a_1^k > \frac{j+1}{2}$ for all $1 \leq k \leq \tilde{K}^*_{j+1}$ by Lemma 2. Suppose there is some $K \geq \tilde{K}^*_{j+1} + 1$ such that $a_1^k > \frac{j+1}{2}$ for all $1 \leq k \leq K$. We want to show this holds for level $K+1$ player 1. Level $K+1$ player 1 would choose $\frac{j+1}{2} + 1$ if and only if

$$p_0\left[\frac{1}{S+1}(1-(2S-j+2)c)\right] + \left(\sum_{\kappa=1}^{\tilde{K}^*_{j+1}-1} p_\kappa\right)(-2c) + \left(\sum_{\kappa=\tilde{K}^*_{j+1}}^{K} p_\kappa\right)(1-c) \leq 0.$$

Moreover, we can observe that this condition is implied by:

$$p_0\left[\frac{1}{S+1}(1-(2S-j+2)c)\right] + \left(\sum_{\kappa=1}^{\tilde{K}^*_{j+1}-1} p_\kappa\right)(-2c) + \left(1 - p_0 - \sum_{\kappa=1}^{\tilde{K}^*_{j+1}-1} p_\kappa\right)(1-c) \leq 0$$

$$\iff p_0\left[\frac{S}{S+1} - \frac{(j-1)c}{(S+1)(1+c)}\right] + \sum_{\kappa=1}^{\tilde{K}^*_{j+1}-1} p_\kappa \geq \frac{1-c}{1+c}.$$

By our assumption, we can conclude that the optimal choice for level $(K+1)$ player 1 is $\frac{j+1}{2}+1$,[35] which completes the only if part of the proof.

If: For any $1 \leq j \leq 2S-2$, suppose

$$p_0\left(\frac{S}{S+1} - \frac{2\lfloor \frac{j}{2} \rfloor c}{(S+1)(1+c)}\right) + \sum_{\kappa=1}^{\tilde{K}^*_{j+1}-1} p_\kappa < \frac{1-c}{1+c},$$

then there exists N^* where $\tilde{K}^*_{j+1} + 1 \leq N^* < \infty$ such that

$$\mu_0^{N^*}\left(\frac{S}{S+1} - \frac{2\lfloor \frac{j}{2} \rfloor c}{(S+1)(1+c)}\right) + \frac{\sum_{\kappa=1}^{\tilde{K}^*_{j+1}-1} p_\kappa}{\sum_{\kappa=0}^{N^*-1} p_\kappa} < \frac{1-c}{1+c}.$$

[35] If j is even, then by the same argument, we can obtain level $(K+1)$ player 2 would choose $\frac{j}{2}+1$ as

$$p_0\left(\frac{S}{S+1} - \frac{jc}{(S+1)(1+c)}\right) + \sum_{l=1}^{\tilde{K}^*_{j+1}-1} p_l \geq \frac{1-c}{1+c}.$$

Therefore, by previous calculation and the existence of such $N^* < \infty$, we can obtain that

$$\tilde{K}_j^* = \arg\min_{N^*} \left\{ \mu_0^{N^*} \left(\frac{S}{S+1} - \frac{2\lfloor \frac{j}{2} \rfloor c}{(S+1)(1+c)} \right) + \frac{\sum_{\kappa=1}^{\tilde{K}_{j+1}^*-1} p_\kappa}{\sum_{\kappa=0}^{N^*-1} p_\kappa} < \frac{1-c}{1+c} \right\} < \infty,$$

which is the lowest level of player who would take at no later than stage j. ■

Proof of Theorem 1

Step 1: By Lemma 1 and Lemma 2, we can obtain that $1 = K_{2S}^* \leq \tilde{K}_{2S}^* = 1$, suggesting that the inequality holds at stage $2S$.

Step 2: By Proposition 6 and 8, we know K_{2S-1}^* and $\tilde{K}_{2S-1}^*$ are the lowest levels such that

$$\frac{p_0}{\sum_{\kappa=0}^{K_{2S-1}^*-1} p_\kappa} < \frac{2^S}{2^S + \left(\frac{-1+3c}{1-c}\right)}, \text{ and}$$
$$\frac{p_0}{\sum_{\kappa=0}^{\tilde{K}_{2S-1}^*-1} p_\kappa} < \frac{S+1}{(S+1) + \left(\frac{-1+3c}{1-c}\right)}, \text{ respectively.}$$

We can observe that $\frac{S+1}{(S+1)+\left(\frac{-1+3c}{1-c}\right)} < \frac{2^S}{2^S+\left(\frac{-1+3c}{1-c}\right)}$, suggesting the inequality for the dynamic model is less stringent. Hence, we can obtain that $K_{2S-1}^* \leq \tilde{K}_{2S-1}^*$.

Step 3: We can finish the proof by induction on the stages. At stage $2S-2$, as we rearrange the condition from Proposition 7, we can obtain K_{2S-2}^* is the lowest level such that

$$\sum_{\kappa=1}^{K_{2S-2}^*-1} p_\kappa > p_0 \left(\frac{1}{2}\right)^S \left(\frac{-1+3c}{1-c}\right) + \left(\sum_{\kappa=1}^{K_{2S-1}^*-1} p_\kappa\right) \left(\frac{1+c}{1-c}\right). \tag{5}$$

Similarly, as we rearrange the necessary and sufficient condition from Proposition 9, we can find that $\tilde{K}_{2S-2}^*$ is the lowest level such that

$$\sum_{\kappa=1}^{\tilde{K}_{2S-2}^*-1} p_\kappa > p_0 \left(\frac{1}{S+1}\right) \left(\frac{-1+3c}{1-c}\right) + \left(\sum_{\kappa=1}^{\tilde{K}_{2S-1}^*-1} p_\kappa\right) \left(\frac{1+c}{1-c}\right). \tag{6}$$

It suffices to prove $K_{2S-2}^* \leq \tilde{K}_{2S-2}^*$ by showing the right-hand side of Condition (5) is smaller than the right-hand side of (6). This holds because $\left(\frac{1}{2}\right)^S < \frac{1}{S+1}$ for all $S \geq 2$ and $K_{2S-1}^* \leq \tilde{K}_{2S-1}^*$ as we have shown in *step 2.*

Step 4: Consider any j where $3 \leq j \leq 2S-1$ and suppose $K_{2S-i}^* \leq \tilde{K}_{2S-i}^*$ for all $0 \leq i \leq j-1$. We want to show $K_{2S-j}^* \leq \tilde{K}_{2S-j}^*$. Without loss of generality, we consider an odd j. That is, player 1 owns stage $2S-j$. By Proposition 7, we know K_{2S-j}^* is the lowest level such that

$$\sum_{\kappa=1}^{K_{2S-j}^*-1} p_\kappa > p_0 \left(\frac{1}{2}\right)^{S-\frac{j+1}{2}+1} \left(\frac{-1+3c}{1-c}\right) + \left(\sum_{\kappa=1}^{K_{2S-j+1}^*-1} p_\kappa\right) \left(\frac{1+c}{1-c}\right). \tag{7}$$

Similarly, as we rearrange the necessary and sufficient condition from Proposition 9, we can obtain that $\tilde{K}^*_{2S-j}$ is the lowest level such that

$$\sum_{\kappa=1}^{\tilde{K}^*_{2S-j}-1} p_\kappa > p_0 \left(\frac{1}{S+1}\right)\left[\frac{-1+(j+2)c}{1-c}\right] + \left(\sum_{\kappa=1}^{\tilde{K}^*_{2S-j+1}-1} p_\kappa\right)\left(\frac{1+c}{1-c}\right). \tag{8}$$

Similar to the previous step, we can finish the proof by showing the right-hand side of Condition (7) is smaller than the right-hand side of (8). The induction hypothesis implies the second term of (8) is larger than the second term of (7). Hence, the only thing left to show is

$$\left(\frac{1}{2}\right)^{S-\frac{j+1}{2}+1}\left(\frac{-1+3c}{1-c}\right) < \left(\frac{1}{S+1}\right)\left[\frac{-1+(j+2)c}{1-c}\right].$$

Or equivalently,

$$(S+1)(-1+3c) < 2^{S-\frac{j+1}{2}+1}(-1+(j+2)c). \tag{9}$$

Since $3 \leq j \leq 2S-1$, there is nothing to show if $S < \frac{j+1}{2}$. When $S \geq \frac{j+1}{2}$, we know (9) would hold in the following three different cases.

- *Case 1:* If $S+1 = 2^{S-\frac{j+1}{2}+1}$, then (9) becomes $-1+3c < -1+(j+2)c \iff j > 1$.
- *Case 2:* If $S+1 < 2^{S-\frac{j+1}{2}+1}$, then (9) is equivalent to

$$2^{S-\frac{j+1}{2}+1}-(S+1) < \left[(j+2)2^{S-\frac{j+1}{2}+1} - 3(S+1)\right]c \iff 1 < \left[3 + \underbrace{\frac{(j-1)2^{S-\frac{j+1}{2}+1}}{2^{S-\frac{j+1}{2}+1}-(S+1)}}_{>0 \text{ as } j\geq 3}\right]c,$$

  which holds under our assumption $c > \frac{1}{3}$.
- *Case 3:* If $S+1 > 2^{S-\frac{j+1}{2}+1}$, then (9) can be rearranged as

$$(S+1)-2^{S-\frac{j+1}{2}+1} > \left[3(S+1)-(j+2)2^{S-\frac{j+1}{2}+1}\right]c \iff 1 > \left[3 - \frac{j-1}{\left(\frac{S+1}{2^{S-\frac{j+1}{2}+1}}\right)-1}\right]c.$$

  The right-hand side of the inequality is negative since

$$3 - \frac{j-1}{\left(\frac{S+1}{2^{S-\frac{j+1}{2}+1}}\right)-1} \leq 3 - \frac{j-1}{\left(\frac{\frac{j+1}{2}+1}{2}\right)-1} = -1.$$

This completes the proof. ■

Proof of Proposition 10

By Proposition 6, we know

$$K^*_{2S-1} < \infty \iff p_0 < \frac{2^S}{2^S\left(\frac{-1+3c}{1-c}\right)}.$$

As the prior distribution follows Poisson(λ), the condition becomes

$$\begin{aligned} K^*_{2S-1}(\lambda) < \infty &\iff e^{-\lambda} < \frac{2^S}{2^S\left(\frac{-1+3c}{1-c}\right)} \\ &\iff \lambda > ln\left[1 + \left(\frac{1}{2}\right)^S\left(\frac{-1+3c}{1-c}\right)\right]. \end{aligned}$$

Similarly, by Proposition 8, we know

$$\tilde{K}^*_{2S-1} < \infty \iff p_0 < \frac{S+1}{(S+1)\left(\frac{-1+3c}{1-c}\right)},$$

which can be rearranged to the following expression when the prior distribution follows Poisson(λ):

$$\begin{aligned} \tilde{K}^*_{2S-1}(\lambda) < \infty &\iff e^{-\lambda} < \frac{S+1}{(S+1)\left(\frac{-1+3c}{1-c}\right)} \\ &\iff \lambda > ln\left[1 + \left(\frac{1}{S+1}\right)\left(\frac{-1+3c}{1-c}\right)\right]. \end{aligned}$$

This completes the proof. ■

Proof of Proposition 11

Here we show the existence of λ^*. The existence of $\tilde{\lambda}^*$ can be proven by the same argument.

Step 1: By Proposition 1, we know for all $\lambda > 0$, $K^*_{2S}(\lambda) = 1$.

Step 2: By Proposition 10, we know

$$K^*_{2S-1}(\lambda) < \infty \iff \lambda > ln\left[1 + \left(\frac{1}{2}\right)^S\left(\frac{-1+3c}{1-c}\right)\right] \equiv \lambda^*_{2S-1}.$$

Since $\lambda^*_{2S-1} < \infty$, we know $K^*_{2S-1}(\lambda) < \infty \iff \lambda > \lambda^*_{2S-1}$.

Step 3: By Proposition 7, we know

$$\begin{aligned} K^*_{2S-2}(\lambda) < \infty &\iff \frac{e^{-\lambda}\left(\frac{1}{2}\right)^S + e^{-\lambda}\sum_{l=1}^{K^*_{2S-1}(\lambda)-1}\frac{\lambda^l}{l!}}{e^{-\lambda}\left(\frac{1}{2}\right)^{S-1} + (1-e^{-\lambda})} < \frac{1-c}{1+c} \\ &\iff 1 - e^{-\lambda}\left[1 + \left(\frac{1}{2}\right)^S\left(\frac{-1+3c}{1-c}\right)\right] - e^{-\lambda}\sum_{\kappa=1}^{K^*_{2S-1}(\lambda)-1}\frac{\lambda^\kappa}{\kappa!}\left(\frac{1+c}{1-c}\right) > 0. \end{aligned}$$

Notice that by **step 2**, we know there exists some $M < \infty$, such that for all $\lambda > \lambda^*_{2S-1}$, $K^*_{2S-1}(\lambda) < M$. Moreover, by Proposition 4, we know $K^*_{2S-1}(\lambda) \geq 2$. Hence,

$$0 = \lim_{\lambda\to\infty} \frac{\lambda}{e^\lambda} \leq \lim_{\lambda\to\infty} e^{-\lambda} \sum_{\kappa=1}^{K^*_{2S-1}(\lambda)-1} \frac{\lambda^\kappa}{\kappa!} \leq \lim_{\lambda\to\infty} e^{-\lambda} \sum_{\kappa=1}^{M-1} \frac{\lambda^\kappa}{\kappa!} = 0.$$

Coupled with the fact that $\lim_{\lambda\to\infty} e^{-\lambda} = 0$, we can conclude that there exists λ^*_{2S-2} such that $\lambda^*_{2S-1} < \lambda^*_{2S-2} < \infty$ and $K^*_{2S-2}(\lambda) < \infty \iff \lambda > \lambda^*_{2S-2}$.

Step 4: Now we can prove this statement by induction on each stage. Consider any j where $3 \leq j \leq 2S-1$ and suppose there exists $\lambda^*_{2S-j+1} < \infty$ such that $K^*_{2S-j+1}(\lambda) < \infty$ for all $\lambda > \lambda^*_{2S-j+1}$. By Proposition 7, we know

$$K^*_{2S-j}(\lambda) < \infty \iff \frac{e^{-\lambda}\left(\frac{1}{2}\right)^{\lfloor\frac{2S-j}{2}\rfloor+1} + e^{-\lambda}\sum_{\kappa=1}^{K^*_{2S-j+1}(\lambda)-1}\frac{\lambda^\kappa}{\kappa!}}{e^{-\lambda}\left(\frac{1}{2}\right)^{\lfloor\frac{2S-j}{2}\rfloor} + (1-e^{-\lambda})} < \frac{1-c}{1+c}$$

$$\iff 1 - e^{-\lambda}\left[1 + \left(\frac{1}{2}\right)^{\lfloor\frac{2S-j}{2}\rfloor+1}\left(\frac{-1+3c}{1-c}\right)\right] - e^{-\lambda}\sum_{\kappa=1}^{K^*_{2S-j+1}(\lambda)-1}\frac{\lambda^\kappa}{\kappa!}\left(\frac{1+c}{1-c}\right) > 0.$$

By the induction hypothesis, we know there exists some $L < \infty$ such that for all $\lambda > \lambda^*_{2S-j+1}$, $K^*_{2S-j+1}(\lambda) < L$. Proposition 4 gives us $K^*_{2S-j+1}(\lambda) \geq j$, and hence,

$$0 = \lim_{\lambda\to\infty} e^{-\lambda}\left(\sum_{\kappa=1}^{j-1}\frac{\lambda^\kappa}{\kappa!}\right) \leq \lim_{\lambda\to\infty} e^{-\lambda}\sum_{\kappa=1}^{K^*_{2S-j+1}(\lambda)-1}\frac{\lambda^\kappa}{\kappa!} \leq \lim_{\lambda\to\infty} e^{-\lambda}\sum_{\kappa=1}^{L-1}\frac{\lambda^\kappa}{\kappa!} = 0.$$

Combined with the fact $\lim_{\lambda\to\infty} e^{-\lambda} = 0$, we have proved that there exists λ^*_{2S-j} such that $\lambda^*_{2S-j+1} < \lambda^*_{2S-j} < \infty$ and $K^*_{2S-j}(\lambda) < \infty \iff \lambda > \lambda^*_{2S-j}$. Thus, λ^*_1 is the desired λ^*. ■

Proof of Proposition 12

The proofs of Propositions 6 through 9 provide a recipe to derive the necessary and sufficient conditions for complete unraveling at each stage. That is, when the prior distribution follows Poisson distribution, we can compute the minimum λ for both models such that the predictions coincide with the standard level-k model. In the dynamic model, we can obtain from Proposition 6 and 7 that for any stage $2S-j$ where $1 \leq j \leq 2S-1$,

$$K^*_{2S-1}(\lambda) = 2 \iff \frac{e^{-\lambda}}{e^{-\lambda} + \lambda e^{-\lambda}} < \frac{2^S}{2^S + \left(\frac{-1+3c}{1-c}\right)} \iff \lambda > \left(\frac{1}{2}\right)^S\left(\frac{-1+3c}{1-c}\right) \equiv \lambda^{**}_{2S-1}, \text{ and}$$

$$K^*_{2S-j}(\lambda) = j+1 \iff \sum_{\kappa=1}^{j}\frac{\lambda^\kappa e^{-\lambda}}{\kappa!} > e^{-\lambda}\left(\frac{1}{2}\right)^{S-\lfloor\frac{j+1}{2}\rfloor+1}\left(\frac{-1+3c}{1-c}\right) + \left(\sum_{\kappa=1}^{j-1}\frac{\lambda^\kappa e^{-\lambda}}{\kappa!}\right)\left(\frac{1+c}{1-c}\right)$$

$$\iff \frac{1}{j!}\lambda^j - \left(\frac{2c}{1-c}\right)\left(\sum_{\kappa=1}^{j-1}\frac{\lambda^\kappa}{\kappa!}\right) > \left(\frac{1}{2}\right)^{S-\lfloor\frac{j+1}{2}\rfloor+1}\left(\frac{-1+3c}{1-c}\right) \equiv M^{**}_{2S-j}.$$

Similarly, we know from Proposition 8 and Proposition 9 that for any stage $2S-j$ where $1 \leq j \leq 2S-1$,

$$\tilde{K}^*_{2S-1}(\lambda) = 2 \iff \frac{e^{-\lambda}}{e^{-\lambda}+\lambda e^{-\lambda}} < \frac{S+1}{(S+1)+\left(\frac{-1+3c}{1-c}\right)} \iff \lambda > \frac{-1+3c}{(S+1)(1-c)} \equiv \tilde{\lambda}^{**}_{2S-1}, \text{ and}$$

$$\begin{aligned}\tilde{K}^*_{2S-j}(\lambda) = j+1 &\iff \sum_{\kappa=1}^{j} \frac{\lambda^\kappa e^{-\lambda}}{\kappa!} > e^{-\lambda}\left(\frac{-1+\left(2\lfloor\frac{j+1}{2}\rfloor+1\right)c}{(S+1)(1-c)}\right) + \left(\sum_{\kappa=1}^{j-1}\frac{\lambda^\kappa e^{-\lambda}}{\kappa!}\right)\left(\frac{1+c}{1-c}\right) \\ &\iff \frac{1}{j!}\lambda^j - \left(\frac{2c}{1-c}\right)\left(\sum_{\kappa=1}^{j-1}\frac{\lambda^\kappa}{\kappa!}\right) > \left(\frac{-1+\left(2\lfloor\frac{j+1}{2}\rfloor+1\right)c}{(S+1)(1-c)}\right) \equiv \tilde{M}^{**}_{2S-j}.\end{aligned}$$

First, we can find $\lambda^{**}_{2S-1} < \tilde{\lambda}^{**}_{2S-1}$ since $(\frac{1}{2})^S < \frac{1}{S+1}$. Moreover, because the LHS of each inequality is a degree of j polynomial of λ, it has only one positive root by Descartes' rule of signs. Hence, it suffices to prove $M^{**}_{2S-j} < \tilde{M}^{**}_{2S-j}$, or equivalently, $\frac{\tilde{M}^{**}_{2S-j}}{M^{**}_{2S-j}} > 1$, for all $2 \leq j \leq 2S-1$. Due to the property of floor functions, we can focus on odd j without loss of generality. Also, we can observe that this ratio is decreasing in j since for any odd j where $3 \leq j \leq 2S-3$,

$$\begin{aligned}\frac{\tilde{M}^{**}_{2S-(j+2)}}{M^{**}_{2S-(j+2)}} &= \left(\frac{2^{S-\frac{j+1}{2}}}{S+1}\right)\left(\frac{-1+(j+4)c}{-1+3c}\right) \\ &= \frac{1}{2}\left(\frac{\tilde{M}^{**}_{2S-j}}{M^{**}_{2S-j}}\right) + \frac{1}{2}\left[\left(\frac{2^{S-\frac{j-1}{2}}}{S+1}\right)\left(\frac{2c}{-1+3c}\right)\right] < \frac{\tilde{M}^{**}_{2S-j}}{M^{**}_{2S-j}} \\ \iff 2c < -1+(j+2)c &\iff 1 < jc,\end{aligned}$$

which holds because of the assumption $c > \frac{1}{3}$. The monotonicity implies that the ratio is minimized when $j = 2S-1$, and we can obtain the conclusion by showing $\frac{\tilde{M}^{**}_1}{M^{**}_1} > 1$:

$$\frac{\tilde{M}^{**}_1}{M^{**}_1} = \left(\frac{2}{S+1}\right)\left(\frac{-1+(2S+1)c}{-1+3c}\right) > 1 \iff (S-1)(1+c) > 0,$$

as desired. ■

Proof of Proposition 13

Here we only provide the proof for the dynamic model. A very similar argument can be applied to the static model. First of all, by Proposition 1, we know $K^*_{2S}(\lambda) = 1$ for all $\lambda > 0$. Therefore, it is weakly decreasing in λ.

To show the monotonicity of $K^*_{2S-1}(\lambda)$, we need to introduce the function $F_k(\lambda) : \mathbb{R}_{++} \to \mathbb{R}$ where $k \in \mathbb{N}$ and $F_k(\lambda) = \sum_{\kappa=1}^{k} \frac{\lambda^\kappa}{\kappa!}$. Notice that $F_{k+1}(\lambda) > F_k(\lambda)$ for all $\lambda > 0$, and $F_k(\lambda)$ is strictly increasing since $F'_k(\lambda) = \sum_{\kappa=0}^{k-1} \frac{\lambda^\kappa}{\kappa!} > 0$ for all $\lambda > 0$. We prove the monotonicity toward contradiction. By Proposition 6, we know $K^*_{2S-1}(\lambda)$ is the lowest level such that

$$F_{K^*_{2S-1}(\lambda)-1}(\lambda) > \left(\frac{1}{2}\right)^S \left(\frac{-1+3c}{1-c}\right).$$

If $K^*_{2S-1}(\lambda)$ is not weakly decreasing in λ, then there exists $\lambda' > \lambda$ such that $K^*_{2S-1}(\lambda') > K^*_{2S-1}(\lambda)$. By the construction and the monotonicity of $F_k(\lambda)$, we can find that

$$F_{K^*_{2S-1}(\lambda)-1}(\lambda') > F_{K^*_{2S-1}(\lambda)-1}(\lambda) > \left(\frac{1}{2}\right)^S \left(\frac{-1+3c}{1-c}\right).$$

Also, $K^*_{2S-1}(\lambda')$ is the lowest level such that

$$F_{K^*_{2S-1}(\lambda')-1}(\lambda') > \left(\frac{1}{2}\right)^S \left(\frac{-1+3c}{1-c}\right),$$

implying that

$$F_{K^*_{2S-1}(\lambda)-1}(\lambda') > F_{K^*_{2S-1}(\lambda')-1}(\lambda') > \left(\frac{1}{2}\right)^S \left(\frac{-1+3c}{1-c}\right).$$

This contradicts the assumption that $K^*_{2S-1}(\lambda') > K^*_{2S-1}(\lambda)$. ■

Appendix C: Table for Figure 8

Table 3: The cumulative probabilities and the conditional take probabilities of each stage

Game		1	2	3	4	5	6
CG 1	Direct Response	0.156 (0.156)	0.444 (0.342)	0.711 (0.480)	0.933 (0.769)	1.000 (1.000)	1.000 —
	Strategy Method	0.040 (0.040)	0.219 (0.187)	0.486 (0.342)	0.655 (0.328)	0.878 (0.646)	0.991 (0.927)
CG 2	Direct Response	0.600 (0.600)	0.889 (0.722)	1.000 (1.000)	1.000 —	1.000 —	1.000 —
	Strategy Method	0.224 (0.224)	0.721 (0.640)	0.934 (0.763)	0.983 (0.741)	0.994 (0.643)	1.000 (1.000)
CG 3	Direct Response	0.146 (0.146)	0.561 (0.486)	0.634 (0.167)	0.854 (0.600)	0.927 (0.500)	0.976 (0.667)
	Strategy Method	0.158 (0.158)	0.641 (0.573)	0.731 (0.250)	0.882 (0.562)	0.929 (0.396)	0.995 (0.929)
CG 4	Direct Response	0.489 (0.489)	0.822 (0.652)	0.956 (0.750)	1.000 (1.000)	1.000 —	1.000 —
	Strategy Method	0.395 (0.395)	0.677 (0.467)	0.895 (0.674)	0.953 (0.550)	0.978 (0.533)	1.000 (1.000)

Note: The conditional take probabilities are reported in parentheses.

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