Parameterized Barrier Functions to Guarantee Safety under Uncertainty

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Abstract—Deploying safety-critical controllers in practice necessitates the ability to modulate uncertainties in control systems. In this context, robust control barrier functions—in a variety of forms—have been used to obtain safety guarantees for uncertain systems. Yet the differing types of uncertainty experienced in practice have resulted in a fractured landscape of robustification—with a variety of instantiations depending on the structure of the uncertainty. This paper proposes a framework for generalizing these variations into a single form: parameterized barrier functions (PBFs), which yield safety guarantees for a wide spectrum of uncertainty types. This leads to controllers that enforce robust safety guarantees while their conservativeness scales by the parameterization. To illustrate the generality of this approach, we show that input-to-state safety (ISSf) is a special case of the PBF framework, whereby improved safety guarantees can be given relative to ISSf.

I. INTRODUCTION

Control barrier functions (CBFs) [1] have become a popular tool for synthesizing safety-critical controllers due to their generality and relative ease of synthesis and implementation. Safety is encoded by a single scalar inequality constraint, by which CBFs provide an easy-to-compute condition that implies the safety of the system when satisfied. The efficacy of this approach has been demonstrated in a variety of applications such as multi-agent systems [2], robotic manipulators [3], autonomous vessels [4] and autonomous trucks [5]. One of the main challenges in obtaining formal safety guarantees with CBFs in practice is uncertainty: both of the internal model used to synthesize the CBF controller, and the external environment with which the system interacts. Since CBFs use models to calculate safe actions, a mismatch between a system and its model can lead to degradations in safety [6].

Robustness against uncertainties is typically achieved by introducing a robustifying term in the CBF condition. In one of the first works on robust CBFs [7], a robustifying term was added based upon a bound on the uncertainties with the result being robust safety. Later, different observer and identification techniques have been proposed to alleviate the conservativeness of robust controllers by estimating the uncertainty, or considering specific classes of uncertainties [8]–[13]. Data-driven methods account for uncertainties in a similar fashion where a sufficient condition for the safety is found using properties of uncertainties [14]–[16]. Learning can also be utilized to estimate a robustifying term in an episodic fashion [17]—this has been deployed successfully on robotic systems [18]. In the case of stochastic estimation techniques, probabilistic safety guarantees are obtained using chance constraints with the standard deviation of the process used as an upper confidence bound [19], [20].

To quantify the effect of uncertainties on safety, it is important to characterize how adding a robustness term impacts the ability to satisfy a safety constraint. On one hand, various studies focused on ensuring safety w.r.t. a more conservative constraint for robustness [9]–[11]. Similar concepts were investigated to improve the feasibility of a stability problem with control Lyapunov functions before in [21], and of a safety problem with CBFs more recently in [22]. On the other hand, input-to-state safety (ISSf) [23], inspired from input-to-state stability [24], enforces safety w.r.t. a relaxed safety constraint that is parameterized by the robustifying term. Arbitrarily small relaxation in the safety constraint can be achieved, and the robustness-safety trade-off can be improved with less conservative “tunable” ISSf conditions [25]. ISSf has proven to be useful when implementing CBFs in practice; for example, in safety aware control of quadrupeds [26], and control of full-scale trucks [5]. However, the fact that ISSf relaxes the safety constraint prevents the analysis of conservative controller performance that may occur when robustness terms overcome the uncertainty.

The first goal of this paper is to generalize the concept of robust CBFs across various forms of robustification methods in the literature. Second, we formulate parameterized barrier functions (PBFs), where safety guarantees are parameterized relative to a given safety constraint with a generalized robust CBF. The parameterization allows for the relaxation of the strict safety condition expressed in the robust CBF formulation. This gives us flexibility to establish safety guarantees for other levels sets of the CBF in the case that the nominal robust CBF conditions are not met. Third, we connect the ISSf framework to PBFs and show that it is possible to obtain improved safety guarantees for ISSf-CBF-based controllers. An inverted pendulum example is used throughout to illustrate the key concepts.

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II. BACKGROUND

Consider a nonlinear control system of the form:
\[ \dot{x} = f(t, x) + g(t, x)u, \]
with state \( x \in \mathbb{R}^n \) and input \( u \in \mathbb{R}^m \). The functions \( f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) and \( g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n \times m} \) are locally Lipschitz continuous in \( x \) and piece-wise continuous in \( t \). A feedback controller \( k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m \), \( u = k(t, x) \), that is locally Lipschitz continuous in \( x \) and piece-wise continuous in \( t \) implies there exists a time interval \( I(t_0, x_0) \subseteq \mathbb{R} \) for each initial condition \( x(t_0) = x_0 \) such that the closed loop system has a unique solution \( x(t) \) for all \( t \in I(t_0, x_0) \) [27]. For convenience we take \( t_0 = 0 \) and assume that the solution exists for all time, that is, \( I(0, x_0) = [0, \infty) \) for all \( x_0 \in \mathbb{R}^n \).

Safety is formally defined as the forward invariance of a set in the state space. We define the 0-superlevel set of a continuously differentiable function \( h : \mathbb{R}^n \to \mathcal{H}, \mathcal{H} \subseteq \mathbb{R}^m \):
\[ \mathcal{S} = \{ x \in \mathbb{R}^n \mid h(x) \geq 0 \}, \]
(2)
such that \( \mathcal{S} \) is nonempty and has no isolated points, and we say that the system (1) with a controller \( u = k(t, x) \) is safe w.r.t. the set \( \mathcal{S} \) if \( x_0 \in \mathcal{S} \implies x(t) \in \mathcal{S} \) for all \( t \geq 0 \) and \( x_0 \in \mathcal{S} \). Control barrier functions [1] give us tools to synthesize controllers with safety guarantees.

Definition 1 ([11]). A continuously differentiable function \( h \) is a control barrier function (CBF) for (1) on \( \mathcal{S} \) if it is a regular value\(^1\) and there exists a function \( \alpha \in \mathcal{K}^\infty \)\(^2\) such that the following holds for all \( t \geq 0 \) and \( x \in \mathcal{S} \):
\[ \sup_{u \in \mathbb{R}^m} [\nabla h(x)f(t, x) + \nabla h(x)g(t, x)u] > -\alpha(h(x)). \]
(3)
The existence of a CBF implies that the set of controllers:
\[ \mathcal{K}_{\text{CBF}}(t, x) = \{ u \in \mathbb{R}^m \mid \nabla h(x)f(t, x) + \nabla h(x)g(t, x)u \geq -\alpha(h(x)) \} \]
(4)
is not empty, and the main result in [11] states that controllers taking values in \( \mathcal{K}_{\text{CBF}} \) ensure safety:

Theorem 1 ([11]). Let \( h \) be a CBF for (1) on \( \mathcal{S} \). Then, any controller \( u = k(t, x) \in \mathcal{K}_{\text{CBF}}(t, x) \) renders (1) safe w.r.t. \( \mathcal{S} \).

III. ROBUST CONTROL BARRIER FUNCTIONS

Safety guarantees established by CBFs may deteriorate in the presence of an uncertainty in the model. Consider:
\[ \dot{x} = f(t, x) + g(t, x)u + \tilde{f}(t, x) + \tilde{g}(t, x)u, \]
(5)
where the unknown functions \( \tilde{f} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n \) and \( \tilde{g} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^{n \times m} \) are assumed to be locally Lipschitz in \( x \) and piece-wise continuous in \( t \). Uncertainties \( \tilde{f} \) and \( \tilde{g} \) are often called as additive and multiplicative uncertainties, respectively, emphasizing their relationship with the input \( u \)

1If \( h(x) = q \implies \nabla h(x) \neq 0 \), then \( q \) is a regular value of \( h \).
2Function \( \alpha : \mathbb{R} \to \mathbb{R} \) belongs to extended class-\( \mathcal{K}^\infty \) (\( \alpha \in \mathcal{K}^\infty \)) if it is continuous, strictly increasing, \( \alpha(0) = 0 \), and \( \lim_{r \to \pm \infty} \alpha(r) = \pm \infty \).

in the dynamics. Their effect on safety is seen in \( \dot{h} \):
\[ \dot{h}(t, x, u) = \overline{\nabla h(x)f(t, x) + \nabla h(x)g(t, x)u} + \nabla h(x)\tilde{f}(t, x) + \nabla h(x)\tilde{g}(t, x)u, \]
(6)
\[ L_1 h(t, x) \]
\[ L_2 h(t, x) \]
where \( \dot{h} \) denotes the known portion of \( h \) while the Lie derivatives \( L_1 h(t, x) \) and \( L_2 h(t, x) \) are unknown. A controller \( u = k(t, x) \in \mathcal{K}_{\text{RCBF}}(t, x) \) yields:
\[ \dot{h}(t, x, k(t, x)) \geq -\alpha(h(x)) + L_1 h(t, x) + L_2 h(t, x)k(t, x), \]
(7)
and no longer satisfies the condition \( \dot{h} \geq -\alpha(h) \).

In the literature this problem is often addressed by adding a compensation term to the safety constraint (3) for robustness against the uncertainty. To capture this term for a variety of approaches, we generalize the notion of robust CBF, which was first proposed in [7] using a specific compensation term for a specific type of uncertainty.

Definition 2. A continuously differentiable function \( h \) is a robust control barrier function (RCBF) for (5) on \( \mathcal{S} \) if it is a regular value of \( h \) and there exist functions \( \sigma : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) and \( \alpha \in \mathcal{K}^\infty \) such that the following holds for all \( t \geq 0 \) and \( x \in \mathcal{S} \):
\[ \sup_{u \in \mathbb{R}^m} \left[ h_n(t, x, u) - \sigma(t, x, u) \right] > -\alpha(h(x)). \]
(8)
The compensation term \( \sigma \) allows one to cancel the undesired effects of uncertainties on safety. Similar to CBFs, the existence of a RCBF yields a set of robustly safe controllers:
\[ \mathcal{K}_{\text{RCBF}}(t, x) = \{ u \in \mathbb{R}^m \mid h_n(t, x, u) - \sigma(t, x, u) \geq -\alpha(h(x)) \}, \]
(9)
and the following theorem, generalized from [7], gives a sufficient condition to obtain robust safety results:

Theorem 2. Let \( h \) be a RCBF for (5) on \( \mathcal{S} \) with \( \sigma \) satisfying:
\[ L_1 h(t, x) + L_2 h(t, x)u + \sigma(t, x, u) \geq 0, \]
(10)
for all \( t \geq 0, x \in \partial \mathcal{S} \) and \( u \in \mathbb{R}^m \). Then, any controller \( u = k(t, x) \in \mathcal{K}_{\text{RCBF}}(t, x) \) renders (5) safe w.r.t. \( \mathcal{S} \).

Remark 1. Robust safety-critical controller design is often formulated as the optimization problem:
\[ k(t, x) = \arg\min_{u \in \mathbb{R}^m} \| u - k_d(t, x) \|^2 \]
(11)
s.t. \( h_n(t, x, u) - \sigma(t, x, u) \geq -\alpha(h(x)) \),
where \( k_d : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m \) is a desired controller.

A plethora of methods has been proposed in the literature to design \( \sigma \); see a list in Table I for RCBF-based methods as well as an input-to-state safety-based method that will be described in Section V. To illustrate robust safety we now consider a certain class of uncertainty with bounded additive term and no multiplicative term:
\[ \| \tilde{f}(t, x) \| \leq p, \quad \tilde{g}(t, x) = 0, \]
(12)
for all \( t \geq 0 \), \( x \in \mathbb{R}^n \) with some bound \( p \geq 0 \). Inspired by [7], we will use a compensation term of the form:

\[
\sigma(t, x, u) = \|\nabla h(x)\| p,
\]

which, along with (12), implies that (10) is satisfied, and thus the controller (11) with (13) keeps the set \( S \) safe. Results obtained through Theorem 2 are illustrated using an inverted pendulum example with a time varying uncertainty.

**Example 1.** Consider the inverted pendulum in Fig.2(a) that consists of a massless rod of length \( l \) and a concentrated mass \( m \). The pendulum is actuated with a torque \( u \), while an unknown external force \( F(t) \) is acting horizontally on the mass. With the angle \( \theta \), angular velocity \( \dot{\theta} \), and state \( x = \begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}^T \), the equation of motion of the pendulum reads:

\[
\dot{x} = \begin{bmatrix} x_2 \\ \frac{q}{l} \sin x_1 + \frac{1}{m l^2} u + \frac{0}{m l} \end{bmatrix} \begin{bmatrix} f(t, x) \\ g(t, x) \end{bmatrix},
\]

where the parameter \( q \) is the gravitational acceleration. All the parameters used in this example are given in Table II.

The external force \( F(t) \) yields an additive uncertainty \( \tilde{f} \) such as \( \|\tilde{f}(t)\| = 0 \). We assume that there exists an upper bound \( \bar{F} \) such that \( |F(t)| \leq \bar{F} \), \( \forall t \geq 0 \), which yields \( p = \frac{\bar{F}}{m} \). A piece-wise continuous force is considered for simulations:

\[
F(t) = \begin{cases} 0 & \text{if } f(t, x) = 0 \end{cases}
\]

where \( s \) is the Heaviside function.

We seek to design a control torque \( u \) such that we keep the pendulum upright within a given safe region of angles, even with the disturbance \( F(t) \). The set \( S \) is defined using:

\[
h(x) = 1 - \frac{1}{2} x^T A x,
\]

with parameters \( q_1, q_2 > 0 \) given in Table II. The resulting set \( S \) is the black ellipse in Fig. 2(b). Note that \( \nabla h(x) = 0 \) only if \( x = 0 \), while \( h(0) = 1 \), thus \( 0 \) is a regular value of \( h \). A desired controller is selected as:

\[
k_d(x) = ml^2 \left( -g/l \sin x_1 - K_p x_1 - K_d x_2 \right)
\]

with parameters \( K_p, K_d > 0 \). We use (11) as robust safety-critical controller with \( \sigma \) in (13) and \( \alpha(r) = \alpha_c r \), \( \alpha_c > 0 \). Simulation results are depicted in Fig. 2(b) as a blue curve. The controller successfully keeps the system safe w.r.t. \( S \).

To achieve robust safety, the compensation term \( \sigma \) is typically designed based on certain properties of \( \tilde{f} \) and \( \tilde{g} \) such as the upper bound \( p \) in (12); see Table I. In practice, these properties may be hard to estimate, thus the compensation (10) required for robust safety may not be realized. For example, if \( p \) in (12) is not known precisely, one may rely on an estimation \( \hat{p} \) of \( p \) instead, with the compensation term:

\[
\sigma(t, x, u) = \|\nabla h(x)\| \hat{p}.
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\[
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\[
\sigma(t, x, u) = \|\nabla h(x)\| \hat{p}.
\]
and \( u = \frac{1}{2} \) for all \( t \geq 0 \). Theorem 3. Let \( H \) be a PBF for (5) on \( S^* \) with \( h^* \in H \) and \( \sigma \) satisfying:

\[
L \dot{h}(t,x) + L_h h(t,x) u + \sigma(t,x,u) \geq \alpha(h^*),
\]

for all \( t \geq 0 \), \( x \in \partial S^* \) and \( u \in \mathbb{R}^m \). Then, any controller \( u = k(t,x) \in K_{RCBF}(t,x) \) renders (5) safe w.r.t. \( S^* \).

**Proof.** \( H \) is continuously differentiable since \( h \) is a RCBF and \( h^* \) is a constant, and we have:

\[
\dot{H}(t,x,u) = h_n(t,x,u) + L \dot{h}(t,x) + L_h h(t,x) u.
\]

For any controller \( u = k(t,x) \in K_{RCBF}(t,x) \) this yields:

\[
\dot{H}(t,x,k(t,x)) \geq -\alpha(h^*) + \sigma(t,x,k(t,x)) + L \dot{h}(t,x) + L_h h(t,x) k(t,x).
\]

Considering \( x \in \partial S^* \), i.e., \( h(x) = h^* \), (22) implies that

\[
\dot{H}(t,x,k(t,x)) \geq 0.
\]

Since \( h^* \) is a regular value of \( h \) we have that 0 is a regular value of \( H \). Thus, the rest of the proof follows from [29].

**Remark 2.** Theorem 3 relaxes the sufficient condition (10), which is based on certain known properties of the unknown functions \( \hat{f} \) and \( \hat{g} \), and consequently establishes more accurate safety guarantees. Indeed, Theorem 2 is a special case of Theorem 3 with \( h^* = 0 \). If (22) holds with \( h^* < 0 \), condition (10) may not hold and Theorem 2 cannot establish safety. Still, Theorem 3 provides safety guarantees w.r.t. \( S^* \supset S \), hence it quantifies safety degradation. If (22) holds with \( h^* > 0 \), condition (10) also holds, and Theorem 2 establishes safety w.r.t. \( S \). However, Theorem 3 provides safety w.r.t. \( S^* \subset S \), hence it quantifies conservativeness.

More accurate safety guarantees can be established for the compensation term in (18) through Theorem 3 as follows.

**Corollary 1.** Consider (5) with (12) and (18). Assume that there exist \( \delta, \tilde{\delta} : H \to \mathbb{R}_{\geq 0} \) such that for any \( h^* \in H \):

\[
\hat{\delta}(h^*) \leq \| \nabla h(x) \| \leq \tilde{\delta}(h^*),
\]

\[
\forall x \in \partial S^*. \text{ If } H \text{ is a PBF for (5) on } S^* \text{ with } h^* \text{ defined by:}
\]

\[
\alpha(h^*) = \begin{cases} 
\delta(h^*)(\hat{p} - p) & \text{if } \hat{p} < p, \\
\delta(h^*)(\hat{p} - p) & \text{if } \hat{p} > p,
\end{cases}
\]

then \( u = k(t,x) \in K_{RCBF}(t,x) \) renders (5) safe w.r.t. \( S^* \).

**Proof.** The choice (18) of the robustifying term \( \sigma \) implies:

\[
L \dot{h}(t,x) + L_h h(t,x) u + \sigma(t,x,u) \geq \| \nabla h(x) \|(\hat{p} - p).
\]

This leads to (22) by using (26) and (27), and the rest of the proof follows from Theorem 3.

**Remark 3.** The value of \( h^* \) given by (27) quantifies safety degradation and conservativeness. If the uncertainty is under-approximated (\( \hat{p} < p \), (27) yields \( h^* < 0 \) and \( S^* \supset S \), while over-approximation (\( \hat{p} > p \)) leads to \( h^* > 0 \) and \( S^* \subset S \).

**Example 3.** Consider the setup of Example 2. Based on (16), we get \( \nabla h(x) = -Ax \), and it can be shown that (26) holds for any \( h^* \in H = (-\infty, 1) \) and for all \( x \in \partial S^* \) with \( \hat{\delta}(h^*) = \sqrt{2\lambda(1 - h^*)} \) and \( \delta(h^*) = \sqrt{2\lambda(1 - h^*)} \), where \( 0 < \lambda \leq \tilde{\lambda} \) are the eigenvalues of \( A \). Since any \( h^* < 1 \) is a regular value of \( h \), Corollary 1 establishes safety w.r.t. the set \( S^* \) with \( h^* \) given by (27).

The value of \( h^* \) is depicted in Fig. 3 with dashed line along with \( h(x(t)) \) corresponding to the simulated trajec-
In the next theorem, we link ISSf-CBFs to the PBF framework and establish the same result via PBFs.

**Theorem 4.** If $h$ is an ISSf-CBF for (5) with (12), then $H$ is a PBF for this system on $S^*$ with:

$$
\sigma(t, x, u) = \frac{\|\nabla h(x)\|^2}{\epsilon(h(x))},
$$

and $h^*$ is the unique solution of:

$$
h^* - \alpha^{-1} \left( -\epsilon(h^*)p^2/4 \right) = 0.
$$

Furthermore, any controller $u = k(t, x) \in K_{ISS}(t, x)$ renders (5) safe w.r.t. $S^* = S_{ISSF} \supset S$.

**Proof.** First, we observe that (33) has a unique solution $h^*$ based on the monotonicity properties of $\alpha^{-1}$ and $\epsilon$. Furthermore, $S^* = S_{ISSF}$ based on (29) and (33), while the property $\epsilon(r) > 0$ for all $r \in \mathcal{H}$ yields $h^* < 0$ and $S^* \supset S$. Moreover, $h^*$ is a regular value of $h$ thanks to the strict inequality in (30); please refer to the proof of Theorem 3 in [25] for details. Hence, comparing (30) with (8) and (32) establishes that $H$ is a PBF. Finally, by noticing that $\|\nabla h(x)\|^2 - \|\nabla h(x)\|p \geq -\epsilon(h(x))p^2/4$, (12) and (32) yield:

$$
L_h h(t, x) u + L_f h(t, x) + \sigma(t, x, u) \geq -\epsilon(h(x))p^2/4.
$$

This inequality and (33) imply that condition (22) in Theorem 3 holds, therefore (5) is safe w.r.t. $S^*$.

Next, we derive more accurate safety guarantees for ISSf-CBF-based controllers via the PBF framework.

**Corollary 2.** Consider (5) with (12) and (32). Assume that there exists $\delta : \mathcal{H} \rightarrow \mathbb{R}_{\geq 0}$ such that for any $h^* \in \mathcal{H}$:

$$
\delta(h^*) \leq \|\nabla h(x)\|,
$$

\forall x \in \partial S^*. If $H$ is a PBF for (5) on $S^*$ with $h^*$ satisfying:

$$
h^* - \alpha^{-1} \left( \delta(h^*)^2/\epsilon(h^*) - \delta(h^*)p \right) = 0,
$$

and $\epsilon(h^*) \leq 2\delta(h^*)/p$ holds, then $u = k(t, x) \in K_{RCBF}(t, x)$ renders (5) safe w.r.t. $S^* \subseteq S_{ISSF}$.

**Proof.** Based on (12), (32) and $\epsilon(h^*) \leq 2\delta(h^*)/p$, it can be shown that the following holds for all $x \in \partial S^*$:

$$
L_h h(t, x) u + L_f h(t, x) + \sigma(t, x, u) \geq \frac{\delta(h^*)^2}{\epsilon(h^*)} - \delta(h^*)p
$$

Using (36) gives (22), and Theorem 3 yields safety w.r.t. $S^*$. Moreover, $h^* \geq \alpha^{-1} \left( -\epsilon(h^*)p^2/4 \right)$ holds, thus $S^* \subseteq S_{ISSF}$.}

**Remark 5.** Since $S^* \subseteq S_{ISSF}$, the PBF framework provides a tighter safety guarantee than ISSF theory. Indeed, all cases of $h^* < 0$, $h^* = 0$ and $h^* > 0$ can occur in (36), corresponding to safety degradation, safety and conservativeness.

**Example 4.** Consider the inverted pendulum problem in Example 1. We utilize the controller (11) with $\sigma$ in (32), $\epsilon(r) = \epsilon_0 \exp^{\lambda r}$, $\epsilon_0 > 0$ and $\lambda \geq 0$. Two simulation results are given in Fig. 4, with $\epsilon_0 = 1$, $\lambda = 0$ (orange dashed-dotted curve), and $\epsilon_0 = 1$, $\lambda = 4$ (brown dashed curve). Both
 simulated trajectories stay within $S$. Indeed, while the former parameter pair yields a more conservative result, introducing $\lambda$ alleviates the conservativeness as discussed in Remark 4.

Boundaries of the corresponding $S_{\text{ISSf}}$ sets, calculated by solving (33), are also plotted by cyan solid and red dashed ellipses. As expected, these sets obtained from the ISSf theory fail to evaluate the conservativeness. The boundaries of the sets $S^*$, after solving (36), are plotted by orange and brown solid lines in Fig. 4. Indeed, they are more accurate bounds on the trajectories of the system. This shows that the PBF framework provides flexibility to quantify conservativeness.

VI. CONCLUSION

This work focused on establishing safety guarantees for control systems with uncertainties. We proposed parameterized barrier functions (PBFs) that generalize existing robust control barrier function (RCBF) formulations. We highlighted that the PBF framework offers flexibility to evaluate not only safety, but safety degradation and conservativeness of RCBF-based controllers. Moreover, we showed that input-to-state safety (ISSf) can be viewed as a special case of the PBF framework, and we derived improved safety guarantees for ISSf-CBF-based controllers. Future research may extend the PBF framework to sampled-data systems to analyze robustness against inter-sampling effects in continuous-time.

REFERENCES


