ENTANGLEMENT ENTROPY AND THE SPLIT PROPERTY OF QUANTUM SPIN CHAINS, AFTER MATSUI

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Abstract. In this note we review the work of Matsui [5] establishing the split property of 1d quantum spin systems from boundedness of entanglement entropy.

1. Introduction

Consider a spin chain with $n$-dimensional Hilbert spaces. For a finite subset $\Lambda \subseteq \mathbb{Z}$ we define

$$U_{\Lambda} = \bigotimes_{j \in \Lambda} M_n(\mathbb{C}).$$

For $\Lambda \subseteq \Lambda'$ we have an inclusion $U_{\Lambda} \subseteq U_{\Lambda'}$. We define the algebra of local observables, denoted $U_{\text{loc}}$, by

$$U_{\text{loc}} = \colim_{\Lambda \subseteq \mathbb{Z}, |\Lambda| < \infty} U_{\Lambda}.$$

We denote by $U$ the completion of $U_{\text{loc}}$ in the norm topology. For each subset $\Lambda \subseteq \mathbb{Z}$ we denote by $U_{\Lambda}$ the $C^*$-algebra of $U$ generated by $U_{\{j\}}$ for $j \in \Lambda$. We also denote

$$U_R = U_{[1, \infty]}, \quad U_L = U_{(-\infty, 0)}.$$

We begin with some reminders and preliminaries.

1.1. States, time evolution, and spectral gap. Recall from two weeks ago that a state of a $C^*$ algebra is a positive linear functional of norm 1. That is, a linear functional $\varphi : U \rightarrow \mathbb{C}$ such that $\varphi(A^*A) \geq 0$ for all $A \in U$ and $||\varphi|| = 1$. Also recall that for any state $\varphi$, there is a unique (up to unique unitary isomorphism) representation $(\pi_{\varphi}, \varphi, \Omega_{\varphi})$ of $U$ such that

$$\varphi(A) = \langle \Omega_{\varphi}, \pi_{\varphi}(A)\Omega_{\varphi} \rangle$$

In particular, we can take $(\pi_{\varphi}, \varphi, \Omega_{\varphi})$ to be the GNS construction. The set of states is a pre-compact convex set under the weak-* topology and the set of extremal points consists of 0 and the set of pure states, see [2, p. 2.3.15].

Proposition 1.1.1. A state is pure if and only if it has an associated cyclic representation which is irreducible.

For the proof see [2, p. 2.3.19]. The essential uniqueness implies that any cyclic representation associated to $\varphi$ would be irreducible.

In the special case of quasi-local observables, we define $\varphi_{\Lambda}$ to be the restriction of $\varphi$ to $U_{\Lambda}$. We denote by $\mathfrak{M}_{\Lambda}$ the bicommutant $\pi_{\varphi}(U_{\Lambda})''$. It is the von Neumann algebra generated by $\pi_{\varphi}(U_{\Lambda})$. 

Definition 1.1.2. An interaction is an assignment $X \mapsto \Psi(X)$ for every finite subset $X$ of $\mathbb{Z}$ to a self-adjoint element $\Psi(X)$ of $U$. 

Definition 1.1.3. An interaction $\Psi$ is said to be of finite range if there exists an $r > 0$ such that $\Psi(X) = 0$ for any finite subset $X \subseteq \mathbb{Z}$ such that the radius of $X$ is bigger than $r$.

Definition 1.1.4. An interaction $\Psi$ is called bounded if

$$\sup_{j \in \mathbb{Z}} \sum_{j \in X} \frac{||\Psi(X)||}{|X|} < \infty.$$ 

For any $j \in \mathbb{Z}$ let $\tau_j$ be the translation automorphism of $U$. It is determined uniquely by the requirement that $\tau_j(U_{\Lambda}) \cong U_{\Lambda + j}$ is given by the canonical identification for any finite $\Lambda \subseteq \mathbb{Z}$.

Definition 1.1.5. An interaction $\Psi$ is called translation invariant if $\tau_j(\Psi(X)) = \Psi(X + j)$ for any $X \subseteq \mathbb{Z}$ and $j \in \mathbb{Z}$.

Given an interaction we define a time evolution as follows. For a fixed $N \in \mathbb{Z}$ let

$$H_N = \sum_{X \subseteq [-N,N]} \Psi(X)$$

Given $Q \in U_{\text{loc}}$ we define

$$\alpha_t(Q) = \lim_{N \to \infty} e^{i t H_N} Q e^{-i t H_N}$$

$$[H, Q] = \lim_{N \to \infty} [H_N, Q]$$

Note that these limits exist as the sequences stabilize. The linear map $Q \mapsto \alpha_t(Q)$ is bounded and therefore extends uniquely to a linear map on $U$. Explicitly, we have

$$\alpha_t(Q) = \lim_{N \to \infty} e^{i t H_N} Q e^{-i t H_N}$$

for any $Q \in U$.

Remark 1.1.6. For $Q \in U_{\text{loc}}$ the map $t \mapsto \alpha_t(Q)$ extends to an analytic function $z \mapsto \alpha_z(Q)$ from $\mathbb{Z}$ to $U$.

The essential uniqueness of a cyclic representation associated to $\varphi$ implies the following. Given any automorphism $\tau : U \to U$ such that $\varphi(\tau(A)) = \varphi(A)$ there is a unique unitary operator $U_\varphi$ on $h_\varphi$ such that

$$U_\varphi \pi_\varphi(A) U_\varphi^{-1} = \pi_\varphi(\tau(A))$$

$$U_\varphi \Omega_\varphi = \Omega_\varphi.$$ 

In particular, given a time evolution $\{\alpha_t\}_{t \in \mathbb{R}}$ we get an unitary operator $U_\varphi(t)$ on $h_\varphi$ for every $t \in \mathbb{R}$. These operators constitute a strongly continuous one parameter group of unitary operators on $h_\varphi$ fixing $\Omega_\varphi$. That is, the map

$$\mathbb{R} \times h_\varphi \to h_\varphi$$

$$(t, v) \mapsto U_\varphi(t)v$$

is continuous, and for all $t, s \in \mathbb{R}$

$$U_\varphi(t + s) = U_\varphi(t) U_\varphi(s)$$

$$U_\varphi(t) \Omega_\varphi = \Omega_\varphi.$$
By Stone’s theorem, there is a densely-defined self-adjoint operator \( H_\varphi \) on \( \mathfrak{h}_\varphi \) generating the group \( \{ U_\varphi(t) \} \). That is, for all \( t \in \mathbb{R} \), we have \( U_\varphi(t) = e^{itH_\varphi} \).

**Definition 1.1.7.** Let \( \mathcal{U} \), \( \alpha_t \) be as above and let \( \varphi \) be a state of \( \mathcal{U} \). \( \varphi \) is called a ground state with respect to \( \alpha_t \) if the corresponding self adjoint operator \( H_\varphi \) is positive semi-definite.

**Proposition 1.1.8.** Let \( \alpha_t \) be a time evolution on \( \mathcal{U} \) and let \( \varphi \) be a state. Then \( \varphi \) is a ground state with respect to \( \alpha_t \) if and only if for every \( Q \in \mathcal{U}_{\text{loc}} \)

\[
\varphi(Q^*[H, Q]) \geq 0
\]

**Remark 1.1.9.** Note that for \( Q \in \mathcal{U}_{\text{loc}} \),

\[
\varphi(Q^*[H, \alpha_t(Q)]) = -i \frac{d}{dt} \varphi(Q^*\alpha_t(Q)).
\]

**Proof.** A full proof can be found at [1, prop. 5.3.19.]. We will give a sketch ignoring some technicalities regarding domains of definition of unbounded self-adjoint operators.

Assume \( \varphi \) is a state invariant under time evolution. Let \( (\pi_\varphi, \mathfrak{h}_\varphi, \Omega_\varphi) \) be the corresponding representation and let \( H_\varphi \) be the generator of the time evolution (given by invariance). Since \( H_\varphi \Omega_\varphi = 0 \) we have

\[
\varphi(Q^*[H, Q]) = \langle \pi_\varphi(Q)\Omega_\varphi, [H_\varphi, \pi_\varphi(Q)]\Omega_\varphi \rangle = \langle \pi_\varphi(Q)\Omega_\varphi, H_\varphi \pi_\varphi(Q)\Omega_\varphi \rangle.
\]

Then if \( H_\varphi \) is positive semi-definite we get that \( \varphi(Q^*[H, Q]) \geq 0 \). Conversely, if \( \varphi(Q^*[H, Q]) \geq 0 \) then

\[
\langle \pi_\varphi(Q)\Omega_\varphi, H_\varphi \pi_\varphi(Q)\Omega_\varphi \rangle \geq 0
\]

for every \( Q \in \mathcal{U}_{\text{loc}} \) and by cyclicity of \( \Omega_\varphi \) we deduce that \( H_\varphi \geq 0 \). Thus to finish the proof, we only need to show that the condition \( \varphi(Q^*[H, Q]) \geq 0 \) implies that \( \varphi \) is invariant under time translation.

Let \( Q = Q^* \) be a self-adjoint element of \( \mathcal{U}_{\text{loc}} \). Then \( \varphi(Q[H, Q]) \in i\mathbb{R} \) so

\[
\varphi([H, Q]) = \varphi(Q[H, Q]) = -\varphi(Q[H, Q]).
\]

By the derivation property of the commutator

\[
\varphi([H, Q^2]) = \varphi([H, Q]Q) + \varphi(Q[H, Q]) = 0.
\]

This implies that for self adjoint \( Q \),

\[
\varphi(\alpha_t(Q^2)) = i \int_0^t \varphi([H, \alpha_s(Q^2)])ds = i \int_0^t \varphi([H, \alpha_s(Q^2)])ds = 0
\]

That is, \( \varphi(\alpha_t(Q^2)) = \varphi(Q^2) \) for all \( t \) and self-adjoint \( Q \in \mathcal{U}_{\text{loc}} \). As every positive element is a square of a positive-definite self-adjoint operator the invariance is true for positive elements. This implies \( \varphi \) is \( \alpha_t \)-invariant as every element can be expressed as a linear combination of four positive elements. Indeed, every self adjoint element \( Q \) can be written as a difference of positive elements by taking

\[
Q_{\pm} = \frac{1}{2}(|Q| \pm Q)
\]

where \( |Q| \) is the unique positive square root of \( Q^2 \). Thus, \( \varphi \) is invariant under time evolution.

A key property for us is the spectral gap of a state.

**Definition 1.1.10.** We say that \( H_\varphi \) has a spectral gap if 0 is a non-degenerate eigenvalue of \( H_\varphi \) and there exists a \( M > 0 \) such that the spectrum of \( H_\varphi \) does not intersect \((0, M)\).
1.2. \textbf{Entanglement entropy}. Let $\Lambda \subseteq \mathbb{Z}$ be a finite subset. Let $\rho_\Lambda$ denote the density matrix of $\phi_\Lambda$ considered as a state of the finite dimensional algebra $\Omega_\Lambda$. Let

$$s_\Lambda = -\text{tr}_\Lambda (\rho_\Lambda \log \rho_\Lambda).$$

$s(\phi_\Lambda)$ is called the entanglement entropy of $\phi_\Lambda$. We say that a state has bounded entanglement entropy if there exists some $S > 0$ such that $s_\Lambda \leq S$ for all finite subsets $\Lambda \subseteq \mathbb{Z}$.

1.3. \textbf{Main results}. Let $\mathfrak{h}$ be a Hilbert space and $\mathfrak{M}_1, \mathfrak{M}_2$ von Neumann subalgebras of $B(\mathfrak{h})$ such that $\mathfrak{M}_1 \subseteq \mathfrak{M}_2$. The pair $\mathfrak{M}_1, \mathfrak{M}_2$ is said to satisfy the split property if there exists a type I factor $\mathfrak{N}$ such that $\mathfrak{M}_1 \subseteq \mathfrak{N} \subseteq \mathfrak{M}_2$. In this case, $N = B(\mathfrak{h}_1)$ for some $\mathfrak{h}_1$ and we get a factorization $\mathfrak{h} \cong \mathfrak{h}_1 \otimes \mathfrak{h}_2$ and subalgebras $\mathfrak{M}_1 \subseteq B(\mathfrak{h}_1), \mathfrak{M}_2 \subseteq B(\mathfrak{h}_2)$ such that under the identification of $\mathfrak{h}$ with $\mathfrak{h}_1 \otimes \mathfrak{h}_2$ we have

$$\mathfrak{M}_1 = \mathfrak{M}_1 \otimes 1_{\mathfrak{h}_1}, \quad \mathfrak{M}_2 = 1_{\mathfrak{h}_2} \otimes \mathfrak{M}_2.$$ 

When $\mathfrak{M}_1, \mathfrak{M}_2$ generate $B(\mathfrak{h})$ we get equalities $\mathfrak{M}_1 = B(\mathfrak{h}_1)$ and $\mathfrak{M}_2 = B(\mathfrak{h}_2)$. We can now state the main result we will discuss today. We keep the notations $\mathfrak{H}, \mathfrak{H}_\Lambda, \mathfrak{H}_L, \mathfrak{H}_R$ as above.

\textbf{Theorem 1.3.1}. Let $\phi$ be a pure state of $\mathfrak{H}$ with bounded entanglement entropy. Then $\phi$ satisfies the split property.

It was proven by Hastings \cite{Hastings} that if the interaction is bounded and of finite range, then a pure ground state with a spectral gap has bounded entanglement entropy. This implies the following corollary.

\textbf{Corollary 1.3.2}. Let $\Psi$ be a bounded, finite range interaction with corresponding Hamiltonian $H$. Let $\phi$ be a pure ground state such that $H_\phi$ has a spectral gap. Then $\phi$ satisfies the split property.

\textbf{Remark 1.3.3}. In particular, in the situation of theorem 1.3.1 Haag duality holds for $R$ and $L$. That is,

$$\pi_\phi(\mathfrak{H}_L)' = \pi_\phi(\mathfrak{H}_R)'.$$
In this section we will prove theorem 1.3.1. Before we turn to the proof, we will review the notion of quasi-equivalence of states and some of its useful characterizations.

2.1. Digression - quasi-equivalences.

**Definition 2.1.1.** A state $\omega$ on a von Neumann algebra $M$ is called normal if there exists a positive trace class operator $\rho$ with $\text{tr}(\rho) = 1$ such that
$$\omega(A) = \text{tr}(\rho A).$$

**Remark 2.1.2.** $\omega$ is normal if and only if it is continuous with respect to the ultra-weak topology.

**Definition 2.1.3.** Let $\pi$ be a representation of a C$^*$-algebra $U$. A state $\omega$ of $\pi$ is said to be normal if there exists a normal state $\rho$ of $\pi(U)' \cap \pi(U)$ such that
$$\omega(A) = \rho(\pi(A)).$$

**Definition 2.1.4.** Let $U$ be a C$^*$-algebra. Two representations $\pi_1, \pi_2$ of $U$ are called quasi-equivalent, denoted $\pi_1 \sim \pi_2$, if they have the same set of normal states. That is, a state is $\pi_1$-normal if and only if it is $\pi_2$-normal.

**Theorem 2.1.5** (2 2.4.26.). Let $U$ be a C$^*$-algebra and $(\pi_1, h_1), (\pi_2, h_2)$ representations of $U$. The following are equivalent

1. There exists an isomorphism $\tau : \pi_1(U)'' \to \pi_2(U)''$ such that $\tau \circ \pi_1 = \pi_2$.
2. $\pi_1$ and $\pi_2$ are quasi-equivalent.
3. There exist $n, m \in \mathbb{N}$ and projections $E_1 \in n\pi_1(U)'$, $E_2 \in m\pi_2(U)'$ and unitary operators $U_1 : h_1 \to E_2(mh_2)$ and $U_2 : h_2 \to E_1(nh_1)$ such that
$$U_1\pi_1(A)U_1^* = m\pi_2(A)E_2$$
$$U_2\pi_2(A)U_2^* = n\pi_1(A)E_1$$
4. There exists an $n \in \mathbb{N}$ such that $n\pi_1 \cong n\pi_2$, i.e. $\pi_1$ and $\pi_2$ are unitary equivalent up to multiplicity.

**Corollary 2.1.6.** If $\pi_1$ and $\pi_2$ are irreducible then being quasi-equivalent implies equivalence.

**Proposition 2.1.7.** Every subrepresentation of a factor is quasi-equivalent to the ambient representation. In particular, factor states are either disjoint or quasi-equivalent.

**Proof.** Let $\pi$ be a factor representation and let $\rho$ be a a subrepresentation. Let $E$ be an orthogonal projection to $h_\rho$. Invariance of $h_\rho$ means $E \in \pi(U)'$ and therefore Every element in $\pi(U)''$ commutes with $E$ and therefore leaves $h_\rho$ invariant. We thus get a continuous (in ultraweak? even norm!) surjection $p : \pi(U)' \to \rho(U)'$ by restriction. The kernel $\mathcal{J}$ of $p$ is a two sided ultra-weakly closed ideal. By [2, p. 2.4.22], there is a projection $E_p \in \pi(U)'' \cap \pi(U)'$ such that $\mathcal{J} = E_p\pi(U)'$. As $\pi$ is factor, this implies $E_p$ must be trivial or an isomorphism. Since the image is non-trivial we must have $E_p = 0$ so $p$ is an isomorphism. □

We will use the following criterion to detect quasi-equivalences in quasi-local hyperfinite algebras.
Proposition 2.1.8 (2.6.11.). Let $\Omega$ be as in the introduction and let $\omega_1$ and $\omega_2$ be factor states. Then $\omega_1$ and $\omega_2$ are quasi-equivalent if and only if for all $\epsilon > 0$ there exists an $r > 0$ such that for all $B \in \mathcal{U}_{\text{loc}}$ supported outside $[-r,r]$ we have

$$||\omega_1(B) - \omega_2(B)|| \leq \epsilon ||B||.$$ 

2.2. Boundedness implies split property. We now turn to the proof of theorem 1.3.1. The main idea is to use the boundedness of entanglement entropy to approximate the state $\varphi_L \otimes \varphi_R$ by states coming from factorizations determined by the finite set $[1,N]$ of sites and its complement $[1,N]^c$ in a uniform way. Let’s make this precise.

As $\varphi$ is pure, it is enough to prove that it is quasi-equivalent to a state of the form $\psi_L \otimes \psi_R$, since then it would have to be an irreducible component. Indeed, in that case $\pi_\varphi \simeq \pi_L \otimes \pi_R$ with $\pi_L$ (resp. $\pi_R$) a representation of $\mathcal{U}_L$ (resp $\mathcal{U}_R$). As $\varphi$ is irreducible, each of these representations is irreducible so the corresponding factors are of type I.

In fact, we will show that $\varphi$ is quasi-equivalent to $\varphi_L \otimes \varphi_R$.

First, note that as $\varphi$ is irreducible it is factor by Schur’s lemma. Moreover, $\varphi_L$ and $\varphi_R$ are also factor. Indeed, the center $Z_L$ of $\mathcal{M}_L$ satisfies

$$Z_L \cap M_L' = (Z_L \cup M_L')' \cap M_L'$$

but since $M_R \subseteq M_L'$ we have

$$B(h) = (M_L \cup M_R)' \subseteq Z_L'$$

and so the center of $\mathcal{M}_L$ is trivial. The same logic applies to $\mathcal{M}_R$.

We denote $\pi_\varphi(\mathcal{U}_\Lambda)'$ by $\mathcal{M}_\Lambda$. Note that if $|\Lambda| < \infty$ then $\mathcal{U}_\Lambda \simeq \mathcal{M}_\Lambda \simeq B(h_\Lambda)$ with $\dim(h_\Lambda) = n^{|\Lambda|}$. Given a finite $\Lambda \subseteq \mathbb{Z}$ the inclusion of type I factors $B(h_\Lambda) \subseteq B(h_\varphi)$ induces a decomposition

$$h_\varphi \simeq h_\Lambda \otimes h_\Lambda^c.$$

Using the Schmidt decomposition in this splitting, we can write the unit vector $\Omega_\varphi$ as

$$\Omega_\varphi = \sum_{i=1}^{l_\Lambda} \sqrt{\mu_i^\Lambda} e_i^\Lambda \otimes f_i^\Lambda.$$

With $0 < \mu_1 \leq \mu_{l-1} \leq \cdots \leq \mu_1 \leq 1$ and such that for all $i,j$ we have

$$\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = \delta_{i,j}.$$ 

Note that $\sum \mu_i = 1$. In this case, the entropy $s(\varphi_\Lambda)$ is given by

$$s(\varphi_\Lambda) = -\sum_{i=1}^{l_\Lambda} \mu_i^\Lambda \log(\mu_i^\Lambda).$$

For $N \in \mathbb{N}$, we denote by $\mu_i^N$, $e_i^N$, $f_i^N$ the corresponding elements for $\Lambda = [1,N]$.

Lemma 2.2.1. Let $S = \sup_N s(\varphi_{1,N})$. Let $0 < \epsilon < 1$ and let $k$ be the least integer such that

$$\sum_{i=k+1}^{l(N)} \mu_i^N < \epsilon.$$

Then $k \leq \exp(\frac{S}{\epsilon})$ and $\mu_1^N \geq \exp(-\frac{S}{\epsilon})$. 
Proof. We omit $N$ from the notation in the proof, as it is fixed. Since, by the definition of $k$
\[ \sum_{i=k}^{l} \mu_i \geq \epsilon \]
and $-\log(\mu_i) \leq -\log(\mu_{i+m})$ for all $i$ and $m > 0$ we have
\[ -\epsilon \log(\mu_k) \leq \sum_{i=k}^{l} -\mu_i \log(\mu_k) \leq s(\varphi|_{[1,N]}) \leq S. \]
Then $\mu_k \leq \exp\left(-\frac{S}{\epsilon}\right)$. On the other hand,
\[ k\mu_k \leq \sum_{i=1}^{k} \mu_i \leq 1 \]
so $k \leq \exp\left(\frac{S}{\epsilon}\right)$. \qed

**Lemma 2.2.2.** Let $\omega, \eta$ be states and assume there is some $C > 0$ with $\eta \leq C\omega$. If $\omega$ is a factor then $\eta$ and $\omega$ are quasi-equivalent.

Proof. For any $A \in \mathfrak{A}$, we have $\eta(A^*A) \leq C\omega(A^*A)$. This implies that $I_\omega \subseteq I_\varphi$. This gives a map $A/I_\omega \to A/I_\varphi$ of pre-Hilbert spaces of norm less than or equal to $C$. This gives a (non-unitary) bounded surjective map $h_\omega \to h_\varphi$ of spaces intertwining the representations. Thus, the kernel $K$ of this map is an invariant subspace, and therefore has a complement $K'$. This gives a splitting of our map. Any such splitting then realizes $h_\varphi$ as a sub-representation of $h_\omega$. As $\omega$ is a factor, any sub-representation is quasi-equivalent to it. This proves the claim. \qed

We can now turn to the proof of the main result.

**Proof of theorem 1.3.1** The states $\varphi$ and $\varphi_L \otimes \varphi_R$ are both factor states. Thus, by [1, p. 2.6.11] (using the fact we have hyperfinite quasi-local algebras) it is enough to show that for every $\epsilon > 0$ there exists $r > 0$ such that $|\varphi(A) - \varphi_L \otimes \varphi_R(A)| < \epsilon||A||$ for $A \in \mathfrak{A}_{\text{loc}}$ supported outside $[-r,r]$.

Fix $\epsilon > 0$ and we assume $\epsilon < \frac{1}{2}$. Fix $S$ as before. Let $L$ be the least integer which is less than or equal to $\exp\left(\frac{S}{\epsilon}\right)$. Given $N \in \mathbb{N}$, define
\[ \tilde{\Omega}(N) = \sum_{i=1}^{L} \sqrt{\mu_i^N} e_i^N \otimes f_i^N. \]
Then, by lemma 2.2.1 we have
\[ ||\tilde{\Omega}(N) - \Omega_\varphi|| \leq \sum_{i=L+1}^{L(N)} \mu_i^N \leq \epsilon \]
\[ ||\tilde{\Omega}(N)||^2 = \sum_{i=1}^{L} \mu_i^N \geq 1 - \epsilon. \]
Denote by $\Omega(N)$ the normalization of $\tilde{\Omega}(N)$. Then,

$$||\Omega(N) - \Omega_N||^2 \leq \left( \frac{1}{||\Omega(N)||^2} - 1 \right) ||\tilde{\Omega}(N)|| + ||\tilde{\Omega}(N) - \Omega||$$

$$\leq \frac{\epsilon}{1 - \epsilon} + \epsilon \leq 3\epsilon.$$ 

Let $\eta_N$ denote the state corresponding to the vector $\Omega(N)$. The inequalities derived above imply that for every $N$, we have $||\eta_N - \varphi|| \leq 2\sqrt{3}\epsilon$.

Let $1 \leq i \leq L$. Denote by $\rho_{i,N}$ a state of $\mathcal{U}_R$ extending the state determined by the vector $e_i^N$ and by $\psi_{i,N}$ a state of $\mathcal{U}_L$ extending the state determined by $f_i^N$. By compactness (weak-* compactness of the set of states) we can choose a sequence $N_m$ of integers such that $\{\eta_{N_m}\}$, $\{\rho_{i,N_m}\}$, $\{\psi_{i,N_m}\}$ (for all $1 \leq i \leq L$) converge in the corresponding weak-* topologies and that the sequence $\mu^N_m$ converges for all $i$. We denote by $\eta$, $\rho_i$, $\psi_i$, $\mu_i$ the corresponding limits. Since balls are weak-* closed we still have

$$||\eta - \varphi|| < 2\sqrt{3}\epsilon.$$ 

By the Cauchy-Schwartz inequality, for any $A \in \mathcal{U}_{[1,M]}$ with $M \leq N$, we have

$$\eta_{N}(A^*A) = \sum_{i,j=1}^{L} \mu_i^N \mu_j^N (e_i^N \otimes f_i^N, A^*A(e_i^N \otimes f_i^N))$$

$$= \sum_{i,j=1}^{L} \mu_i^N \mu_j^N (e_i^N \otimes f_i^N, ((A^*A)e_i^N) \otimes f_i^N))$$

$$\leq \sum_{i,j=1}^{L} \mu_i^N \mu_j^N (e_i^N, (A^*A)e_i^N) \langle f_i^N, f_i^N \rangle$$

$$= L \sum_{i=1}^{L} \mu_i^N \langle e_i^N, (A^*A)e_i^N \rangle \leq L \sum_{i=1}^{L} \mu_i^N (\rho_{i,N} \otimes \psi_{i,N})(A^*A).$$

That is, for every $N$ we have

$$\eta_N \leq L \sum_{i=1}^{L} \mu_i^N \rho_{i,N} \otimes \psi_{i,N},$$

when we restrict to $\mathcal{U}_{[1,N]}$. Passing to the limit (everything is done pointwise) we get

$$\eta \leq L \sum_{i=1}^{L} \mu_i \rho_i \otimes \psi_i.$$ 

By lemma 2.2.1 we have $\mu_i^N \geq \exp(-\frac{c}{\epsilon})$ so at least one of the $\mu_i$ is not zero. Let $\tilde{\varphi}$ be a state and $C \in \mathbb{R}$ such that

$$C \tilde{\varphi} = L \sum_{i=1}^{L} \mu_i \rho_i \otimes \psi_i.$$ 

By lemma 2.2.3 all the states with non-zero coefficient are quasi-equivalent to $\varphi_R \otimes \varphi_L$ and thus $\tilde{\varphi}$ is quasi-equivalent to $\varphi_L \otimes \varphi_R$. In particular, $\tilde{\varphi}$ is a factor state. Then by lemma 2.2.2 the state $\eta_N$ is quasi-equivalent to $\varphi_L \otimes \varphi_R$. In particular, there exists an $r > 0$ such that for all $A \in \mathcal{U}_{loc}$ with support outside $[-r,r]$ we have

$$||\eta(A) - (\varphi_L \otimes \varphi_R)(A)|| \leq \epsilon.$$
Then, for such $A$ we also have
\[ ||\varphi(A) - (\varphi_L \otimes \varphi_R)(A)|| \leq \epsilon + 2\sqrt{3}\epsilon. \]

\[ \square \]

Lemma 2.2.3. If $\mu_i \neq 0$, then the state $\rho_i$ is quasi-equivalent to $\varphi_R$ and $\psi_i$ is quasi-equivalent to $\varphi_L$.

Proof. We prove the claim for $\varphi_R$, the proof for $\varphi_L$ is similar. For $M > 0$ and $Q \in \mathfrak{u}[1,M]$ we have that
\[ \pi \varphi(Q) \in \mathcal{B}([1,M]). \]

Note that for $\varphi_L$ the argument is slightly different, in this case we have that $\pi \varphi(Q) \in \mathcal{B}([1,M])'$ and by splitting of Type I factors we have $B([1,M])' = B([1,M]^c)$.

For $N > M$, using the orthogonality relations on $f_i^N$, that
\[ \varphi_R(Q) = (\sqrt{\mu_i^N} e_i^N \otimes f_i^N, \sqrt{\mu_i^N} \varphi_L^N \otimes f_i^N) = \sum_{i=1}^{l(N)} \mu_i^N \rho_i(Q). \]

In particular, $\varphi_R \geq \mu_i^N \rho_{i,N}(Q)$ for all $i = 1, 2, \ldots, L$ if we restrict the states to $\mathfrak{u}[1,M]$. Passing to the limit gives $\mu_i \rho_i$. As the state $\varphi_R$ is factor, if $\mu_i \neq 0$ by lemma 2.2.2 $\rho_i$ is quasi-equivalent to $\varphi_i$. In particular it is a factor state. \[ \square \]

3. Bounding Entanglement Entropy, after Hastings

It was shown by Hastings [4] (see also [3]) that for a time evolution of a ground state with a spectral gap, the entropy $s_{[1,N]}$ is bounded.

Let $P_0 = \Omega^*_\varphi \otimes \Omega^*_\varphi$ be the projection to the ground state. The key technical result is the ability to following. For all $n$ and $l < \frac{n}{2}$ there exists projections $O_L(n,l), O_R(n,l)$, a positive self adjoint operator $O_B(n,l)$ such that
\[ O_R(n,l) \in \pi \varphi([0,n-1],[0,n-1]^c) \]
\[ O_L(n,l) \in \pi \varphi([0,n-1])' \]
\[ O_B(n,l) \in \pi \varphi([n-l,n-l]) \]

and such that there exists constants $C_1, C_2$ such that
\[ ||O_B(n,l)O_L(n,l)O_R(n,l) - P_0|| \leq C_1 \exp(-C_2l) \]

The idea of the construction is to decompose the hamiltonian into a sum $H = H_L + H_B + H_R$ and then using the Lieb-Robinson bound to approximate by operators which are supported locally. Then we can approximate $P_0$ by using the spectral gap to bound the difference between $P_0$ and operators of the form
\[ P_\eta = \frac{M}{\sqrt{2\pi \eta}} \int \exp(iHt) \exp(-(tM)^2)dt. \]
References


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