MA 191A: SELECTED TOPICS IN MATHEMATICS
SECTION 2: LOW-DIMENSIONAL EMBEDDINGS OF DOUBLING METRICS

FALL 2024

Instructor: Seung-Yeon Ryoo (she/her, `sryoo@caltech.edu`)
Lectures: Tuesday and Thursday, 9:00 AM – 10:25 AM, Linde 289
Office hours: Thursday 10:30 AM – 11:30 AM, Linde 258

Course Description: Consider the following (purposefully vaguely stated) question:

How do we characterize the metric spaces that embed bilipschitzly into some Euclidean space?

Here, a metric space (X, d_X) is said to embed bilipschitzly into a Euclidean space $\mathbb{R}^d$ if there exists a function $f : X \to \mathbb{R}^d$ and a real number $D \geqslant 1$ (the infimum over which is called the distortion) such that

$$d_X(x, y) \leqslant |f(x) - f(y)| \leqslant D d_X(x, y), \quad \forall x, y \in X.$$

So far, we do not have a complete and informative answer to the above question.

If we relax the codomain to be infinite-dimensional, then we have some useful characterizations. Long ago, Schoenberg [Sch38b, Sch38a] has shown that a metric space (X, d_X) embeds isometrically (i.e., bilipschitzly with $D = 1$)[1] into a Hilbert space if and only if for every $n \in \mathbb{N}$ and every positive semidefinite symmetric $n \times n$ matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ all of whose rows sum to zero, the following inequality

$$\sum_{i,j=1}^{n} a_{ij} d_X(x_i, x_j)^2 \leqslant 0, \quad \forall x_1, \cdots, x_n \in X$$

holds. More recently, Linial, London, and Rabinovich [LLR95] have shown that a metric space (X, d_X) embeds D-bilipschitzly into Hilbert space if and only if for every $n \in \mathbb{N}$ and $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ as above, the following inequality

$$\sum_{i,j=1}^{n} a_{ij} d_X(x_i, x_j)^2 \leqslant \frac{D^2 - 1}{D^2 + 1} \sum_{i,j=1}^{n} |a_{ij}| d_X(x_i, x_j)^2, \quad \forall x_1, \cdots, x_n \in X$$

holds. This geometric inequality gives a necessary condition for bilipschitz embeddability into a Euclidean space, as every Euclidean space is Hilbertian.

How would we take into consideration the fact that the codomain is finite-dimensional? One way is to require the domain X to be doubling, i.e., there exists an integer $K \geqslant 2$ such that for any $x \in X$ and $r > 0$ there exist $y_1, \cdots, y_K \in X$ such that[2]

$$B_r(x) \subseteq \bigcup_{i=1}^{K} B_{r/2}(y_i).$$

If one wants to stress the particular K, one says that X is K-doubling. A straightforward volumetric argument gives that if X embeds D-bilipschitzly into $\mathbb{R}^d$, then X must be $(4D + 1)^d$-doubling.

In the 1980's, the doubling condition was considered a candidate answer to the above characterization question, and Assouad [Ass83] showed that given a metric space (X, d_X), the following three statements are equivalent:

[1] I must stress, especially for those from differential geometry backgrounds, that this nomenclature of "isometric embedding" is qualitatively different from its common usage in the differential geometry literature. There, an isometric embedding of a Riemannian manifold into Euclidean space means that the pullback of the Euclidean metric is the Riemannian metric. This does not give an isometric embedding as in the sense we are describing, since we are using the ambient metric of $\mathbb{R}^d$, not the geodesic metric on the image.

[2] Here, $B_r(x) \coloneqq \{y \in X : d_X(x, y) \leqslant r\}$ denotes the closed ball of radius r centered at x.

(1) The metric space (X, d_X) is doubling.
(2) For some $0 < \alpha < 1$, the metric space (X, d_X^α) embeds bilipschitzly into some $\mathbb{R}^d$.
(3) For all $0 < \alpha < 1$, the metric space (X, d_X^α) embeds bilipschitzly into some $\mathbb{R}^d$.

This falls short of answering the characterization question, since the parameter α is not allowed to be 1. Semmes [Sem96], employing Pansu's differentiation theorem [Pan89], showed that it is impossible to take $\alpha = 1$, i.e., there exist doubling metric spaces, such as the 3-dimensional Heisenberg group $\mathbb{H}^3$, that fail to embed bilipschitzly into Euclidean space. Cheeger and Kleiner [CK06] and Lee and Naor [LN06] independently observed that Semmes' argument also works when the codomain is Hilbert space, i.e., $\mathbb{H}^3$ fails to embed bilipschitzly into Hilbert space.

So far, all examples of doubling metric spaces that fail to embed bilipschitzly into Euclidean space also fail to embed bilipschitzly into Hilbert space. This led Lang and Plaut [LP01] to ask the following question:

> If a doubling metric space embeds bilipschitzly into a Hilbert space, does it embed bilipschitzly into a Euclidean space? Equivalently, does a doubling subset of Hilbert space embed bilipschitzly into Euclidean space?

An affirmative answer would be a complete answer to the above-stated characterization question. It is equally interesting to find a counterexample that negatively answers Lang and Plaut's question.

From the point of view of Theoretical Computer Science, this type of question belongs to the area of metric dimension reduction, the most well-known result of which is the Johnson–Lindenstrauss lemma: if $X \subseteq \ell_2$ has cardinality n, then there exists a linear mapping $L : X \to \mathbb{R}^{O(\log n)}$ with distortion $O(1)$. Coupled with the question by Lang and Plaut, we may ask whether a (necessarily nonlinear) version of the Johnson-Lindenstrauss lemma holds:

> Does a K-doubling subset of Hilbert space embed $O(1)$-bilipschitzly into a Euclidean space of dimension $O(\log K)$?

An affirmative answer to this question would be useful, as it would serve as an antidote to the commonly known "curse of dimensionality" for algorithms concerning the metric structure of the dataset.

In this class, we will first review the classical works in this area, such as those described above, to establish notation and commonly known facts; I estimate this to take 3-5 weeks. Then we will closely examine the recent developments. This can take a number of directions, such as functional analysis, differential geometry, and theoretical computer science. I encourage you to choose topics which you find to be the most interesting and present them in class.

Prerequisites: Ma 108 and Ma 109 or equivalent, or instructor's permission.

References

[Ass83] Patrice Assouad. Plongements lipschitziens dans $\mathbb{R}^n$. *Bulletin de la Société Mathématique de France*, 111:429–448, 1983.

[CK06] Jeff Cheeger and Bruce Kleiner. On the differentiability of Lipschitz maps from metric measure spaces to Banach spaces. In *Inspired By SS Chern: A Memorial Volume in Honor of A Great Mathematician*, pages 129–152. World Scientific, 2006.

[LLR95] Nathan Linial, Eran London, and Yuri Rabinovich. The geometry of graphs and some of its algorithmic applications. *Combinatorica*, 15:215–245, 1995.

[LN06] James R Lee and Assaf Naor. L^p metrics on the Heisenberg group and the Goemans-Linial conjecture. In *2006 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS'06)*, pages 99–108. IEEE, 2006.

[LP01] Urs Lang and Conrad Plaut. Bilipschitz embeddings of metric spaces into space forms. *Geom. Dedicata*, 87(1):285–307, 2001.

[Pan89] Pierre Pansu. Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math.*, pages 1–60, 1989.

[Sch38a] I. J. Schoenberg. Metric spaces and completely monotone functions. *Ann. of Math. (2)*, 39(4):811–841, 1938.

[Sch38b] I. J. Schoenberg. Metric spaces and positive definite functions. *Trans. Amer. Math. Soc.*, 44(3):522–536, 1938.

[Sem96] Stephen Semmes. On the nonexistence of bilipschitz parameterizations and geometric problems about A_∞-weights. *Rev. Mat. Iberoam.*, 12(2):337–410, 1996.

(Last updated September 22, 2024)