RESEARCH STATEMENT

SUNGHYUK PARK

1. Introduction and motivation

"If a smooth 4-manifold is homeomorphic to S^4, then does it follow that it is diffeomorphic to S^4?" This is a major open problem in topology known as the smooth Poincaré conjecture in dimension four. The fact that such a fundamental question remains open to date shows how little is known about the differential topology in dimension four. In order to develop an understanding of smooth 4-manifolds, it is vital to construct an invariant of smooth 4-manifolds that is strong and easily computable.

A *categorification* of an invariant of n-manifolds assigns a graded vector space $\mathcal{H}(Y) = \oplus_{i\in\mathbb{Z}}\mathcal{H}_i(Y)$, instead of a number, to each n-manifold Y in such a way that the Euler characteristic $\sum_{i\in\mathbb{Z}}(-1)^i \dim \mathcal{H}_i(Y)$ of the graded vector space recovers the original invariant. For each cobordism X from Y_1 to Y_2, the categorification assigns a linear map from $\mathcal{H}(Y_1)$ to $\mathcal{H}(Y_2)$. In particular, it assigns a number to each closed $(n+1)$-manifold X, which can be considered as a cobordism from an empty manifold to itself. The idea of categorification has revolutionized the study of low-dimensional topology, as it provides a way to construct much stronger invariants of manifolds. For instance, Rasmussen's s-invariant of a knot [Ras10], which provides a lower bound to the slice genus of the knot, can be derived from Khovanov homology [Kho00], a categorification of the Jones polynomial.

To date, however, most research on categorification has been limited to invariants of knots, instead of 3-manifolds. While categorification of knot invariants has provided new insights on smooth surfaces in the 4-ball, it does not tell us much about general smooth 4-manifolds. In order to construct novel invariants of smooth 4-manifolds, we need to categorify topological invariants of 3-manifolds, instead of knots. The main challenge has been to find the right candidate of an invariant of 3-manifolds that can be categorified.

Physics provides a promising answer to this question, at the physical level of rigor. There is a construction in string theory and M-theory that takes an n-dimensional manifold Y and an ADE type Lie algebra $\mathfrak{g}$ as an input and produces a $(6-n)$-dimensional quantum field theory $T_{\mathfrak{g}}[Y]$ that contains wealth of information about the manifold Y itself. When the manifold Y is 3-dimensional, there is an invariant $\hat{Z}^{\mathfrak{g}}(Y)$ of Y derived from $T_{\mathfrak{g}}[Y]$ known as the *homological block* [GPV17, GPPV20]. Physically, the homological block is a certain count of what is known as the *Bogomol'nyi–Prasad–Sommerfield (BPS) states*. This means that the vector space of BPS states provides a natural categorification of the homological block of 3-dimensional manifolds. In other words, the homological block $\hat{Z}^{\mathfrak{g}}$ is the sought-after invariant of 3-dimensional manifolds that can be categorified.

My research program is to construct a novel invariant of smooth 4-dimensional manifolds by categorifying the homological block, in a mathematically rigorous way.

Problem 1. Mathematically construct and categorify the homological block. This problem can be divided into two parts:

(1) Mathematically construct the homological block $\hat{Z}^{\mathfrak{g}}$ as an invariant of 3-dimensional manifolds.
(2) Mathematically construct the vector space of BPS states as a categorification of the homological block.

It should be noted that it is a highly non-trivial problem to "translate" the physical construction into mathematics. This is because, for one thing, quantum field theory is not yet mathematically rigorously defined, and for another, not much is known about the physics of the quantum field theory $T_{\mathfrak{g}}[Y]$ for a general 3-manifold Y. Hence, completing this research problem would not only be impactful in low-dimensional topology but also shed light on the physics of the quantum field theory $T_{\mathfrak{g}}[Y]$.

2. Overview of my research

2.1. Toward a mathematical definition of the homological block $\hat{Z}$. In the past few years, there has been significant progress on the first part of this problem – finding a mathematical definition for the homological block $\hat{Z}^{\mathfrak{g}}$. Let us focus on the case $\mathfrak{g} = \mathfrak{sl}(2)$ for simplicity. According to the prediction from physics [GPPV20, GM21], the homological block $\hat{Z}^{\mathfrak{sl}(2)}$ should assign a q-series with integer coefficients to each 3-manifold Y decorated with a spin^c-structure $\mathfrak{s}$:

$$(Y, \mathfrak{s}) \mapsto \hat{Z}^{\mathfrak{sl}(2)}_{\mathfrak{s}}(Y) \in q^{\Delta}\mathbb{Z}[[q]], \quad \text{for some } \Delta \in \mathbb{Q}.$$

It is a well-known theorem in low-dimensional topology that any 3-manifold can be obtained by a Dehn surgery on a link in S^3. Therefore, if we can define the homological block for any link complement and if we understand how it behaves under gluing solid tori along the boundary, then we can mathematically construct the homological block for any 3-manifold. This is the approach outlined in a paper by Gukov and Manolescu [GM21] where the authors computed the homological block for complements of torus knots and the figure-eight knot and conjectured some gluing formulas. Following their notation, we denote by $F^{\mathfrak{sl}(2)}_L := \hat{Z}^{\mathfrak{sl}(2)}(S^3 \setminus L)$ the homological block $\hat{Z}^{\mathfrak{sl}(2)}$ for the complement of a link $L \subset S^3$. If L has l components, then F_L is a power series in $x_1, \cdots, x_l$, where $x_1, \cdots, x_l$ are formal variables dual to the spin^c-structures on the link complement, with coefficients in $q^{\Delta}\mathbb{Z}[[q]]$ for some $\Delta \in \mathbb{Q}$.

One of the main results of my work [Par20b, Par21] is to construct the homological block for a big class of link complements.

Theorem 1 ([Par20b, Par21])**.** *There is a well-defined invariant $F^{\mathfrak{sl}(2)}_L := \hat{Z}^{\mathfrak{sl}(2)}(S^3 \setminus L)$ of a big class of link complements that contains all fibered knots up to 10 crossings as well as all homogeneous braid knots.*

More precisely, if L is such a link, the Melvin-Morton-Rozansky expansion [MM95, BNG96, Roz98] of the colored Jones polynomials of L can be re-summed into a power series

$$F_L(x_1, \cdots, x_l, q) \in \mathbb{Z}[q, q^{-1}][[x_1, \cdots, x_l]]$$

in $x_1, \cdots, x_l$ with coefficients in $\mathbb{Z}[q, q^{-1}]$.

The main idea in this construction is to use the *large-color R-matrix.* The large-color R-matrix is the limit of the R-matrices for the colored Jones polynomials where we send the "color" n to infinity, while keeping $x := q^n$ fixed. In other words, it is the universal R-matrix for the quantum group $U_q(\mathfrak{sl}_2)$ applied to Verma modules with a generic complex weight. In this limit, the new variable x can be naturally interpreted as the meridional holonomy eigenvalue of an abelian $SL(2,\mathbb{C})$ flat connection on the link complement, as depicted in Fig. 1.

$$\check{R}(x_1,x_2)_{i,j}^{i',j'} =$$

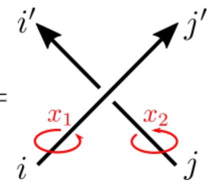

FIGURE 1. A large-color R-matrix element and its diagram

In another direction of development, it has been studied how the homological block behaves under the change of Lie algebra $\mathfrak{g}$, for a certain class of 3-manifolds. In the paper by Gukov, Pei, Putrov and Vafa [GPPV20], the authors gave a mathematical definition of the homological block $\hat{Z}^{\mathfrak{sl}(2)}$ for a class of 3-manifolds known as the *negative definite plumbed 3-manifolds*. This is a class of 3-manifolds which can be obtained from a tree-shaped link, such as the one in Fig. 2, by a Dehn surgery with framing coefficients such that the linking matrix is negative definite. Their construction has been generalized in my paper to arbitrary

FIGURE 2. A tree-shaped link

Lie algebra $\mathfrak{g}$.

Theorem 2 ([Par20a])**.** *For any Lie algebra $\mathfrak{g}$, there is a well-defined invariant $\hat{Z}^{\mathfrak{g}}$ for negative definite plumbed 3-manifolds decorated with a structure which is a higher rank analogue of spin^c-structures. Moreover, there is an explicit expression for $\hat{Z}^{\mathfrak{g}}$ for torus knot complements.*

2.2. **Connection to enumerative geometry.** In [Par20a], it was observed that $\hat{Z}^{\mathfrak{sl}(N)}$ for torus knot complements behaves regularly under the change of N. Based on this observation, I have conjectured that there is a 1-parameter deformation of the homological block for knot complements.

Conjecture 1 ([Par20a, EGG+20])**.** *There is a knot invariant $F_K(x,a,q)$ which associates to each knot K a 3-variable series that satisfies*

$$\begin{aligned}
F_K(x,q^N,q) &= F_K^{\mathfrak{sl}_N}(x,q) \quad \forall N \geq 2,\\
F_K(x,a,q) &= F_K(a^{-1}x^{-1},a,q) \quad \textit{(Weyl symmetry)},\\
\lim_{q\to 1} F_K(x,q^N,q) &= \Delta_K(x)^{1-N},\\
\hat{A}_K(\hat{x},\hat{y},a,q)F_K(x,a,q) &= 0.
\end{aligned}$$

Here, $\Delta_K(x)$ denotes the Alexander polynomial, and $\hat{A}_K(\hat{x},\hat{y},a,q)$ denotes the quantum A-polynomial.

In [EGG+20], based on the physical duality between Chern-Simons theory and topological string theory [Wit95, GV99, OV00], we have argued that this 3-variables series invariant can be interpreted as a certain count of holomorphic curves.

Conjecture 2 ([EGG+20])**.** *Let X be the resolved conifold, and let $M_K \subset X$ denote the Lagrangian submanifold obtained by the Lagrange surgery along the clean intersection between the knot conormal Lagrangian and S^3; M_K is diffeomorphic to $S^3 \setminus K$. Then,*

$$F_K(x,a,q) = \exp\left(\sum_{k\geq -1} g_s^k U_K^k(x,a)\right),$$

where $q = e^{g_s}$, and $U_K^k(x,a)$ counts connected curves of Euler characteristic $\chi = -k$ in X with boundary on M_K. (The curves are naturally graded by $H_2(X, M_K)$, and x and a represent certain generators of $H_2(X, M_K)$.)

In the sequel [EGG+21], we continued our study of $F_K(x,a,q)$ and made two main conjectures. The first one is that there is an analogous three-variable series invariant associated to each branch of the A-polynomial, with the one associated to the abelian branch being the one previously studied.

Conjecture 3 ([EGG+21])**.** *Given a knot K, let $y^{(\alpha)}(x,a)$ be a branch of y near $x = 0$ (or $x = \infty$) of the a-deformed A-polynomial of K, $A_K(x,y,a)$.*

(1) There exists a wave function $F_K^{(\alpha)}(x,a,q)$ associated to this branch in a sense that

$$\langle \hat{y} \rangle := \lim_{q\to 1} \frac{F_K^{(\alpha)}(qx,a,q)}{F_K^{(\alpha)}(x,a,q)} = y^{(\alpha)}(x,a)$$

and this wave function is annihilated by the quantum a-deformed A-polynomial $\widehat{A}_K(\hat{x},\hat{y},a,q)$ (which is the same for all branches $y^{(\alpha)}(x,a)$)

$$\widehat{A}_K(\hat{x},\hat{y},a,q) F_K^{(\alpha)}(x,a,q) = 0.$$

(2) The wave function $F_K^{(\alpha)}(x,a,q)$ has a quiver form, in the sense of knots-quivers correspondence [KRSS17, KRSS19, Kuc20].

Moreover, we find that for each knot K there is a naturally associated holomorphic Lagrangian that generalizes the A-polynomial.

Conjecture 4 ([EGG+21])**.** *Let's endow $(\mathbb{C}^*)^4$ with the holomorphic symplectic form*

$$\Omega := d\log x \wedge d\log y + d\log a \wedge d\log b, \quad x,y,a,b \in \mathbb{C}^*.$$

For every knot K, there is a holomorphic Lagrangian subvariety $\Gamma_K \subset (\mathbb{C}^*)^4$ with the following properties:

(1) This holomorphic Lagrangian is preserved under the Weyl symmetry

$$x \mapsto a^{-1}x^{-1}, \quad y \mapsto y^{-1}, \quad a \mapsto a, \quad b \mapsto y^{-1}b.$$

(2) The projection of Γ_K on $(\mathbb{C}^*)^3_{x,y,a}$ is the A-polynomial of K.

(3) Moreover, if $\hat{x},\hat{y},\hat{a},\hat{b}$ are operators such that

$$\hat{y}\hat{x} = q\hat{x}\hat{y}, \quad \hat{b}\hat{a} = q\hat{a}\hat{b},$$

and all the other pairs commute, then the ideal defining Γ_K can be quantized to a left ideal $\hat{\Gamma}_K \subset \mathbb{C}[\hat{x}^{\pm 1},\hat{y}^{\pm 1},\hat{a}^{\pm 1},\hat{b}^{\pm 1}]$ that annihilates $F_K(x,a,q)$.

3. Proposed research

There are a number of fascinating directions of research regarding the homological block. The long-term goal of this research program was already outlined in Problem 1, so here I list some research problems that are expected be achieved in a shorter time scale.

Based on the behavior of the homological block under cutting and gluing along torus boundary, it is expected that the homological block is a decorated $(2+1)$-dimensional topological quantum field theory (TQFT).

Problem 2. Construct $\hat{Z}^{\mathfrak{sl}_2}$ as a decorated $(2+1)$d TQFT. Study the (projective) representations of the mapping class groups coming from this TQFT.

Once one obtains this representation, it would be natural to study how it is related to well-known representations of mapping class groups in quantum topology (e.g. those coming from skein algebras, quantum Teichmüller theory, or the CGP invariant [BCGPM16]). When the surface is an n-punctured disk, then the corresponding representation of the braid group is the one coming from the large-color R-matrix. This is essentially the Lawrence representation, which is known to be faithful. It would be interesting to study if the representations of the mapping class groups of other surfaces are faithful as well.

The connection to enumerative geometry also provides a number of intriguing future research directions. It is well-known [AENV14] that the A-polynomial of a knot can be understood as the augmentation polynomial of the differential graded algebra (DGA) whose homology is known as the *knot contact homology*. Hence, it is a very interesting question if the holomorphic Lagrangian, which is a lift of the A-polynomial, can be understood in this context as well.

Problem 3. Understand the holomorphic Lagrangian Γ_K of Conjecture 4 in the context of knot contact homology. Is there a generalization of the knot DGA that incorporates the new variable b whose augmentation variety is Γ_K?

Another important problem is to verify the q-holonomicity in variable a predicted in Conjecture 4. This can be stated in terms of (symmetrically) colored HOMFLYPT polynomials which are mathematically well-defined objects, as opposed to $F_K(x, a, q)$.

Problem 4. Prove mathematically that the colored HOMFLYPT generating function is q-holonomic in variable a; that is, it is annihilated by a q-difference operator $\hat{B}_K(\hat{a}, \hat{b}, y, q)$.

Note, this is different from the q-holonomicity of the colored HOMFLYPT generating function in variable y, which was proven in [GLL18].

Last but not least, while Conjecture 3 simply states that there are three-variable series $F_K^{(\alpha)}(x, a, q)$ associated to each branch of the A-polynomial, these series are not disparate objects. For instance, they all satisfy the same recurrence relation given by the quantum A-polynomial or its generalization, which is the quantized ideal $\hat{\Gamma}$ in Conjecture 4. Thus, it would be great if we can understand how to obtain one series from another or, even better, unify them.

Problem 5. Unify the three-variable series invariant $F_K^{(\alpha)}(x, a, q)$ of Conjecture 3 associated to each branch of the A-polynomial.

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