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ORIGINAL PAPER

Robustness to manipulations in school choice

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Abstract
We study the school choice problem and propose a new criterion for comparing non-strategy-proof mechanisms: *robustness to manipulations*. Mechanism A is *more robust* than mechanism B if each student (given any preferences of this student and any profile of schools' priorities) can potentially access a smaller set of schools via a profitable manipulation under mechanism A than under mechanism B. This criterion strengthens the two independent criteria proposed by Bonkoungou and Nesterov (Theor Econ 16(3):881–909, 2021) and Decerf and Van der Linden (J Econ Theory 197:105313, 2021). We then show that all results obtained with these two criteria, as well as with the original criterion proposed by Pathak and Sönmez (Am Econ Rev 103(1):80–106, 2013), can also be obtained using *robustness*. Our results provide a stronger rationalization for a wide range of reforms in school choice and college admissions system.

1 Introduction

We study matching mechanisms which are widely used around the world in centralized school and college admission systems. Examples include versions of the well-known Boston mechanism and the Deferred Acceptance mechanism in the US and Eng-

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land (Pathak and Sönmez 2013) as well as hybrids of these two mechanisms: the First-Preference-First mechanism in England and Wales, and the Chinese Parallel mechanism in China (Chen and Kesten 2017).[1]

In each of these mechanisms, at the beginning of the enrollment process the students are asked to report their preferences over schools, usually in a limited number. However, under most of these mechanisms, reporting preferences truthfully may be risky because there is a chance to end up with a less desirable school or even unassigned. For this reason, students have an incentive to misreport their preferences: to order schools differently from their true preferences or exclude some more desirable schools from their rankings and include less desirable schools instead. We call such strategic misreports *manipulations*. We call a mechanism *manipulable* if it allows students to profitably manipulate, that is, to obtain a seat in a more preferred school than under a truthful report.

Strategic behavior of students appears to be a complicated policy issue. In recent years, a wave of reforms took place in the US, England, Wales, and China (Table 1 in Pathak and Sönmez 2013; Table 2 in Bonkoungou and Nesterov 2021). The old mechanisms were replaced with the new ones, and manipulability was cited as a major motivation for these changes: the policymakers and parents stated that in the old admission systems a good strategy is worthier than the exam score (Chen and Kesten 2017). Nevertheless, the new mechanisms also remain manipulable, and in order to evaluate the reforms, one needs a useful tool to compare mechanisms according to their manipulability. In this paper we provide a new such tool.

There are several existing ways to measure manipulability of mechanisms that can be applied to the problem at hand: by the inclusion of preference profiles vulnerable to manipulations (Pathak and Sönmez 2013),[2] by the inclusion of schools that can be accessed strategically (Bonkoungou and Nesterov 2021), and by the inclusion of truthful dominant strategies (Decerf and Van der Linden 2021). All these concepts are meaningful, logically independent (see Fig. 1), and useful for evaluation of the reforms albeit not universal. For instance, *the inclusion of preference profiles* can not compare the constrained versions of the First-Preference-First and the Deferred Acceptance mechanisms, and hence, important reforms in England and Wales can not be evaluated in terms of this criterion (Bonkoungou and Nesterov 2021).

In this paper, we study the other two criteria and argue that both *the inclusion of strategically accessible schools* and *the inclusion of truthful dominant strategies* can be captured by a single stronger criterion that we call *robustness to manipulations*. We introduce this criterion (Definition 2) and show that the results previously obtained for the two weaker criteria also hold for this new criterion.

Our criterion works as follows. Consider some student j and let him know his preferences over schools and the environment: the set of students and schools, the schools' capacities and priorities over all students, but not the preferences of other students. With this information, what can student j deduce regarding his manipulation opportunities? If there exists a profile of other students' preferences such that some

[1] The definitions of the mechanisms are given in "Mechanisms" section.

[2] Pathak and Sönmez (2013) provide three logically related criteria but only the weakest can be used to compare school choice mechanisms.

manipulation gives student j a school better than his outcome under the truthful report, this school is called *strategically accessible to student j at his true preference relation*. If for every environment and every student j's preference relation, a change from an old mechanism to a new mechanism does not provide student j with new strategically accessible schools, and if it is true for each student, then we say that the new mechanism is *at least as robust to manipulations* as the old one. If, in addition, at least one student loses at least one of his strategically accessible schools, the new mechanism is *more robust to manipulations*.

Why is robustness a good measure of manipulability? First, robustness accounts for all information available to a student—his preferences and the environment—and looks at a possible preference profile of other students, as in a best-case (or worst-case for the designer) scenario.[3] Second, and most importantly, robustness does a fine comparison for many mechanisms (see Table 1).

Related literature and our contribution

The first attempt to compare manipulable matching mechanisms was made by Pathak and Sönmez (2013). A preference profile is called *vulnerable* if at least one student can profitably manipulate. One mechanism is said to be *more manipulable* (henceforth *more PS-manipulable*) than another if the set of vulnerable profiles under the former mechanism is a strict subset of the vulnerable profiles under the latter mechanism. It was proved that the constrained *DA* with the possibility to rank up to k schools, DA^k, is more PS-manipulable than DA^{k+1}. That is, the more schools a student can report, the less PS-manipulable the mechanism is. Similarly, the constrained Boston, β^k, is more PS-manipulable than β^{k+1} (Decerf and Van der Linden 2021). The same criterion can be applied to compare different versions of Chinese Parallel mechanisms ($Ch^{(k)}$) (Chen and Kesten 2017), and the constrained Boston with the constrained *DA* (Table 1).[4] As shown in Bonkoungou and Nesterov (2021), PS-manipulability is unable to rank the constrained First-Preference-First (*FPF*) and the *DA* mechanisms, while β is more PS-manipulable than *DA*.[5]

Another criterion, based on truthful dominant strategies, was designed to compare median voter schemes (Arribillaga and Massó 2016) and applied to the comparison of voting by committees (Arribillaga and Massó 2017). It was then adopted for the comparison of school choice mechanisms (Decerf and Van der Linden 2021). Given an environment, a student has a *truthful dominant strategy* if there does not exist a

[3] This inspired the term robustness similar to the hypothetical worst-case scenario in the robust mechanism design literature (Bergemann and Morris 2005; Andreyanov and Sadzik 2021; Suzdaltsev 2022).

[4] The results on the constrained Boston and *DA* have been strengthened using a stronger criterion of counting the number of students with an incentive to manipulate (Bonkoungou and Nesterov 2023; Imamura and Tomoeda 2022). Chen et al. (2016) compare stable mechanisms using a stronger criterion.

[5] There have been other attempts to improve upon the Boston mechanism in terms of incentive properties that can be evaluated using PS-manipulability and that we do not consider in the paper: the Boston-with-skips mechanism (Alcalde 1996; Miralles 2009; Harless 2019; Dur 2019; Mennle and Seuken 2021), also known as Modified Boston Mechanism or Adaptive Boston Mechanism, and the Secure Boston mechanism (Dur et al. 2019). The recent modification called the Neutralized Boston mechanism (Decerf 2023) is compared to the Boston mechanism using a criterion specific to this mechanism.

profile where manipulation can be profitable for him. Following Arribillaga and Massó (2016), one mechanism is said to be *at least as AM-manipulable* as another mechanism if the set of problems for which a student has a truthful dominant strategy under the former mechanism is a subset of problems for which a student has a truthful dominant strategy under the latter. If this inclusion is strict, at least for one student, then we call the first mechanism *more AM-manipulable* than the second.[6] It was proved that the constrained Boston mechanism, β^k, is more AM-manipulable than DA^k, which, in turn, is more AM-manipulable than DA^{k+1}, whereas different constrained versions of the Boston mechanisms are equally AM-manipulable (Decerf and Van der Linden 2021). Furthermore, DA^k and $Ch^{(k-1)}$ are more AM-manipulable than $Ch^{(k)}$, and, somewhat surprisingly, DA^k is less AM-manipulable than β^l for any $l > k$.

The closest criterion to the robustness to manipulation is based on strategically accessible schools (Bonkoungou and Nesterov 2021). A school is *strategically accessible* to a student if there exists a preference profile at which it can be obtained via a profitable manipulation. The difference with our notion is that the true preference relation is not fixed. One mechanism is said to be at least as strategically accessible as another (henceforth *at least as BN-manipulable*) if the set of strategically accessible schools for each student under the former mechanism is a subset of the strategically accessible schools under the latter. The existence of the strict inclusion, at least for one student, implies that the first mechanism is *more BN-manipulable* than the second.[7] The results of the mechanisms comparison are the same as by AM-manipulability (Table 1).

The rest of the paper is organized as follows: in "Model" section, we introduce the model and provide the central definitions. Furthermore, we show the logical relation between robustness and AM-, BN-manipulability. We also formally define the mechanisms. In "Results" section, we compare mechanisms by the robustness criterion and show that the recent reforms resulted in the adoption of more robust mechanisms. Then, we add the comparison between some mechanisms according to AM-, BN-, PS-manipulability. In "Discussion and conclusion" section, we discuss the results and the relation of robustness to manipulation to other criteria. All proofs are presented in the "Appendix".

2 Model

The school choice problem was introduced by Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003). They formulated a mechanism design problem and analyzed some of the existing school choice plans. The school choice problem in essence modifies the college admission problem of Gale and Shapley (1962) by disregarding schools' behavior and preferences.

There is a finite set of schools S and a finite set of students J, the generic elements of which we denote by s and j correspondingly. The size of the sets are denoted by $|S|$ and $|J|$ correspondingly. We typically assume that $|J| \geq 2$ and $|S| \geq 2$. The number

[6] For a formal statement of AM-manipulability, see Definition 3.

[7] For a formal statement of BN-manipulability, see Definition 4.

Table 1 Manipulability of mechanisms by the four criteria

From	Less PS-manipulable?	Less AM-manipulable?	Less BN-manipulable?	More robust?	To
β	Yes (PS)	Yes (DL)	Yes (BN)	**Yes**	DA
β^k	Yes (DL)	The same[a] (DL)	**The same**[a]	**Yes**	β^{k+1}
DA^{k-1}	Yes (PS)	Yes (DL)	Yes (BN)	**Yes**	DA^k
β^k	Yes (PS′)	Yes (DL)	Yes (BN)	**Yes**	DA^k
$\beta^l\,(l > k)$	Not comparable (DL)	Yes (DL)	**Yes**[a,b]	**Not comparable**	DA^k
FPF^k	Not comparable (BN)	**Yes**	Yes (BN)	**Yes**	DA^k
$Ch^{(k-1)}$	Yes (CK)	Yes(DL)	Yes (BN)	**Yes**	$Ch^{(k)}$
DA^k	**Yes**	Yes(DL)	**Yes**	**Yes**	$Ch^{(k)}$
FPF^k	**Not comparable**	**Yes**	**Yes**	**Yes**	$Ch^{(k)}$

The table evaluates the change from the mechanisms in the first column to the mechanism in the last column line by line. The criteria are given in the columns: manipulability by Pathak and Sönmez (2013) (PS-manipulability), manipulability by Arribillaga and Massó (2016) (AM-manipulability), manipulability by Bonkoungou and Nesterov (2021) (BN-manipulability), and robustness to manipulations. If mechanism φ is more AM-/BN-/PS-manipulable than mechanism ψ, then we equivalently say that ψ is less AM-/BN-/PS-manipulable than φ. The results are true for constraint $k > 1$. Notes in parentheses refer to the paper where the result is proved. (PS—Pathak and Sönmez 2013, PS′—Pathak and Sönmez 2011, 2020; DL—Decerf and Van der Linden 2021; BN—Bonkoungou and Nesterov 2021; CK—Chen and Kesten 2017). The other results highlighted in bold are obtained in our paper

[a] Under the additional assumption: no two schools can accept all students

[b] Under the additional assumption: each school prefers a student to an empty seat

of students each school s can accept is limited by its capacity q_s. Student selection for school s is determined by its priority relation $\succ_s$ over students. A set of priority relations of all schools $\succ = (\succ_s)_{s \in S}$ we call a priority profile.

Similarly, each student j has a strict preference relation P_j over schools and the option of being unassigned—the outside option $\emptyset$. Denote the set of possible preference relations as $\mathscr{P}$. A list of preference relations of all students $P = (P_j)_{j \in J}$ is called a preference profile. For convenience, while focusing on one student j we write $P = (P_j, P_{-j})$, where P_{-j} denotes a preference profile excluding preference relation of student j.

Any preference report of student j that differs from his true preference relation P_j we denote by P'_j and call a manipulation of student j. In order to make a comparison of different outcomes resulting from manipulations sensible, we need to fix some parameters. When we fix a set of students, set of schools, their priority relations, and capacities, we call it environment and denote by $(\succ, q)$. If, in addition, we fix a preference profile, then $(P, \succ, q)$ represents what is called the school choice problem, or simply a problem. We will regularly assume that the environment is fixed and denote a problem by a profile P alone.

By μ we denote a matching which maps the set of students to the set of schools or the outside option, μ: $J \to S \cup \{\emptyset\}$. The outcome of matching for student j is μ_j. A function mapping problems into matchings is called a mechanism. The outcome of a mechanism φ for student j at some profile P we denote as $\varphi_j(P)$.

The following notion will be useful to formalize the three comparison criteria studied in the paper. We consider schools that a given student can profitably access via misreports, we use the set of such schools as a measure of how much this student can manipulate a mechanism at a given environment and a preference relation.

Definition 1 Consider an environment $(\succ, q)$. School s is *strategically accessible to student* $\boldsymbol{j}$ *at preference relation* $\boldsymbol{P}_j$ via mechanism φ if there exists a manipulation P'_j and a list of preferences P_{-j} such that

$$\varphi_j(P'_j, P_{-j}) = s \; P_j \; \varphi_j(P_j, P_{-j}).$$

Every school that does not satisfy this definition we call ***not strategically accessible*** *to student* $\boldsymbol{j}$ *at preference relation* $\boldsymbol{P}_j$ via mechanism φ.

In other words, we consider some student j who knows only his preferences and the environment, and look over all possible preference relations of other students. If we find a profile at which student j is strictly better off with a manipulation that gives him school s, then we call school s strategically accessible to this student at preference relation P_j. If, regardless of other students' preferences, there is no profitable manipulation that gives this student school s, then we call such school not strategically accessible to student j at preference relation P_j.

The *set of strategically accessible schools* to student j at preference relation P_j via some mechanism φ we denote by $S_j(P_j, \varphi)$.

When the set of strategically accessible schools is empty, we can be sure about the student's incentives. Specifically, if $S_j(P_j, \varphi) = \emptyset$, then we say that j has a *truthful* *dominant strategy* P_j, meaning that in the preference-revelation game induced by φ at the environment $(\succ, q)$, P_j is a j's weakly dominant strategy.

We are ready to define our comparison criterion. If for each student and each preference relation the new mechanism generates the set of strategically accessible schools which is included in the set of strategically accessible schools generated by the old mechanism, then we call the new mechanism at least as robust to manipulations. If, in addition, the set inclusion is strict for at least one agent, then it is more robust to manipulations (for convenience, we sometimes write "more robust" throughout the text).

Definition 2 Mechanism ψ is *at least as robust to manipulations* as mechanism φ if for each environment $(\succ, q)$ whenever school s is strategically accessible to some student j at a preference relation P_j via mechanism ψ, it is also strategically accessible to student j at the preference relation P_j via mechanism φ,

$$\forall j \in J, P_j \in \mathscr{P}\colon S_j(P_j, \psi) \subseteq S_j(P_j, \varphi).$$

Mechanism ψ is *more robust to manipulations* than mechanism φ if mechanism ψ is at least as robust as mechanism, and the converse is not true. That is, there exists a student i, and preference relation P_i, such that some school is strategically accessible to i at P_i via φ, but not via ψ,

$$\exists i \in J, P_i \in \mathscr{P}\colon\ S_i(P_i, \psi) \subsetneq S_i(P_i, \varphi).$$

If mechanism ψ is more robust than mechanism φ, we equivalently say that φ is less robust than ψ.

Criteria Comparison

Now let us discuss the two alternative ways of comparing mechanisms proposed in the literature. The first concept is based on the inclusion of truthful dominant strategies (Decerf and Van der Linden 2021). To compare it with our criterion, we restate the definition of more manipulable mechanisms in the sense of Arribillaga and Massó (2016) using our notations. Informally, the AM-manipulability criterion is the same as robustness but is blind regarding the set of strategically accessible schools; it only notices if this set for a given student with a given preference relation is empty or not.

Definition 3 Mechanism φ is *at least as AM-manipulable* as mechanism ψ if for each environment $(\succ, q)$ whenever student j has a truthful dominant strategy via mechanism φ, he also has a truthful dominant strategy via mechanism ψ,

$$\forall j \in J, P_j \in \mathscr{P}\colon S_j(P_j, \varphi) = \emptyset \Rightarrow S_j(P_j, \psi) = \emptyset$$

Mechanism φ is *more AM-manipulable* than mechanism ψ if mechanism φ is at least as AM-manipulable as mechanism ψ and the converse is not true.

Taking into account the definitions of *more robust* and *at least as AM-manipulable* mechanisms, we obtain the immediate proposition which relates the two concepts. Intuitively, if we replace a mechanism with a more robust one, then each student receives a weakly lower number of strategically accessible schools. Thus, if some student does not have strategically accessible schools initially (so he has a truthful dominant strategy), then under the new mechanism he also can not manipulate profitably.

Remark 1 If mechanism ψ is at least as robust as mechanism φ, then mechanism φ is at least as AM-manipulable as mechanism ψ.[8]

Thus, robustness is a finer criterion than AM-manipulability (in the "at least as" sense)[9] as it measures to what extent a student can potentially manipulate a mechanism. Whenever two mechanisms are comparable via robustness, they are also comparable via AM-manipulability. The reverse is not necessarily true: when two mechanisms are comparable via AM-manipulability, some student might have sets of strategically accessible schools under both mechanisms that are not contained in each other, and thus the mechanisms are not comparable via robustness.

The second method of comparing mechanisms is based on the inclusion of strategically accessible schools (Bonkoungou and Nesterov 2021) but, compared to robustness, disregards the preference relation at which these schools are strategically accessible.

Definition 4 Mechanism φ is *at least as BN-manipulable* as mechanism ψ if for each environment $(\succ, q)$ and for each student j whenever there exists a preference relation P_j such that some school s is strategically accessible at preference relation P_j to student j via mechanism ψ, there exists a preference relation P'_j such that school s is also strategically accessible at preference relation P'_j to student j via mechanism φ,

$$\forall j \in J\colon \bigcup_{P_j \in \mathscr{P}} S_j(P_j, \psi) \subseteq \bigcup_{P_j \in \mathscr{P}} S_j(P_j, \varphi)$$

Mechanism φ is *more BN-manipulable* than mechanism ψ if mechanism φ is at least as BN-manipulable as mechanism ψ and the converse is not true.

Compared to this approach, in the robustness criterion we fix the student's preferences and, as result, obtain a stronger criterion.

Remark 2 If mechanism ψ is at least as robust as mechanism φ, then mechanism φ is at least as BN-manipulable as mechanism ψ.[10]

[8] The converse is not true. Indeed, see Table 1: β^{k+1} is at least as AM-manipulable as β^k, but the same is not true by robustness.

[9] Note: if two mechanisms are comparable by robustness, this does not imply the strict comparison by AM-manipulability.

[10] Indeed, since for each student at fixed preference relation the set of strategically accessible schools via mechanism ψ is weakly included in the set of strategically accessible schools via mechanism φ, the union of the strategically accessible schools over all preference relations via ψ is also included in the union of the strategically accessible schools over all preference relations via φ.

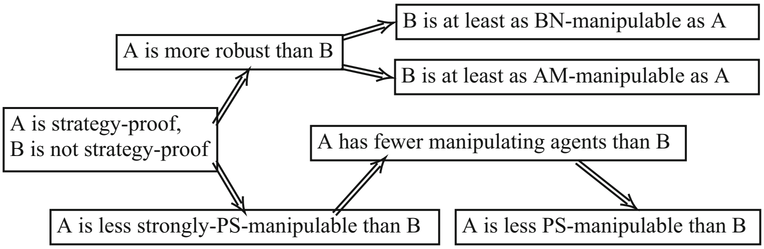

Fig. 1 Logical relations between manipulability criteria. *Notes*. The figure shows different criteria to compare mechanisms A and B, from strongest to weakest. Strong-PS-manipulability and PS-manipulability are defined in Pathak and Sönmez (2013), counting manipulating agents is defined in Bonkoungou and Nesterov (2023), robustness is defined in the present paper, BN-manipulability is defined in Bonkoungou and Nesterov (2021), AM-manipulability is defined in Arribillaga and Massó (2016)

Overall, robustness is stronger than both AM-manipulability and BN-manipulability comparison criteria (in the "at least as" sense) as it combines the distinct features of both. Like AM-manipulability, robustness accounts for all information available to a student—the environment and his preference relation, while in the BN-manipulability the student is assumed to know only the environment. Like BN-manipulability, robustness measures the manipulation possibility precisely by the set of strategically accessible schools, while AM-manipulability only cares whether any strategically accessible school exists or not. See Fig. 1 for the logical relations between these and other manipulability criteria. Next, we define the mechanisms that are to be compared using these criteria.

Mechanisms

Abdulkadiroğlu and Sönmez (2003) describe the mechanism used in Boston at that time to assign students to schools.

1. *Boston mechanism* is also known as the immediate acceptance mechanism. Formally, it is implemented in the following way.

Students submit their preferences: they rank acceptable schools from the most desirable to the least desirable. Schools submit their priorities over students.

- In the 1st round each student sends his request to the school that is ranked at the first position in his preference list. Schools consider the applicants and reject students with lower priority in case the number of entrants exceeds the quota. Other students are accepted to their first choices and can not be rejected in the next rounds.

  …
- In the lth round students remaining unassigned send their requests to the school ranked at the lth position (if any) in their preference list. Schools consider the new applicants and reject students with lower priority in case the number of entrants exceeds the quota. Other students are accepted to the lth choices and can not be rejected in the next rounds.

The algorithm terminates when all students are assigned, or there are some unassigned students who were rejected by every school preferred to the outside option.

We also need to define a constrained version of this mechanism. The constrained school choice was first studied by (Haeringer and Klijn 2009). A mechanism β^k is a Boston mechanism that considers only first k reported schools.

2. *Deferred Acceptance* proposed by Gale and Shapley (1962) has the following realization.

Students submit their preferences: they rank acceptable schools from the most desirable to the least desirable. Schools also submit priorities over students.

- In the 1st round each student sends his request to the most preferred school. Each school *temporarily* accepts students up to its capacity according to its priority relation, which means that in the next rounds these students can be rejected in case students with higher priority apply.

  …
- In the lth round students remaining unassigned send their requests to the next reported school (if any) that has not been considered in their preference list. Schools regard new applicants and previously accepted students and reject those with lower priorities in case the number of entrants exceeds capacity.

The algorithm terminates when all students are assigned, or there are some unassigned students who were rejected by every school better than the outside option.

Similarly, a constrained version of this mechanism, DA^k, proceeds the same way but considers only first k choices.

3. *First-Preference-First* (*FPF*) mechanism is a hybrid between the Boston and the Deferred Acceptance mechanisms. The set of schools is exogenously partitioned into two categories: the *first-preference-first schools* that make their decisions regarding students' acceptance accounting for the order of ranked schools, as in the Boston mechanism, and the *equal preference schools* that do not account for the order of ranked schools and accept students only according to their priorities, as in the Deferred Acceptance. The partitioning of schools is another input of the problem (see Pathak and Sönmez 2013; Bonkoungou and Nesterov 2021 for details). When all schools are first-preference-first schools, then *FPF* is equivalent to β. When all schools are equal preference schools, then *FPF* is equivalent to *DA*. The constrained version is denoted by FPF^k.
4. *Chinese Parallel* mechanism with parameter k ($Ch^{(k)}$)[11] is another hybrid of the deferred and immediate acceptance mechanisms, and it can be seen as a sequential version of constrained Deferred Acceptance mechanism. Students submit their preferences, schools submit their priorities. Initially students are distributed via DA^k. The matched students and seats leave the market, the others proceed to the next round where DA^k is applied again. The process repeats until there is neither vacant seats in schools nor unassigned students who were rejected by all schools. $Ch^{(1)}$ is equivalent to β, $Ch^{(|S|)}$ is equivalent to *DA*.

[11] Note that the parameter k in Chinese Parallel mechanism has a different meaning than in the rest of the mechanisms. For this reason, we write this parameter in parentheses.

Whenever we consider a constrained version of a mechanism, φ^k, or a Chinese Parallel mechanism with parameter k, $Ch^{(k)}$, we assume that there are at least k schools: $|S| \geq k$.

3 Results

We begin by comparing the constrained versions of the Deferred Acceptance mechanism according to the robustness criterion. We show that for each student the set of strategically accessible schools at a fixed preference relation decreases as we relax the constraint.

Theorem 1 *Let* $k > l \geq 1$. *Then* DA^k *is more robust than* DA^l.

Theorem 1 suggests that extension of students' preference lists has decreased the number of schools that students can get through strategic behavior. For the most part, this happened in Chicago (2010), some English cities (2010) and secondary schools in Ghana (2007, 2008), see Pathak and Sönmez (2013).

The same conclusion is applicable to the constrained Boston mechanisms: the increase in the number of considered students' preferences makes the Boston mechanism less manipulable in terms of robustness.

Theorem 2 *Let* $k > l \geq 1$. *Then* β^k *is more robust than* β^l.

Although none of the widely known reforms included replacement of the constrained Boston mechanism with less constrained one, this result is particularly useful to illustrate differences between the criteria for comparing mechanisms. As can be seen from Table 1, Boston mechanisms of different constraint length are indistinguishable by AM- and BN-manipulability (Proposition 2.1), while PS-manipulability and robustness favor the less constrained Boston mechanisms in terms of manipulability. To see whether the difference in manipulability of Boston mechanisms is justified, we provide an example.

Example 1 Suppose there are 3 students and 3 schools. Each school has capacity 1. Consider the following preference profile (truthful preference relations) and the priority relations:

P_1	P_2	P_3	$\succ_s$ $(\forall s \in S)$
s_1	s_2	s_1	3
s_2	$\emptyset$	s_2	1
s_3		s_3	2

Take mechanisms β^2 and β^3. Both are not manipulable by students 2 and 3, but manipulable by student 1: for example, school s_2 can be accessed strategically by putting it as the first choice. Both under β^2 and β^3 there is no truthful dominant strategy for student 1 (given preference relation P_1). The mechanisms are equally AM-manipulable.

As for BN-manipulability, it does not distinguish the mechanisms because the profile can be arbitrary. For instance, schools s_2 and s_3 are strategically accessible to student 1 at the Profile 1 via β^2, but only school s_2—via β^3. However, we can easily construct a new profile by changing s_2 and s_3 in the preferences of all students, so that s_3 is also strategically accessible to student 1 via β^3.

In contrast, the robustness criterion does not enable modification of the true preference relation. Hence, the set of strategically accessible schools for student 1 at preference relation P_1 declines from $\{s_2, s_3\}$ to $\{s_2\}$ as we replace β^2 with β^3.

This example shows that robustness can recognize the difference between two mechanisms when AM- and BN-manipulability cannot. The reason is that robustness criterion is able to somewhat "expand" AM-manipulability and "restrict" BN-manipulability. In situations where the sets of truthful preference relations coincide for the two mechanisms (and hence these are the same in terms of AM-manipulability), robustness can capture the difference by comparing strategically accessible schools at preference relations that are not truthful. Accordingly, in situations where the sets of strategically accessible schools coincide for the two mechanisms (and these are tantamount in terms of BN-manipulability), robustness is capable of catching a more subtle difference by comparing a restricted set of preference profiles—only those where a preference relation of a given student is fixed.

In the late 2000s in most school admission systems of England First-Preference-First mechanisms were replaced with the constrained versions of the Deferred Acceptance mechanism. This reorganization also reduced the number of strategically accessible schools to each student at fixed preference relation. We state this in the next result.

Theorem 3 *Let $k > 1$, and let there be at least one first-preference-first school. Then DA^k is more robust than FPF^k.*

Some English school choice reforms and the reforms in Boston (2005) and Chicago (2009) included the replacement of the (constrained) Boston with the (constrained) Deferred Acceptance. The decisions resulted in more robust mechanisms because β^k is a special case of FPF^k. We obtain the following corollary.

Corollary 1 *Let $k > 1$. Then DA^k is more robust than β^k.*

By 2012, 28 out of 31 provinces in China switched from the Boston mechanism, a special case of the Chinese parallel mechanism with parameter k equal to 1, to different versions of parallel mechanisms with higher parameters. The new mechanisms transpire to be more robust than the old ones.

Theorem 4 *Let $k > l \geq 1$. Then $Ch^{(k)}$ is more robust than $Ch^{(l)}$.*

In the next result we show that when the Chinese parallel mechanisms are not equivalent to the unconstrained Boston and Deferred Acceptance mechanisms, the former are more robust to manipulations than the constrained DA^k.

Theorem 5 *Let $k > 1$. Then $Ch^{(k)}$ is more robust than DA^k.*

Finally, applying Theorems 3 and 5, we get the following corollary.

Corollary 2 *Let $k > 1$. Then $Ch^{(k)}$ is more robust than FPF^k.*

AM-manipulability and BN-manipulability

Our criterion enables ranking a wide range of mechanisms according to robustness to manipulations. Since *more robust* implies *at least as AM- and BN-manipulable*, it is natural to expect similar results of mechanism comparison by these two criteria. Moreover, Remarks 1 and 2 make the proofs relatively easy: it suffices either to construct an example to show the strict inclusion, or prove that it is impossible. As already shown in the original papers (Table 1), the ranking of most mechanisms is the same according to the three criteria. There are two exceptions. The first one occurs between Boston mechanisms of different constraints, the second one—between "long" Boston and "short" DA. We complement the comparisons by AM- and BN-manipulability with mechanisms that are not compared in the two papers, using the implications of our criterion. We also compare β^l and DA^k $(l > k)$ by robustness as a complement to the corresponding comparisons by PS- and AM-manipulability in Decerf and Van der Linden (2021). These findings are summarized in the following propositions.

Proposition 1 *Let $k > 1$. Then,*

1. *FPF^k is more AM-manipulable than DA^k;*
2. *FPF^k is more AM-manipulable than $Ch^{(k)}$.*

Proposition 2 *Let $k > 1$. Then,*

1. *if no two schools can accept all students, β^k is equivalent to β^l in terms of BN-manipulability for each $k > l > 1$;*
2. *DA^k is more BN-manipulable than $Ch^{(k)}$;*
3. *FPF^k is more BN-manipulable than $Ch^{(k)}$.*

Proposition 3 *Let $l > k > 1$. Then,*

1. *β^l and DA^k are not comparable by robustness;*
2. *if no two schools can accept all students, and for each school all students are acceptable, β^l is more BN-manipulable than DA^k.*

Finally, for completeness, we provide the results for PS-manipulability. The criterion introduced by Pathak and Sönmez (2013) is independent of robustness to manipulations. We first restate the main definitions using our notations.

Definition 5 Consider an environment $(\succ, q)$. School s is *strategically accessible* to student j *at preference profile* $\boldsymbol{P}$ via mechanism φ, $s \in S_j(P, \varphi)$, if there exists a manipulation P'_j such that

$$\varphi_j(P'_j, P_{-j}) = s \; P_j \; \varphi_j(P).$$

Definition 6 Consider an environment $(\succ, q)$. A preference profile P is *vulnerable* under mechanism φ if for some student j the set of strategically accessible schools at this profile is non-empty:

$$S_j(P, \varphi) \neq \emptyset.$$

Definition 7 Mechanism φ is *at least as PS-manipulable* as mechanism ψ if for each environment $(\succ, q)$ whenever a preference profile P is vulnerable via mechanism ψ, it also is vulnerable via φ,

$$\exists j \in J \colon S_j(P, \psi) \neq \emptyset \Rightarrow \exists i \in J \colon S_i(P, \varphi) \neq \emptyset.$$

Mechanism φ is *more PS-manipulable* than mechanism ψ if mechanism φ is at least as PS-manipulable as mechanism ψ, and the converse is not true.

As can be seen from Table 1, the results obtained earlier for PS-manipulability go along with our comparisons. The only exception is the First-Preference-First mechanism that can not be compared to the Deferred Acceptance mechanism by the inclusion of vulnerable preference profiles. We complement these findings with the other two results: first, DA^k appears to be more PS-manipulable than $Ch^{(k)}$, analogous to Theorem 5; second, PS-manipulability appears to be a too strong criterion to compare FPF^k with $Ch^{(k)}$.

Proposition 4 *Let $k \geq 2$. Then,*

(1) *If $|S| > k$, DA^k is more PS-manipulable than $Ch^{(k)}$;*
(2) *$Ch^{(k)}$ and FPF^k are not comparable by PS-manipulability.*

Finally, let us note that robustness and PS-manipulability are logically independent. PS-manipulability does not follow from robustness because, for instance, FPF^k and DA^k are comparable via robustness but not via PS-manipulability. Robustness does not follow from PS-manipulability because, for instance, AM-manipulability follows from robustness but is independent from PS manipulability (Decerf and Van der Linden 2021).

4 Discussion and conclusion

In this paper, we presented the new criterion to compare school choice mechanisms in terms of manipulability—robustness to manipulations. Robustness is logically stronger than two of the earlier established concepts (in their weak forms)—AM-manipulability and BN-manipulability—and is independent from PS-manipulability.

Perhaps surprisingly, while robustness refines AM-manipulability and BN-manipulability, there are no logical relations in their comparisons: mechanisms comparable via robustness may be indistinguishable via AM or BN, while mechanisms comparable via AM or BN may not be comparable via robustness (see Table 1). One can also argue that robustness produces more natural comparisons. First, the constrained Boston mechanism has the same level of AM- or BN-manipulability regardless of the constraint. But for robustness, the longer the constraint is, the more robust the Boston mechanism is, same as for PS-manipulability. Second, a more constrained Boston mechanism is less AM- and BN-manipulable than the Deferred Acceptance mechanism with a more relaxed constraint, but they are not comparable via robustness, again same as for PS-manipulability. This is natural as switching from immediate acceptance in Boston to deferred acceptance in DA reduces manipulability (by any criterion), while shortening the constraint in Deferred Acceptance increases manipulability (by any criterion), thus doing these changes together should produce an ambiguous result.

How are the four criteria different at a higher level of analysis? Very roughly, one can highlight two related dimensions, in which the criteria differ one from another: easiness of manipulation and manipulation damage.

Easiness of manipulation refers to the amount of information that agents need to manipulate and is related to the likelihood and the scope of manipulation. In PS-manipulability, the agent is assumed to possess all information: the environment and the preference profile. In robustness and AM-manipulability, the agent knows the same except for preferences of other agents. In BN-manipulability, the agent knows only the environment. Reality is more complex: some information on the environment might not be available (e.g., capacity), but some information on the others' preferences might be at hand (e.g., the expected popularity of schools). A more realistic criterion would allow for this partial information structure.

Manipulation damage refers to the potential negative effect of manipulations once they take place. Ideally, the criterion would measure the number of students that might get worse off and how much, but no criterion goes that far. Robustness and BN-manipulability come closer than the other two criteria: they measure the set of schools that become accessible and thus potentially affected via manipulation.

To conclude, robustness to manipulations complements earlier criteria and corresponding results and allows policy-makers to compare school choice mechanisms using one more reasonable metric, yet there is room for more realistic and more universal criteria.

Appendix: Proofs

Let us first introduce some useful notation. For convenience, we enumerate schools with lowercase indices according to the preference relation of student j: $s_1 \; P_j \; s_2 \; P_j \; \ldots \; P_j \; s_k \; P_j \; \ldots$

In many proofs we work with preference profiles which use similar preference relations. For convenience, we place these preference relations here and refer to them throughout the proofs.

$$\begin{array}{ccc} P_j & P_j^{'} & P_{i \; (\forall i \neq j)} \\ \hline s_1 & s_1 & s_1 \\ \vdots & \vdots & \vdots \\ s_k & s_{k+1} & s_k \\ s_{k+1} & s_k & s_{k+1} \\ \emptyset & \emptyset & \emptyset \end{array} \tag{1}$$

$$\begin{array}{ccc} P_j & P_j^{'} & P_{i \; (\forall i \neq j)} \\ \hline s_1 & s_1 & s_1 \\ \vdots & \vdots & \vdots \\ s_{k-1} & s_k & s_{k-1} \\ s_k & s_{k-1} & s_k \\ \emptyset & \emptyset & \emptyset \end{array} \tag{2}$$

Furthermore, in most proofs the schools' priorities are such that student j is the least preferred:

$$\begin{array}{c} \succ_s \ (\forall s \in S) \\ \hline 1 \\ 2 \\ \vdots \\ j \end{array} \tag{3}$$

Proof of Theorem 1 First, we show that DA^{k+1} is at least as robust as DA^k. Second, we show that DA^{k+1} is more robust than DA^k.

Suppose that school s is strategically accessible to student j at preference relation P_j via mechanism DA^{k+1}. By definition, there exist a manipulation P'_j and a list of preferences P_{-j} such that $DA^{k+1}_j(P'_j, P_{-j}) = s \ P_j \ DA^{k+1}_j(P_j, P_{-j})$. Since $DA^{k+1}(P) = DA(P^{k+1})$[12] is strategy-proof, school s is ranked in the preference relation P_j as the $(k+2)$th choice or with a higher number.[13] Hence, $DA^{k+1}_j(P_j, P_{-j}) = \emptyset$. Since $DA^{k+1}_j(P'_j, P_{-j}) = s$, in manipulation P'_j school s is reported as one of $\{1, \ldots, k+1\}$ first choices. We next use the outcomes of $DA^{k+1}(P_j, P_{-j})$ and construct a list of preferences P^*_{-j} the following way:

- For each student i (not j) such that $DA^{k+1}_i(P_j, P_{-j}) \neq \emptyset$ fill their first row with the outcome $\mu_i = DA^{k+1}_i(P_j, P_{-j})$. Fill their second row with the outside option ($\emptyset$).
- For each student h (not j) who were unassigned, $DA^{k+1}_h(P_j, P_{-j}) = \emptyset$, keep their preferences the same as in P_{-j}.

P'_j	P^*_i	$P^*_h = P_h$
s_1	μ_i	
$\vdots$	$\emptyset$	
s		
$\vdots$		
s_{k+1}		
$\vdots$		

We claim that given such preference list P^*_{-j}, school s must also be strategically accessible to student j at preference relation P_j via mechanism DA^k:

- If student j reports his true preference relation P_j, he remains unassigned via mechanism DA^k, given P^*_{-j}. Since $DA^{k+1}_j(P_j, P_{-j}) = \emptyset$, then schools s_1, s_2, ...,

[12] By P^{k+1} we denote a preference profile that consists of the first $k+1$ rows of the preference profile P.

[13] Lemma 1, Bonkoungou and Nesterov (2021).

s_{k+1} fill their full capacities via $DA^{k+1}(P_j, P_{-j})$. Hence, via $DA^k(P_j, P^*_{-j})$ these schools fill their full capacities with the same students in the first round. Hence, $DA^k_j(P_j, P^*_{-j}) = \emptyset$.

- If student j reports school s as one of the $\{1, \ldots, k\}$ first choices, he obtains s for certain. Since $DA^{k+1}_j(P'_j, P_{-j}) = s$, by the end of the algorithm there were no more than $q_s - 1$ students with higher priorities than j who were accepted to this school. Thus, no more than $q_s - 1$ students with higher priority than j put s in their first row in P^*_{-j}. So in DA^k there always exist a place for student j in school s, given P^*_{-j}. Thus, there exists a manipulation P''_j of student j such that $DA^k_j(P''_j, P^*_{-j}) = s$.

Hence, school s is strategically accessible to student j at preference relation P_j via mechanism DA^k. This proves that mechanism DA^{k+1} is at least as robust as mechanism DA^k.

Now, suppose there are $k + 1$ students ($k > 0$) and $k + 1$ schools with capacities equal to 1. Consider the preference profiles with preferences described in 1 and the priority relation 3.

Then, $DA^k_j(P) = \emptyset$, but $DA^k_j(P^{'}_j, P_{-j}) = s_{k+1}$, so s_{k+1} is strategically accessible to student j at P_j via DA^k. At the same time, there is no strategically accessible school to student j at P_j via DA^{k+1}, which is equivalent to unconstrained version of the DA, because this mechanism is strategy proof. Thus, school s_{k+1} is strategically accessible to student j at preference relation P_j via DA^k, but not via DA^{k+1}.

Then, by induction, for each $k > l$: $\forall j \in J,\ P_j \in \mathscr{P}: S_j(P_j, DA^k) \subseteq S_j(P_j, DA^l)$ & $\exists i \in J,\ P_i \in \mathscr{P}: S_i(P_i, DA^k) \subsetneq S_i(P_i, DA^l)$, so DA^k is more robust than DA^l. □

Proof of Theorem 2 First, we show that β^{k+1} is at least as robust as β^k. Second, we show that β^{k+1} is more robust than β^k.

Suppose school s is strategically accessible to student j at preference relation P_j via β^{k+1}. By definition, there exist a manipulation P'_j and a list of preferences P_{-j} such that $\beta^{k+1}_j(P'_j, P_{-j}) = s\ P_j\ \beta^{k+1}_j(P_j, P_{-j})$. Without loss of generality assume that school s is ranked as the first choice in the report P'_j. Since $\beta^{k+1}_j(P'_j, P_{-j}) = s$, no more than $q_s - 1$ students with higher priority than student j ranked school s as their first choice in P_{-j}. Hence, $\beta^k_j(P'_j, P_{-j}) = s$, as school s has a place for student j in the first round. Moreover, since $s\ P_j\ \beta^{k+1}_j(P_j, P_{-j})$, we have two cases to consider:

- If $\beta^{k+1}_j(P_j, P_{-j}) = \emptyset$, then schools $s_1, s_2, \ldots, s_k$ fill their full capacities via $\beta^{k+1}(P_j, P_{-j})$. Hence, these schools fill their full capacities via $\beta^k(P_j, P_{-j})$. So, $\beta^k_j(P_j, P_{-j}) = \emptyset$.
- If $\beta^{k+1}_j(P_j, P_{-j}) = s_m$, then $s\ P_j\ s_m\ P_j\ s_{k+2}$. All schools $s_1, \ldots, s_{m-1}$ fill their full capacities via $\beta^{k+1}(P_j, P_{-j})$. Hence, these schools fill their full capacities via $\beta^k(P_j, P_{-j})$. So, either $\beta^k_j(P_j, P_{-j}) = s_m$ or $\beta^k_j(P_j, P_{-j}) = \emptyset$.

In both cases, $\beta^k_j(P'_j, P_{-j}) = s\ P_j\ \beta^k_j(P_j, P_{-j})$, so school s is strategically accessible to student j at preference relation P_j via mechanism β^k. Thus, mechanism β^{k+1} is at least as robust as mechanism β^k.

Now, suppose there are $k+1$ students ($k>0$) and $k+1$ schools with capacities equal to 1. Again consider the preference relations 1 and the priority relation 3.

Then, $\beta_j^k(P)=\emptyset$, but $\beta_j^k(P_j^{'}, P_{-j}) = s_{k+1}$, so s_{k+1} is strategically accessible to student j at P_j via β^k. Note that $\beta_j^{k+1}(P)\neq\emptyset$. Thus, it can not be that $s_{k+1}\ P_j\ \beta_j^{k+1}(P)$, so school s_{k+1} is not strategically accessible for student j at preference relation P_j via mechanism β^{k+1}.

Then, by induction, for each $k>l$: $\forall j\in J,\ P_j\in\mathscr{P}$: $S_j(P_j,\beta^k)\subseteq S_j(P_j,\beta^l)$ & $\exists i\in J,\ P_i\in\mathscr{P}$: $S_i(P_i,\beta^k)\subsetneq S_i(P_i,\beta^l)$, so β^k is more robust than β^l. □

Proof of Theorem 3 First, we show that DA^k is at least as robust as FPF^k. Second, we show that DA^k is more robust than FPF^k.

Suppose that school s is strategically accessible to student j at preference relation P_j via mechanism DA^k. By definition, for a given preference relation P_j there exist P_j', P_{-j} such that $DA_j^k(P_j', P_{-j}) = s\ P_j\ DA_j^k(P_j, P_{-j})$. Since $DA^k(P)=DA(P^k)$ is strategy-proof, school s is ranked in the preference relation P_j as the $(k+1)$th choice or with a higher number. Hence, $DA_j^k(P_j, P_{-j})=\emptyset$. Since $DA_j^k(P_j', P_{-j})=s$, in manipulation P_j' school s is reported as one of the $\{1,\ldots,k\}$ first choices. Based on the outcomes of $DA^k(P_j, P_{-j})$, we construct the following profile P_{-j}^* (keeping P_j fixed):

- For each student i (not j) such that $DA_i^k(P_j, P_{-j})\neq\emptyset$ fill their first row with the outcome $\mu_i = DA_i^k(P_j, P_{-j})$. Fill their second row with the outside option (Ø).
- For each student h (not j) who were unassigned, $DA_h^k(P_j, P_{-j})=\emptyset$, keep their preferences the same as in P_{-j}.

P_j''	P_i^*	$P_h^*=P_h$
s	μ_i	
Ø	Ø	

Next, let a manipulation P_j'' be such that s is the only acceptable school. We claim that $FPF_j^k(P_j'', P_{-j}^*) = s\ P_j\ FPF_j^k(P_j, P_{-j}^*)$.

(1) Since $DA_j^k(P_j, P_{-j})=\emptyset$, then schools $s_1,\ldots,s_k$ fill their full capacities via $DA^k(P_j, P_{-j})$. Hence, all these schools fill their full capacities via $FPF^k(P_j, P_{-j}^*)$ with the same students in the first round, as all these students put the outcomes of $DA^k(P_j, P_{-j})$ in their first rows in P_{-j}^*.

(2) Since $DA_j^k(P_j', P_{-j})=s$, then by the end of $DA^k(P_j', P_{-j})$ no more than q_s-1 students had higher priority to school s than student j, and listed them as their first k choices. Then, no more than q_s-1 students with higher priority than student j put s in P_{-j}^* in the first k rows.

Now consider $FPF^k(P_j'', P_{-j}^*)$:

Case 1 School s—a *first-preference-first* school. Due to (1), if student j reports his true preferences, he remains unassigned. $FPF_j^k(P_j, P_{-j}^*)=\emptyset$. Due to (2), no more than $q_s - 1$ students with higher scores put school s as the first choice in P^*_{-j}. Thus, in the first round student j obtains school s with P''_j for certain.

Case 2 School s—an *equal preference* school. Due to (1), if student j reports his true preferences, he remains unassigned. $FPF^k_j(P_j, P^*_{-j}) = \emptyset$. Due to (2), no more than $q_s - 1$ students with higher scores put school s in P^*_{-j}. Thus, during the algorithm student j must get it for certain with P''_j.

Thus, $FPF^k_j(P_j, P^*_{-j}) = \emptyset$, $FPF^k_j(P''_j, P^*_{-j}) = s$, which proves that school s is strategically accessible to student j at P_j via FPF^k. So, mechanism DA^k is at least as robust as mechanism FPF^k.

Now, suppose there are $k > 1$ schools with capacities equal to 1 and $k+1$ students. Also assume that at least one school, for example s_k, is first-preference-first. Consider preferences in 2 and priorities 3.

Then, $FPF^k_j(P) = \emptyset$, all schools reach their full capacity. However, $FPF^k_j(P'_j, P_{-j}) = s_k$, which means that s_k is strategically accessible to student j at P_j via FPF^k.

At the same time, school s_k is not strategically accessible to student j at P_j via DA^k, which is equivalent to unconstrained DA, because this mechanism is strategy proof.

We get that $\forall j \in J,\ P_j \in \mathscr{P}: S_j(P_j, DA^k) \subseteq S_j(P_j, FPF^k)\ \&\ \exists i \in J,\ P_i \in \mathscr{P}: S_i(P_i, DA^k) \subsetneq S_i(P_i, FPF^k)$. □

Proof of Theorem 4. **Lemma 1.** *(Chen and Kesten, 2017). Let k be given. Let P be a preference profile, j —a student, s—a school and P^s_j —a preference relation in which student j has ranked school s first.*

(a) *Suppose that student j is matched to school s under $Ch^{(k)}(P_j, P_{-j})$. Then he is also matched to school s under $Ch^{(k)}(P^s_j, P_{-j})$.*
(b) *Suppose that student j prefers school s to his matching under $Ch^{(k)}(P_j, P_{-j})$* and has ranked it among his top k schools under P. Then he can not obtain a seat at school s by misrepresenting his preferences.

First, we show that $Ch^{(k+1)}$ is at least as robust as $Ch^{(k)}$. Second, we show that $Ch^{(k+1)}$ is more robust than $Ch^{(k)}$.

Suppose that school s is strategically accessible to student j at P_j via $Ch^{(k+1)}$ at P_j, i.e. for a given preference relation P_j there exist P'_j and P_{-j} such that $Ch^{(k+1)}_j(P'_j, P_{-j}) = s\ P_j\ Ch^{(k+1)}_j(P_j, P_{-j}) = \mu_j$.

By Lemma 1, school s is ranked in preference relation P_j as the $(k+2)$th choice or with a higher number. Since school, obtained by student j without manipulation, μ_j (or the outside option), is less preferred by student j than s, this school (or the outside option) must be ranked lower in his preference relation than s. By Lemma 1, we can let $P'_j = P^s_j$. There are two cases how student j could be admitted to school s with the report P'_j:

Case 1 At least one student among those who were matched to school s under $Ch^{(k+1)}(P_j, P_{-j})$ has lower priority than student j under $\succ_s$. Based on the outcomes of $Ch^{(k+1)}(P_j, P_{-j})$, we construct a list of preferences P^*_{-j} the following way:

- For each student i (not j) such that $Ch_i^{(k+1)}(P_j, P_{-j}) \neq \emptyset$ fill their first row with the outcome $\mu_i = Ch_i^{(k+1)}(P_j, P_{-j})$. Fill their second row with the outside option ($\emptyset$).
- For each student h (not j) who were unassigned, $Ch_h^{(k+1)}(P_j, P_{-j}) = \emptyset$, keep their preferences the same as in P_{-j}.

P'_j	P^*_i	$P^*_h = P_h$
s	μ_i	
$\vdots$	$\emptyset$	
s_{k+1}		
$\vdots$		

We claim that given such preference list P^*_{-j}, school s must also be strategically accessible to student j at preference relation P_j via mechanism $Ch^{(k)}$:

- If student j reports his true preference relation P_j, he remains with μ_j via mechanism $Ch^{(k)}$, given P^*_{-j}. Since $Ch_j^{(k+1)}(P_j, P_{-j}) = \mu_j = s_m$, then schools s_1, s_2, ..., $s_{k+1}, \dots, s_{m-1}$ fill their full capacities via $Ch^{(k+1)}(P_j, P_{-j})$. Hence, via $Ch^{(k)}(P_j, P^*_{-j})$ these schools fill their full capacities with the same students in the first round. Hence, $s\ P_j\ Ch_j^{(k)}(P_j, P^*_{-j})$.
- If student j reports $P'_j = P^s_j$, he obtains school s for certain. Since $Ch_j^{(k+1)}(P'_j, P_{-j}) = s$, by the end of the algorithm there were no more than $q_s - 1$ students with higher priorities than j who were accepted to this school. Thus, no more than $q_s - 1$ students with higher priorities than j put s in their first row in P^*_{-j}. So in $Ch^{(k)}$ there always exists a place for student j in school s, given P^*_{-j}. Thus, $Ch_j^{(k)}(P^s_j, P^*_{-j}) = s$.

Case 2 Every student among those who were matched to school s under $Ch^{(k+1)}(P_j, P_{-j})$ has higher priority than j under $\succ_s$. Then, at least one such student has ranked school s as his $(k+2)$th choice or with a higher number. Based on the outcomes of $Ch^{(k+1)}(P_j, P_{-j})$, we construct a list of preferences P^*_{-j} in the following way:

- For each student i (not j) such that either $Ch_i^{(k+1)}(P_j, P_{-j}) \neq \emptyset$ and $Ch_i^{(k+1)}(P_j, P_{-j}) \neq s$, or $Ch_i^{(k+1)}(P_j, P_{-j}) = s$ and s is ranked among first $k+1$ schools in P_i, fill their first row with the outcome $\mu_i = Ch^{(k+1)}(P_j, P_{-j})$. Fill their second row with the outside option ($\emptyset$).
- For each other student h (not j), keep their preferences the same as in P_{-j}. There exist some student h' that keeps ranking school s as $(k+2)$th choice or with a higher number (and who has higher priority than j to s). Denote such student h'.

P'_j	$P^*_{h'}$	P^*_i	$P^*_h = P_h$
s	$\vdots$	μ_i	
$\vdots$	$\vdots$	$\emptyset$	
s_{k+1}	$\vdots$		
$\vdots$	s		

We claim that given such preference list P^*_{-j}, school s must also be strategically accessible to student j at preference relation P_j via mechanism $Ch^{(k)}$:

- If student j reports his true preference relation P_j, he remains with $\mu_j = s_m$ via mechanism $Ch^{(k)}$, given P^*_{-j}. Since $Ch_j^{(k+1)}(P_j, P_{-j}) = s_m$, then schools $s_1, s_2, \ldots, s_{k+1}, \ldots, s_{m-1}$ fill their full capacities via $Ch^{(k+1)}(P_j, P_{-j})$. Hence, via $Ch^{(k)}(P_j, P^*_{-j})$ these schools fill their full capacities with the same students in the first round. Hence, $s\ P_j\ Ch_j^{(k)}(P_j, P^*_{-j})$.
- If student j reports $P'_j = P^s_j$, he obtains school s for certain. Since $Ch_j^{(k+1)}(P'_j, P_{-j}) = s$, by the end of the algorithm there were no more than $q_s - 1$ students with higher priorities than j who were accepted to this school. Thus, no more than $q_s - 1$ students with higher priorities than j put s in their first row in P^*_{-j}. So, in $Ch^{(k)}$ there exists a place for student j in school s, given P^*_{-j}, in the first k rounds, since students h' with higher priority under $\succ_s$ than j put this school as their $(k+2)$th first choice or higher, so they will not be considered by school s in the first k rounds. Thus, $Ch_j^{(k)}(P^s_j, P^*_{-j}) = s$.

In both cases school s is strategically accessible to student j at preference relation P_j via $Ch^{(k)}$. Hence, $Ch^{(k+1)}$ is at least as robust as $Ch^{(k)}$.

Now, suppose there are $k+2$ students ($k > 0$) and $k+1$ schools with capacity equal to 1. Let us consider the preferences 1 and schools' priorities 3:

Then, $Ch_j^{(k)}(P) = \emptyset$, while $Ch_j^{(k)}(P'_j, P_{-j}) = s_{k+1}$. At the same time, s_{k+1} is not strategically accessible to student j at P_j via $Ch^{(k+1)}$, because in this case the mechanism is equivalent to *DA* which is strategy-proof. So, s_{k+1} is not strategically accessible to student j at P_j under $Ch^{(k+1)}$ but under $Ch^{(k)}$. Then, by induction, for each $k > l$: $\forall j \in J,\ P_j \in \mathscr{P}$: $S_j(P_j, Ch^{(k)}) \subseteq S_j(P_j, Ch^{(l)})$ & $\exists i \in J,\ P_i \in \mathscr{P}$: $S_i(P_i, Ch^{(k)}) \subsetneq S_i(P_i, Ch^{(l)})$, so $Ch^{(k)}$ is more robust than $Ch^{(l)}$. □

Proof of Theorem 5. First, we show that mechanism $Ch^{(k)}$ is at least as robust as DA^k. Second, we show that $Ch^{(k)}$ is more robust than DA^k.

Suppose that school s is strategically accessible to j at P_j via $Ch^{(k)}$. This means that there exists a manipulation P'_j and a list of preferences P_{-j} such that

$$Ch_j^{(k)}(P'_j, P_{-j}) = s\ P_j\ Ch_j^{(k)}(P_j, P_{-j}) = \mu_j.$$

The true preference relation P_j is such that school s is not listed among first k schools, because otherwise the student could not obtain s via a manipulation, as in the first k rounds $Ch^{(k)}$ is equivalent to $DA^k = DA(P^k)$ which is strategy-proof. Thus, school s is listed by student j as his $(k+1)$th choice or with a higher number in P_j.

Since at some profile $P = (P_j, P_{-j})$ student j can profitably manipulate to obtain s, he can do it by listing s the first. If he can not do this, then, school s will reach its full capacity at the first round with at least one student that has higher priority than student j. Then, student j can not obtain school s with another manipulations. Thus, we assume further that manipulation P'_j such that s is listed the first and $Ch_j^{(k)}(P'_j, P_{-j}) = s$.

Now we construct a list of preferences P^*_{-j} based on the outcomes of $Ch^{(k)}$ (P_j, P_{-j}):

- Every student who is assigned to school s via $Ch^{(k)}(P_j, P_{-j})$ and every student who is unassigned via $Ch^{(k)}(P_j, P_{-j})$ lists school s_1 as his first choice, and the outside option $\emptyset$ as the second choice.
- Every student i who is assigned to some school $s_i \neq s$ via $Ch^{(k)}(P_j, P_{-j})$ lists this school s_i as his first choice, and the outside option $\emptyset$ as the second choice.

Next, we pick this profile $P^* = (P_j, P^*_{-j})$, conduct the DA^k mechanism, and show that school s is also strategically accessible to student j at P_j.

Case 1 student j reports his true preferences.

Round 1: all students who were assigned to s_1 via $Ch^{(k)}(P_j, P_{-j})$ put s_1 as their first choice in P^*_{-j} and either obtain it via $DA^k(P_j, P^*_{-j})$ or remain unassigned (e.g. in case a person assigned to s in via $Ch^{(k)}(P)$ put s_1 and has a higher priority). Since $Ch_j^{(k)}(P_j, P_{-j}) \neq s_1$, in $Ch^{(k)}(P_j, P_{-j})$ during the first DA^k there are more students with higher priority to s_1 than j who are assigned to this school so that there is no remaining place to student j. Hence, in the first round of $DA^k(P_j, P^*_{-j})$ the same students apply to s_1, so student j is rejected by s_1. Round 2: the only applicant is student j. He is rejected by s_2—his second choice, because by the second round of $Ch^{(k)}(P_j, P_{-j})$ this school has reached its full capacity with more priority students, and in P^* these students listed s_2 as their first choices.

…

Round k: the only applicant is student j. He is rejected by s_k—his kth choice, because by the second round of $Ch^{(k)}(P_j, P_{-j})$ this school has reached its full capacity with students of higher priority, and in P^* these students listed s_k as their first choices.

Thus, with true preferences student j becomes unassigned via DA^k at the profile P^*: $DA^k(P_j, P^*_{-j}) = \emptyset$.

Case 2 Student j reports the manipulation P'_j.

Round 1: student j sends an application to school s and obtains it, as nobody listed this school the first.

Rounds 2-k: no new applicants.

Thus, via the manipulation P_{-j} student j obtains s under DA^k at the profile P^*: $DA^k(P'_j, P^*_{-j}) = s$.

Case 1 and Case 2 imply that school s is strategically accessible to student j at P_j given mechanism DA^k. Hence, $Ch^{(k)}$ is at least as robust as DA^k.

Now, we show that schools that are not strategically accessible to student j at P_j via $Ch^{(k)}$ can be strategically accessible to j at P_j via DA^k (when $k < |S|$, otherwise the two mechanisms are equivalent). To illustrate this, we provide an example.

Suppose there are $k+1$ students ($k > 0$) and $k+1$ schools with capacities equal to 1. Consider preferences given by 1 and the priorities 3.

Then, $DA^k_j(P) = \emptyset$, but $DA^k_j(P^{'}_j, P_{-j}) = s_{k+1}$, so s_{k+1} is strategically accessible to student j at P_j via DA^k. Then, note that $Ch^{(k)}_j(P) \neq \emptyset$ (Otherwise, there exists a school without student, which means that no student including j has applied to this school. This contradicts to preference relation of student j who applied to all schools). Besides, s_{k+1} can be strategically accessible to student j only in the case $s_{k+1}\ P_j\ Ch^{(k)}_j(P)$. The last is true only when $Ch^{(k)}_j(P) = \emptyset$. This contradicts to the previous result. Hence, s_{k+1} is not strategically accessible to student j at P_j via $Ch^{(k)}$, but via DA^k. □

Proof of Proposition 1. (1) By Remark 1 and Theorem 3, FPF^k is at least as AM-manipulable as DA^k. It is left to show that the opposite is not true. Suppose there are $k+1$ students and k schools with capacities equal to 1. Let s_k be a first-preference-first school. Consider preferences given by 2 and the priorities 3.
School s_k is strategically accessible to student j at P_j via FPF^k (e.g. with manipulation P'_j). At the same time, there is no school that is strategically accessible to student j at P_j via DA^k because this mechanism is equivalent to the unconstrained version of the Deferred Acceptance mechanism, which is strategy proof. Hence, student j has a truthful dominant strategy via DA^k, but not via FPF^k.

(2) Immediately follows from Proposition 1.1, and the fact that DA^k is more AM-manipulable than $Ch^{(k)}$ if there are at least $k > 1$ schools (Proposition 7, Decerf and Van der Linden 2021). □

Proof of Proposition 2 (1) By Remark 2 and Theorem 2, β^k is at least as BN-manipulable as β^{k+1}. Let us show that the opposite is also true. In other words, β^k and β^{k+1} are equivalent in terms of BN-manipulability for $k > 1$ (the number of schools is at least 3).

Assume school s is strategically accessible to student j at some P_j via β^k, so there exist P'_j, P_{-j} such that $\beta^k_j(P^{'}_j, P_{-j}) = s\ P_j\ \beta^k_j(P_j, P_{-j})$. Without loss of generality assume that P'_j is such that school s is listed first.

Notice that if $\beta^k_j(P^{'}_j, P_{-j}) = s$, then $\beta^{k+1}_j(P^{'}_j, P_{-j}) = s$, since in the first round of Boston mechanism there are no more than $q_s - 1$ applicants to school s that have higher priority under $\succ_s$ than student j. In case of truthful report P_j (and given the same P_{-j}), the student either has the same assignments under both mechanisms, or student j is unassigned under $\beta^k(P)$, but assigned to school s_{k+1} under $\beta^{k+1}(P)$. Let us consider the two cases separately.

Case 1 Suppose $\beta^{k+1}_j(P) = \beta^k_j(P) = \mu_j$, where $\mu_j \in \{s_3, \ldots, s_k, \emptyset\}$. Note that $\mu_j \neq s_1$, because otherwise s will not be strategically accessible to j at P_j. Also $\mu_j \neq s_2$, as it is impossible to obtain $s_1\ P_j\ s_2$ with any manipulation. Since $\beta^k_j(P^{'}_j, P_{-j}) = s\ P_j\ \mu_j$, we also have $\beta^{k+1}_j(P^{'}_j, P_{-j}) = s\ P_j\ \beta^{k+1}_j(P) = \mu_j$.

Hence, s is also strategically accessible to j at P_j under β^{k+1}. This is true for every $j \in J$ and $P_j \in \mathscr{P}$.

Case 2 Let $\beta_j^{k+1}(P) = s_{k+1}\ P_j\ \beta_j^k(P) = \emptyset$. Here we impose the additional assumption: no two schools can accept all students. We construct a new preference profile $P^* = (P_j^*, P_{-j}^*)$ in the following way:

- P_j^* is the following: school s_1 is ranked first, school s_{k+1} is ranked second, the outside option is ranked third.
- Take the priority relation of school s_1, and choose the top q_{s_1} students under $\succ_{s_1}$. Every such student i put school s_1 as the first choice, school s_{k+1} as the second choice in P_{-j}^*.
- Denote by $\succ'_{s_{k+1}}$ the priority relation of school s_{k+1}, $\succ_{s_{k+1}}$, without top q_{s_1} students under $\succ_{s_1}$. Let T denote the set of the top $q_{s_{k+1}}$ students under $\succ'_{s_{k+1}}$. There are two cases:
  (1) Student j is in T $(j \in T)$. Every student $h \in T\backslash\{j\}$ put school s_{k+1} as their first choice. One student g, who has not been considered yet, put school s_{k+1} as their first choice. Note that such student g always exists, because of the additional assumption.

P_j^*	P_i^*	P_h^*	P_g^*
s_1	s_1	s_{k+1}	s_{k+1}
s_{k+1}	s_{k+1}	$\vdots$	$\vdots$
$\emptyset$	$\vdots$	$\vdots$	$\vdots$

$\succ_{s_1}$	$\succ_{s_{k+1}}$
$\vdots$	$\vdots$
i	j
$\vdots$	$\vdots$
j	g
$\vdots$	$\vdots$

  (2) Student j is not in T $(j \notin T)$. Every student $h \in T$ put school s_1 as their first choice, school s_{k+1} as their second choice.

P_j^*	P_i^*	P_h^*
s_1	s_1	s_1
s_{k+1}	s_{k+1}	s_{k+1}
$\emptyset$	$\vdots$	$\vdots$

$\succ_{s_1}$	$\succ_{s_{k+1}}$
$\vdots$	$\vdots$
i	h
$\vdots$	$\vdots$
j	j
$\vdots$	$\vdots$

- All other students (if any) put s_1 as the first choice, the outside option as the second choice.

We claim that given such profile P^*, school s_{k+1} is also strategically accessible to student j via β^{k+1}. Consider a manipulation of student j such that s_{k+1} is ranked first. P_j'': $s_{k+1}, \ldots$

If (1) follows, then student j is unassigned if he reports his true preferences: $\beta_j^{k+1}(P_j^*, P_{-j}^*) = \emptyset$. Indeed, school s_1 fills its full capacity in the first round with students that have the highest priority under $\succ_{s_1}$. Student j is not among these students, because he was rejected by s_1 in the first round of $\beta^k(P)$. So, in the first round j is rejected by s_1. School s_{k+1} fills its full capacity in the first round, as

$q_{s_{k+1}}$ students put it the first. Hence, in the second round student j is rejected by s_{k+1}. So, $\beta_j^{k+1}(P_j^*, P_{-j}^*) = \emptyset$. If student j reports P_j'', he is accepted to school s_{k+1} in the first round, as $j \in T$. Hence, $\beta_j^{k+1}(P_j'', P_{-j}^*) = s_{k+1}\ P_j^*\ \emptyset$.

If (2) follows, then student j is unassigned if he reports his true preferences: $\beta_j^{k+1}(P_j^*, P_{-j}^*) = \emptyset$. Indeed, school s_1 fills its full capacity in the first round with students that have the highest priority under $\succ_{s_1}$. Student j is not among these students, because he was rejected by s_1 in the first round of $\beta^k(P)$. So, in the first round j is rejected by s_1. School s_{k+1} fills its full capacity in the second round, accepting all students $h \in T$. Since $j \notin T$, student j is rejected by s_{k+1}. So, $\beta_j^{k+1}(P_j^*, P_{-j}^*) = \emptyset$. If student j reports P_j'', he is accepted to school s_{k+1} in the first round, as he is the only applicant to this school. Hence, $\beta_j^{k+1}(P_j'', P_{-j}^*) = s_{k+1}\ P_j^*\ \emptyset$.

So, s_{k+1} is strategically accessible to j at P_j^* via β^{k+1}. Thus, mechanism β^{k+1} is at least as BN-manipulable as β^k.

(2) By Remark 2 and Theorem 5, DA^k is at least as BN-manipulable as $Ch^{(k)}$. It is left to show that the opposite is not true.

Let us refer to the example from the proof of Theorem 5. Suppose there are $k+1$ ($k > 0$) students and $k+1$ schools with capacity 1 each. Consider the preferences given by 1 and the priorities 3.

Recall that school s_{k+1} is strategically accessible to student j at P_j via DA^k, hence it is strategically accessible to j. We show that j has no strategically accessible school via $Ch^{(k)}$.

Consider any preference relation of student j. Then, the first k most preferred schools are not strategically accessible to student j at this preference relation via $Ch^{(k)}$, because during the first k rounds the mechanism works as $DA^k = DA(P^k)$, which makes it impossible to manipulate by the first k preferences. Furthermore, the less preferred school in this preference relation is also not strategically accessible to j via $Ch^{(k)}$. Assume the opposite, then, by definition, student j should receive strictly worse outcome without manipulation, that is no school $\emptyset$. However, since there are $k+1$ rounds in mechanism, $k+1$ students and $k+1$ schools with capacity 1, student j can not end up unassigned, because in his preference relation all schools are preferred to the outside option. So, no school is strategically accessible to j at arbitrarily preference relation via Ch^k. Hence, there are no strategically accessible schools to j via Ch^k, but there are some via DA^k.

(3) Immediately follows from Proposition 2.2 and the fact that FPF^k is more BN-manipulable than DA^k. □

Proof of Proposition 3 (1) To prove that DA^k and β^l ($l > k > 1$) are not comparable according to robustness, we provide environments, in which: (1) school s is strategically accessible to student j at P_j under DA^k, but not under β^l; (2) school s is strategically accessible to student j at P_j under β^l, but not under DA^k.

(1) Let there be l schools, each with capacity 1, and l students. Consider the following preference relations and schools' priorities:

P_j	$P_j^{'}$	$P_{i\ (\forall i \neq j)}$
s_1	s_l	s_1
$\vdots$	$\vdots$	$\vdots$
s_k		s_k
$\vdots$		$\vdots$
s_l		s_l
$\emptyset$		$\emptyset$

$\succ_{s\ (\forall s \in S)}$
1
2
$\vdots$
j

Then $DA_j^k(P_j, P_{-j}) = \emptyset$, but $DA_j^k(P_j', P_{-j}) = s_l$, so school s_l is strategically accessible to j at P_j under DA^k. Under β^l school s_l cannot be strategically accessible to student j at P_j since via β^l with l students (acceptable to all schools) every student must be assigned to some school, so j receives at least s_l if he reports truthfully.

(2) Consider the following preference relations and schools' priorities:

P_j	$P_j^{'}$	P_i	$P_{h\ (\forall h \neq j,i)}$
s_1	s_2	s_1	s_2
s_2	$\vdots$	$\vdots$	s_1
$\vdots$			$\vdots$
s_l			$\vdots$

$\succ_{s\ (\forall s \in S \setminus s_2)}$	s_2
1	j
$\vdots$	$\vdots$
j	

Then $\beta^l(P_j', P_{-j}) = s_2\ P_j\ \beta^l(P_j, P_{-j})$, so s_2 is strategically accessible to j at P_j. School s_2 cannot be strategically accessible to j at P_j because $DA^k(P) = DA^k(P^k)$ (and s_2 is in k first preferences) makes it impossible to manipulate within first k preferences.

(2) Suppose that school s is strategically accessible to student j under DA^k. We want to show that it is also strategically accessible to j under β^l.
First, since s is not strategically accessible to j and for each school each student is better than $\emptyset$ (by assumption), there should be at least k students. Also, school s is not listed among top k preferences of student j. Denote by s_1 student j's most preferred school. Let us construct the following preference profiles:

- Student j: s_1 is the most preferred school, s is the second, s_2 is the third.
- q_{s_1} students, who have higher priorities in s_1 than j (they always exist since s is strategically accessible to j under DA^k): s_1 is the most preferred school.
- Among the remaining students (there are at least q_s remaining students due to the assumption that no two schools have seats for all students), choose $q_s - 1$ students with the least priorities to s. All these students put s as their top choice.

- Among the remaining students: choose one student h with the highest priority to s. If school s prefers j to h, then put s as h's first choice. Otherwise, construct the h's preferences as follows: s_1 is the first, s is the second.
- All the students who are left put s_1 as their first choice, $\emptyset$ as the second.

Under β^l student j is assigned to s_2 if he reports his preferences truthfully. Indeed, s_1 fills all the seats in the first round with students having higher priority to s_1 than student j. School s fills all the places either in the first or in the second round. By construction, student j cannot be accepted to s. If he misreports his preferences by putting s as his first choice then student j is accepted to s, either because there is one vacant seat in the first round or by supplanting a student with lower priority to s. Therefore school s is strategically accessible to student j under β^l.

Now, we provide an example where some school s is strategically accessible to some student j under β^l but not under DA^k. Suppose there are l schools, each with capacity 1 and l students. Consider the following preferences and priority relations as in the proof of Proposition 3 (1), 2).

Then, $\beta^l_j(P'_j, P_{-j}) = s_2 \ P_j \ \beta^l_j(P_j, P_{-j})$, so s_2 is strategically accessible to j under β^l. School s_2 is not strategically accessible to j under DA^k, because whenever s_2 is not in the first k j's preferences, student j obtains some school from the top k preferences under DA^k.

□

Proof of Proposition 4. (1) Assume that some profile P is vulnerable under mechanism $Ch^{(k)}$, that is, there exists student j, school s, and a manipulation $P^{'}_j$ such that $Ch^{(k)}_j(P^{'}_j, P_{-j}) = s \ P_j \ Ch^{(k)}_j(P_j, P_{-j})$. We need to show that this profile P is also vulnerable under mechanism DA^k. In particular, we will show that for the same student j there exists a manipulation $P^{''}_j$ such that $DA^k_j(P^{''}_j, P_{-j}) \ P_j \ DA^k_j(P_j, P_{-j})$.

Since $Ch^{(k)}(P)$ during the first k rounds is equivalent to $DA^k(P) = DA(P^k)$, which is strategy-proof, then school s is ranked in P_j as the $(k+1)$th top choice or with a higher number.

Now, consider $DA^k(P)$. If j reports P_j, he remains unassigned, since schools $s_1, s_2, \ldots, s_k$ rejected student j during the first k rounds of $Ch^{(k)}(P)$. The first k rounds in DA^k are the same. So, $DA^k_j(P) = \emptyset$. Consider P''_j such that school s is ranked the first. Since $Ch^{(k)}_j(P'_j, P_{-j}) = s$, then no more than $q_s - 1$ students with higher priority than j under $\succ_s$ ranked s among their first k top choices. Hence, $DA^k_j(P''_j, P_{-j}) = s \ P_j \ \emptyset$.

So, we have shown that if profile P is vulnerable under mechanism $Ch^{(k)}$, then it is also vulnerable under DA^k.

Now we provide an example, where the opposite is not true. Suppose, there are $k+1$ students (enumerated as 1,2,…,$k+1$) and $k+1$ schools. The preference profile and school priorities are the following:

P_{k+1}	P'_{k+1}	$P_{i\neq k+1}$
s_1	s_{k+1}	s_i
$\vdots$	$\vdots$	$\vdots$
s_k		
s_{k+1}		
$\emptyset$		

$\succ_s$ $(\forall s \in S)$
1
2
$\vdots$
$k+1$

We obtain the following allocations at profile P under mechanisms $Ch^{(k)}$ and DA^k:

$$DA^k(P) = \begin{pmatrix} i & k+1 \\ s_i & \emptyset \end{pmatrix} \quad Ch^{(k)}(P) = \begin{pmatrix} i & k+1 \\ s_i & s_{k+1} \end{pmatrix}$$

Notice that all students, except student $k+1$, obtain his most preferred school under both mechanisms, so they can not benefit by misrepresenting his preferences. As for the student $k+1$, he obtains school s_{k+1} via $Ch^{(k)}$ without manipulation. In other words, he remains unmatched after k rounds. Notice that since the first k rounds of $Ch^{(k)}$ are equivalent to DA^k, the agent does not have a manipulation that gives him a strictly better outcome than s_{k+1}. So, profile P is not vulnerable under $Ch^{(k)}$.

Now assume that student $k+1$ reports preferences P'_{k+1} instead of P_{k+1}:

$$DA^k(P'_{k+1}, P_{-(k+1)}) = \begin{pmatrix} i & k+1 \\ s_i & s_{k+1} \end{pmatrix}$$

Notice that student $k+1$ was unmatched via DA^k at the true preference relation, but obtained school s_{k+1} at manipulation P'_{k+1}. Hence, profile P is vulnerable to manipulations via DA^k but not via $Ch^{(k)}$.

(2) To prove that $Ch^{(k)}$ and FPF^k are not comparable in general we provide two environments, at which: (1) some profile is vulnerable under FPF^1, but not under $Ch^{(1)}$; (2) some profile is vulnerable under $Ch^{(3)}$, but not under FPF^3.

(1) There are 2 students and 2 schools with capacity 1. Both schools are first-preference-first. The preference profile and school priorities are the following:

P_1	P_2	P'_2
s_1	s_1	s_2
s_2	s_2	s_1
$\emptyset$	$\emptyset$	$\emptyset$

$\succ_s$ $(\forall s \in S)$
1
2

We obtain the following allocations at profile P under mechanisms $Ch^{(1)}$ and FPF^1:

$$FPF^1(P) = \begin{pmatrix} 1 & 2 \\ s_1 & \emptyset \end{pmatrix}$$

$$Ch^{(1)}(P) = \begin{pmatrix} 1 & 2 \\ s_1 & s_2 \end{pmatrix}$$

Notice that student 1 obtains his most preferred school under both mechanisms, so he can not benefit by misrepresenting his preferences. As for student 2, he obtains school s_2 via $Ch^{(1)}$ without manipulation. The only school that student 2 prefers to s_2 is s_1. However, no manipulation gives school s_1 to student 2 because the first priority of s_1 is student 1 who reported it as most preferred. Hence, profile P is not vulnerable under $Ch^{(1)}$.
Now assume that student 2 reports preferences $P_2^{'}$ instead of P_2:

$$FPF^1(P_2^{'}, P_{-2}) = \begin{pmatrix} 1 & 2 \\ s_1 & s_2 \end{pmatrix}$$

Notice that student 2 was unmatched via FPF^1 at true preference relation, but obtained school s_2 at manipulation $P_2^{'}$. Hence, profile P is vulnerable to manipulations via FPF^1 but not via $Ch^{(1)}$.

(2) Now let us refer to the Example 1 (Bonkoungou and Nesterov 2021). There are 7 students and 7 schools with capacity 1. The preference profile and school priorities are the following:

$P_1^{'}$	P_1	P_2	P_3	P_4	P_5	P_6	P_7
s_4	s_1	s_1	s_2	s_3	s_5	s_4	s_6
$\emptyset$	s_2	$\emptyset$	$\emptyset$	$\emptyset$	s_7	s_5	s_4
	s_3				$\emptyset$	s_6	$\emptyset$
	s_4					$\emptyset$	
	$\emptyset$						

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$	$\succ_{s_6}$	$\succ_{s_7}$
2	3	4	7	6	6	5
$\vdots$	$\vdots$	$\vdots$	1	5	7	$\vdots$
			6	$\vdots$	$\vdots$	
			$\vdots$			

Assume that school s_5 is the only first-preference-first school. It was shown that profile P is not vulnerable under FPF^3. Now let us check whether it is vulnerable under $Ch^{(3)}$.
We obtain the following allocation at this preference profile under $Ch^{(3)}$:

$$Ch^{(3)}(P) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \emptyset & s_1 & s_2 & s_3 & s_5 & s_4 & s_6 \end{pmatrix}$$

So, each student except student 1 obtains his most preferred school. As a result, these students can not benefit by misrepresenting their preferences.
As for student 1, he ends up without school at preference relation P_1 via $Ch^{(3)}$.
Now, assume that he reports preferences P_1' instead of P_1:

$$Ch^{(3)}(P_1^{'}, P_{-1}) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ s_4 & s_1 & s_2 & s_3 & s_7 & s_5 & s_6 \end{pmatrix}$$

Thus, student 1 obtains s_4 by a manipulation $P_1^{'}$. Hence, profile P is vulnerable under $Ch^{(3)}$, but not under FPF^3.

□

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