Convolution comparison measures

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Convolution $*$

- $(a_0, a_1, \ldots, a_n, \ldots) * (b_0, b_1, \ldots, b_n, \ldots) =$
  $(a_0 b_0, a_0 b_1 + a_1 b_0, \ldots, a_0 b_n + a_1 b_{n-1} + \ldots + a_{n-1} b_1 + a_n b_0, \ldots)$

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- $(f * g)(x) = \int f(y) g(x - y) dy$
- If laws of X and Y are μ and ν, and X and Y are **independent**, then $X + Y$ has law $\mu * \nu$
  - If X and Y are independent dice rolls with laws $d6$, then $X + Y$ has law $d6 * d6$.

Free convolution $\boxplus$

If laws of X and Y are μ and ν, and X and Y are **freely independent**, then $X + Y$ has law $\mu \boxplus \nu$

Free independence

- X and Y are **noncommutative** random variables

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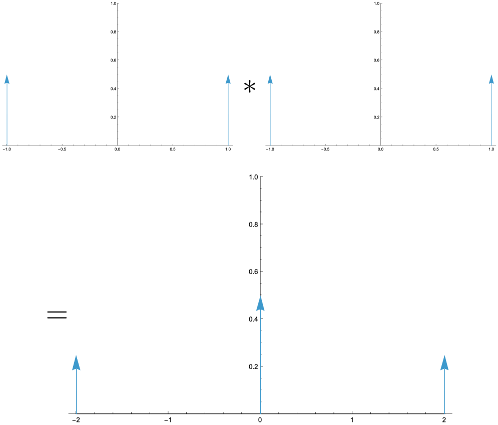
If laws of X and Y are μ and ν, and X and Y are **freely independent**, then $X+Y$ has law $\mu \boxplus \nu$

Rademachers

μ and ν are both $\frac{1}{2}(\delta_{-1} + \delta_1)$



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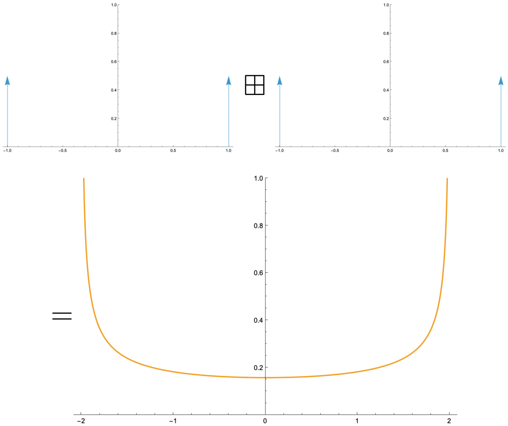

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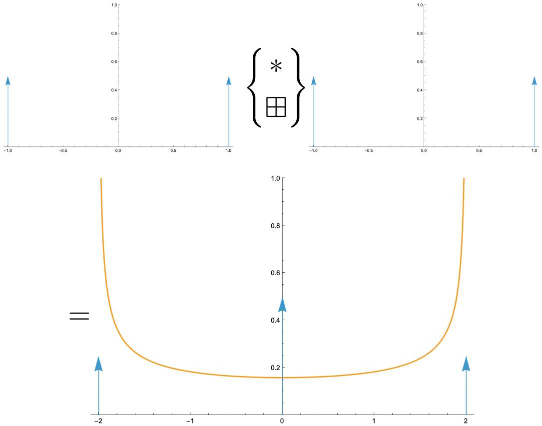
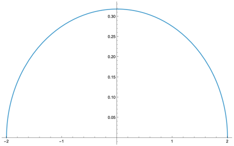
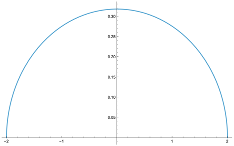


Semicirculars

μ and ν both have density $\frac{1}{2\pi}\sqrt{4-x^2}$ on $[-2,2]$.



$$\left\{ \begin{matrix} * \\ \boxplus \end{matrix} \right\}$$



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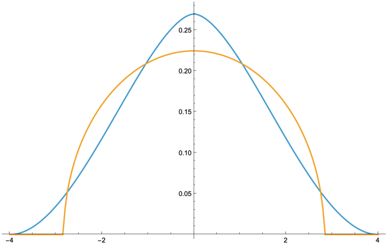
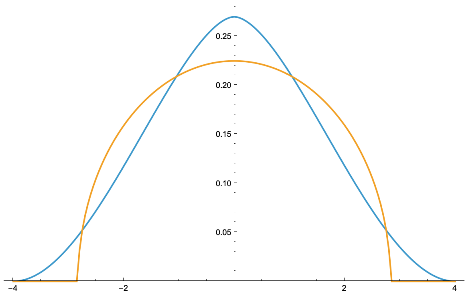


Support inclusion

Always

$$\mathrm{Conv}(\mathrm{supp}(\mu \boxplus \nu)) \subset \mathrm{Conv}(\mathrm{supp}(\mu * \nu))$$



Moment comparison

What about

$$\int_{\mathbb{R}} t^{2k} \, \mathrm{d}(\mu \boxplus \nu)(t) \leq \int_{\mathbb{R}} t^{2k} \, \mathrm{d}(\mu * \nu)(t)?$$

Functional comparison of classical and free convolutions

Theorem (H, 2026)

Let $f \in C^4(\mathbb{R})$ be such that $f^{(4)} \geq 0$. Then for compactly supported μ and ν,

$$\int_{\mathbb{R}} f(t)\,\mathrm{d}(\mu \boxplus \nu)(t) \leq \int_{\mathbb{R}} f(t)\,\mathrm{d}(\mu * \nu)(t). \tag{1}$$

Conversely, if (1) is true for any μ and ν, then $f^{(4)} \geq 0$.

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Since $f^{(4)} \geq 0$ for $f(t) = t^{2k}$, we have

$$\int_{\mathbb{R}} t^{2k}\, \mathrm{d}(\mu \boxplus \nu)(t) \leq \int_{\mathbb{R}} t^{2k}\, \mathrm{d}(\mu * \nu)(t).$$

Special cases

$$\boxed{f^{(4)} \geq 0} \iff \boxed{\int_{\mathbb{R}} f(t)\,\mathrm{d}(\mu \boxplus \nu)(t) \leq \int_{\mathbb{R}} f(t)\,\mathrm{d}(\mu * \nu)(t)}$$

- If $k = 0, 1, 2, 3,$

$$\int_{\mathbb{R}} t^k\,\mathrm{d}(\mu * \nu)(t) = \int_{\mathbb{R}} t^k\,\mathrm{d}(\mu \boxplus \nu)(t).$$

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- If $f(t) = t^4$,

$$\int_{\mathbb{R}} t^4\,\mathrm{d}(\mu * \nu)(t) - \int_{\mathbb{R}} t^4\,\mathrm{d}(\mu \boxplus \nu)(t) = 2\,\mathrm{Var}(\mu)\,\mathrm{Var}(\nu).$$

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- (Arizmendi–Johnston 2023) If $z > \sup(\mathrm{supp}(\mu * \nu))$

$$\int_{\mathbb{R}} \log(z - t)\,\mathrm{d}(\mu \boxplus \nu)(t) \geq \int_{\mathbb{R}} \log(z - t)\,\mathrm{d}(\mu * \nu)(t).$$

Convolution comparison measure

Theorem (H, 2026)

For compactly supported μ, ν there exists a positive measure $\widetilde{m}_{\mu,\nu}$ on $\mathbb{R}^2$ such that for any $a, b \neq 0$ and $f \in C^4(\mathbb{R})$ one has

$$\int_{\mathbb{R}^2} f^{(4)}(ax+by)\,\mathrm{d}\widetilde{m}_{\mu,\nu}(x,y) = \frac{1}{a^2b^2}\left(\int_{\mathbb{R}} f(t)\,\mathrm{d}(a\mu * b\nu)(t) - \int_{\mathbb{R}} f(t)\,\mathrm{d}(a\mu \boxplus b\nu)(t)\right).$$

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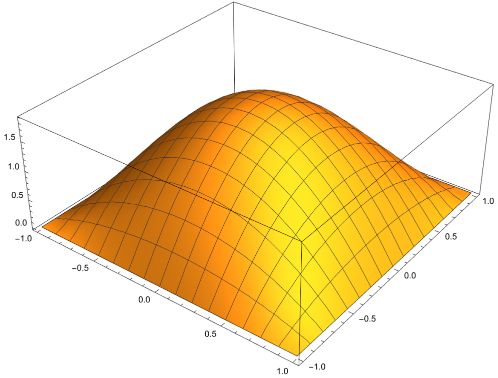
$\widetilde{m}_{\mu,\nu}$ is the **convolution comparison measure** of μ and ν.
Setting $a = b = 1$ gives the comparison between $*$ and $\boxplus$.

Rademachers



Density of $\widetilde{m}_{\mu,\nu}$, when μ and ν are both $\frac{1}{2}(\delta_{-1} + \delta_1)$

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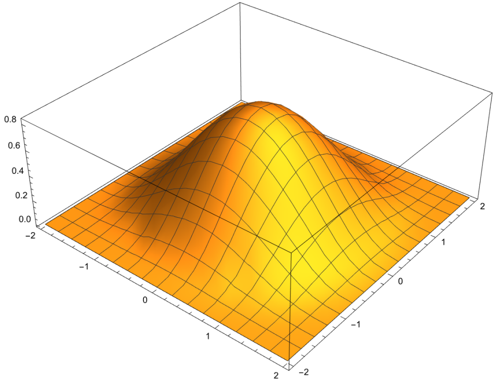

Density of $\widetilde{m}_{\mu,\nu}$, when μ and ν both have density $\frac{1}{2\pi}\sqrt{4-x^2}$ on $[-2,2]$

Properties of convolution comparison measures

1. $\widetilde{m}_{\mu,\nu}$ is supported on $\mathrm{Conv}(\mathrm{supp}(\mu)) \times \mathrm{Conv}(\mathrm{supp}(\nu))$.
2. $\widetilde{m}_{\mu,\nu}$ has total measure $\mathrm{Var}(\mu)\,\mathrm{Var}(\nu)/12$.
3. $\widetilde{m}_{\mu,\nu}$ is absolutely continuous w.r.t. Lebesgue measure on $\mathbb{R}^2$.

Proof elements

The main idea is to find an evidently positive expression for

$$\frac{\mathrm{d}\widetilde{m}_{\mu,\nu}}{\mathrm{d}m_2}.$$

This is done in three steps.

The three steps

1. Use **free cumulants** to manipulate the mixed moments

$$\int_{\mathbb{R}^2} x^n y^m \, \mathrm{d}\widetilde{m}_{\mu,\nu}(x,y) = \sum_{k=0}^{\infty} (2k+3)(-1)^k I_k^{\mu}(t \mapsto t^n) I_k^{\nu}(t \mapsto t^m)$$

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2. Show that

$$I_k^{\mu}(t \mapsto t^n) = \int_{\mathbb{R}\times(-1,1)} t^n C_k^{3/2}(s) \omega_{\mu}(t,s) \, \mathrm{d}t \, \mathrm{d}s,$$

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3. Evaluate a series involving **Gegenbauer polynomials** $C_k^{3/2}(s)$.

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- Free cumulants can be expressed in terms of moments, and vice versa. For instance

$$\kappa_4 = m_4 - 4m_3 m_1 - 2m_2^2 + 10 m_2 m_1^2 - 5 m_1^4$$
$$m_4 = \kappa_4 + 4\kappa_3\kappa_1 + 2\kappa_2^2 + 6\kappa_2\kappa_1^2 + \kappa_1^4.$$

Step 3: Gegenbauer polynomials

- One ends up evaluating for $-1 < x, y < 1$ the series

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- This identity can be checked using the defining properties of Gegenbauer polynomials:
  - $(1-2xt+t^2)^{-3/2} = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} C_k^{3/2}(x) t^k$
  - $\int_{-1}^{1} C_k^{3/2}(x) C_l^{3/2}(x)(1-x^2)\,\mathrm{d}x = \delta_{k,l}\frac{8}{(2k+3)(k+1)(k+2)}$.

Step 2: Some density

What is ω_μ in

$$I_k^{\mu}(t \mapsto t^n) = \int_{\mathbb{R}\times(-1,1)} t^n C_k^{3/2}(s)\omega_\mu(t,s)\,\mathrm{d}t\,\mathrm{d}s?,$$

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What is ω_μ?

Lemma

For Hermitian $A, B \in M_n(\mathbb{C})$,

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\sum_{i=1}^{n}\left|\operatorname{Im}\lambda_i\left((A-aI)(B-bI)\right)\right|\right)\mathrm{d}a\,\mathrm{d}b$$
$$=\operatorname{tr}(A^2B^2)-\operatorname{tr}(ABAB)$$

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$$= \sum_{n=1}^{\min(k,l)+2} \frac{(-1)^{n-1}}{\binom{k+l+1}{n-1}} \sum_{\substack{i_1,\ldots,i_n,j_1,\ldots,j_n \geq 1 \\ i_1+\ldots+i_n=k+2 \\ j_1+\ldots+j_n=l+2}} \mathrm{tr}(A^{i_1}B^{j_1}\cdots A^{i_n}B^{j_n})$$

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$\omega_\mu(a,b) = \frac{1}{2\pi}\sum_{i=1}^{n} |\operatorname{Im}\lambda_i\left((A-aI)(B-bI)\right)|$, where A and B are such that $B = vv^*$ and $\langle A^k v, v\rangle = \int_{\mathbb{R}} t^k \,\mathrm{d}\mu(t)$ for any $k \in \mathbb{N}$.

Tracial joint spectral measures

Proof of the previous Lemma is based on:

Theorem (H, 2023)

For Hermitian $A, B \in M_n(\mathbb{C})$, there exists a unique measure $\mu_{A,B}$ on $\mathbb{R}^2 \setminus \{0\}$ such that for any $x, y \in \mathbb{R}^2$ and any $f : \mathbb{R} \to \mathbb{R}$,

$$\operatorname{tr} H(f)(xA + yB) = \int_{\mathbb{R}^2} f(ax + by)\, \mathrm{d}\mu_{A,B}(a, b),$$

where

$$H(f)(x) = \int_0^1 f(xt) \frac{1-t}{t}\, \mathrm{d}t.$$

Formula for tracial joint spectral measure

Theorem (H, 2023)

Decompose $\mu_{A,B} = \mu_c + \mu_s$ *w.r.t. the Lebesgue measure* *(*$\mu_c \ll m_2$, $\mu_s \perp m_2$*). Then*

$$\frac{\mathrm{d}\mu_c}{\mathrm{d}m_2}(a,b) = \frac{1}{2\pi}\sum_{i=1}^{n}\left|\mathrm{Im}\left(\lambda_i\left(\left(I - \frac{aA+bB}{a^2+b^2}\right)(bA-aB)^{-1}\right)\right)\right|.$$

Recap/Thank you!

Thm 1:

$$\boxed{f^{(4)} \geq 0} \iff \boxed{\int_{\mathbb{R}} f(t)\,\mathrm{d}(\mu \boxplus \nu)(t) \leq \int_{\mathbb{R}} f(t)\,\mathrm{d}(\mu * \nu)(t)}$$

Thm 2: There exists $\widetilde{m}_{\mu,\nu} \geq 0$ such that

$$\int_{\mathbb{R}^2} f^{(4)}(x+y)\,\mathrm{d}\widetilde{m}_{\mu,\nu}(x,y) = \int_{\mathbb{R}} f(t)\,\mathrm{d}(\mu * \nu)(t) - \int_{\mathbb{R}} f(t)\,\mathrm{d}(\mu \boxplus \nu)(t)$$

Identity: For Hermitian A and B

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\sum_{i=1}^{n} |\mathrm{Im}\,\lambda_i\left((A-aI)(B-bI)\right)|\right)\mathrm{d}a\,\mathrm{d}b = \mathrm{tr}(A^2B^2) - \mathrm{tr}(ABAB)$$

Definition of free independence

Bounded self-adjoint elements X and Y of a tracial von Neumann algebra (M, τ) are *freely independent* if for any $f_1, \ldots, f_n, g_1, \ldots, g_n$ with $\tau(f_i(X)) = 0 = \tau(g_i(Y))$ one has

$$\tau\left(f_1(X)g_1(Y)\cdots f_n(X)g_n(Y)\right) = 0.$$

The law of $X + Y$ is the free convolution of the laws of X and Y.