Tracial joint spectral measures

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Abstract

Trace inequalities, that is inequalities between traces of complex matrices, are ubiquitous in various branches of mathematics. While such inequalities are usually easy to state as generalizations of real variable inequalities, proving them often requires deep understanding.

We introduce a new general tool for investigating trace inequalities, namely the tracial joint spectral measure. This positive measure on the plane can be associated to any two Hermitian matrices, and existence of it implies a plethora of non-trivial trace inequalities for these matrices.

In chapter 1, we discuss existence and basic properties of these measures, giving an explicit expression for them along the way. As the first main application, we deduce a new tracial monotonicity property: if f has non-negative k:th derivative, then so does $t \mapsto \operatorname{tr} f(tA + B)$ for any Hermitian A, B with A positive definite.

In chapter 2, we apply the theory of tracial joint spectral measures to Schatten-p trace ideals. In this context, we give a new embedding result: any two-dimensional subspace of Schatten-p is isometric to a subspace of L_p. This result is used to resolve a conjecture of Ball, Carlen, and Lieb on the extension of Hanner's inequality to Schatten-p spaces. Finally, we discuss the ways in which our embedding result fails for more than two matrices/operators and investigate ideas for working in this higher dimensional setting.

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Contents

Chapter 0

Introduction

Trace inequalities are inequalities between traces of matrices or linear operators. Such inequalities appear in a wide variety of contexts: the theory of trace ideals [Sim05], quantum mechanics [Car10], and random matrices [T$^+$15]. While many scalar inequalities have natural trace inequality extensions, methods to prove such generalizations are often quite involved.

The aim of this dissertation is to discuss new methods to prove trace inequalities. My main contribution is the theory of tracial joint spectral measures, introduced in [Hei23], a new powerful structural tool for investigating trace inequalities.

Theorem 1. *Let n be a positive integer and $A, B \in M_n(\mathbb{C})$ be Hermitian. Then, there exists a positive measure $\mu_{A,B}$ on $\mathbb{R}^2$, that we call the* tracial joint spectral measure *of A and B, such that for any positive integer k and $x, y \in \mathbb{R}$ one has,*

$$\operatorname{tr}(xA + yB)^k = k(k+1) \int_{\mathbb{R}^2} (ax + by)^k \, \mathrm{d}\mu_{A,B}(a, b).$$

Mere existence of this measure has the following two striking consequences, the first one being the following monotonicity result.

Theorem 2. *Fix $n \in \mathbb{N}$. Let $A, B \in M_n(\mathbb{C})$ be Hermitian and let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function with non-negative k:th derivative. Consider the function $F : \mathbb{R} \to \mathbb{R}$ given by $F(t) = \operatorname{tr} f(tA + B)$. If k is even, then F is smooth with non-negative k:th derivative. The same holds for odd k if we additionally assume that A is positive semidefinite.*

In Chapter 1, I will prove the existence of tracial joint spectral measures, establish their basic properties, and discuss some of their applications, including Theorem 2.

Denote by S_p the Schatten-p class, the Banach space of compact operators the singular values of which belong to ℓ_p; with the ℓ_p-norm of the singular values as the norm on S_p. My second main application of tracial joint spectral measures is the following embedding result.

Theorem 3. *Let $A, B \in S_p$ for some $1 \leq p \leq \infty$. Then there exists $f, g \in L_p = L_p([0,1], \mathbb{R})$ such that for any $x, y \in \mathbb{R}$,*

$$\|xA + yB\|_{S_p} = \|xf + yg\|_{L_p}.$$

In other words, any two-dimensional real subspace of S_p is linearly isometric to a subspace of L_p.

In Chapter 2, I will investigate the isometric properties of Schatten-p classes. I will prove Theorem 3 and discuss how it relates to uniform convexity of S_p. I will also give alternate arguments for proving Theorem 3 in special cases, based on my earlier work [Hei24]. Finally, I will discuss inequalities going beyond the scope of tracial joint spectral measures, focusing on so-called roundness inequalities of Enflo.

The remainder of the introduction offers historical context and summarizes the two chapters.

0.1 Chapter 1: Tracial joint spectral measure

0.1.1 Joint spectral measures

The spectral theory of self-adjoint operators has a long history, starting from the work of Descartes and Fermat on the principal axis theorem and perhaps culminating in the spectral theorem for unbounded operators by von Neumann (see [Ste73] for a survey).

While the theory for a single self-adjoint operator is largely understood, the story differs for multiple operators. If A and B are *commuting* Hermitian matrices, they can be simultaneously unitarily diagonalized, and as a consequence, for any $f \in C(\mathbb{R}^2)$ one can make good sense of $f(A, B)$.

For non-commuting operators the situation is not so simple, and there have been several attempts to salvage insights from the commutative case. As an example, for Hermitian matrices A and B, simultaneous diagonalization is out of the question, but one can still define functional calculus, so-called Weyl calculus (see [Jef04]) using the Fourier transform with

$$f(A, B) = (2\pi)^{-1/2} \int_{\mathbb{R}^2} \widehat{f}(x, y) e^{ixA + iyB} \, \mathrm{d}x \, \mathrm{d}y. \tag{1}$$

In other words, the usual spectral measure is replaced, at least formally, by the Fourier transform of $(x, y) \mapsto e^{ixA+iyB}$, which we denote by $\mathcal{W}_{A,B}$, and call the *Weyl distribution*, so that

$$f(A,B) = \int_{\mathbb{R}^2} f(a,b)\mathcal{W}_{A,B}(a,b)\,\mathrm{d}a\,\mathrm{d}b =: (\mathcal{W}_{A,B}, f)\,.$$

$\mathcal{W}_{A,B}$ is an operator valued distribution, which is uniquely characterized by the fact that

$$(xA + yB)^k = (\mathcal{W}_{A,B}, (a,b) \mapsto (ax + by)^k) \tag{2}$$

for any $x, y \in \mathbb{R}$ and $k \in \mathbb{N}$. In general, it is compactly supported with support lying in the convex hull of the joint numerical range

$$W(A,B) = \{(\langle Av, v\rangle, \langle Bv, v\rangle) \mid |v| = 1\}.$$

As the name suggests, the Weyl distribution is in general not a measure, so the desirable positivity properties of usual commutative spectral measures are lost. In fact, one has:

Proposition 1 ([And70, Theorem 2])**.** *Let A and B be such that $\mathcal{W}_{A,B}$ is a positive measure. Then A and B commute.*

There is a very good alternate explanation as to why $\mathcal{W}_{A,B}$ cannot in general be a positive measure: it would imply too many inequalities.

Example 1. Consider the polynomial $p(x,y) := -(2x+y)^4+5(x+y)^4+14x^4-y^4 = 3(-x^2+y^2+2xy)^2$ and for Hermitian A, B, examine

$$p(A,B) = -(2A+B)^4 + 5(A+B)^4 + 14A^4 - B^4 = (\mathcal{W}_{A,B}, p)\,.$$

If $\mathcal{W}_{A,B}$ is a positive measure, since p is non-negative, $p(A,B)$ should be positive semidefinite. This is however not always the case, as can be seen by considering say

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}.$$

The underlying reason for this behaviour is the following beautiful result of Helton [Hel02] (see also [BKP16, Theorem 1.30]).

Theorem 4 ([Hel02])**.** *Assume that a non-commutative polynomial p is positive, i.e. for any Hermitian A, B the polynomial $p(A, B)$ is positive semidefinite. Then one can find non-commutative polynomials $(q_i)_{i=1}^N$ such that*

$$p(A, B) = \sum_{i=1}^{N} q_i(A, B)^* q_i(A, B) \tag{3}$$

for any Hermitian A, B.

The issue in Example 1 then was that even though our polynomial was square in commutative variables, it was not a sum of squares in non-commutative ones.

0.1.2 Trace positivity

For trace inequality applications, it is usually not necessary to know that a given matrix/operator is positive, but its trace being positive might suffice. The non-negativity of traces is a much weaker condition, and we will later see (Corollary 2) that the polynomial in Example 1 always has a non-negative trace.

One is now tempted to ask the following question. Is the trace of the Weyl distribution $\mathcal{W}_{A,B}$ always a positive measure? The answer is again no, unless A and B commute (as follows from an argument almost identical to that of the proof of Proposition 1).

One can also give a trace inequality refutation to this question.

Example 2. Consider the polynomial $p(x, y) = 4(x^4 + y^4) + (x + y)^4 + (x - y)^4 - 12(x^2 + y^2) + 6 = 6(x^2 + y^2 - 1)^2$. If $\operatorname{tr} \mathcal{W}_{A,B}$ was always a positive measure, $\operatorname{tr} p(A, B)$ would be positive for any Hermitian A, B. This is however not the case for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It turns out that Helton's theorem fails for trace positivity. Namely, given a non-commutative polynomial p for which $\operatorname{tr} p(A, B) \geq 0$ for any Hermitian A, B, it is not necessarily the case that one can find non-commutative polynomials $q_i(A, B)$ such that (3) holds. This is partly because of a silly reason: adding commutator terms like $A^2B^2 - B^2A^2$ won't affect the trace, but will affect the possibility of the sum-of-squares representation. If one adds commutators to the RHS of (3), the situation is however still not fixed. In fact, consider the non-commutative Motzkin polynomial

$$p(A,B) = AB^4A + BA^4B - 3AB^2A + I.$$

It was observed by [KS08a, Example 4.4] that $\operatorname{tr} p(A,B) \geq 0$ for any Hermitian A, B, but $p(A,B)$ doesn't have a positivity certificate of the form (3) even up to commutators.

While a perfect analogue doesn't exist, there is still a close tracial relative to Helton's result.

Theorem 5 ([KS08a, Theorem 3.12])**.** *Consider a non-commutative polynomial p such that for any finite tracial von Neumann algebra (M,τ) and self-adjoint contractions $A, B \in M$ we have $\tau(p(A,B)) \geq 0$. Then, for every $\varepsilon > 0$, there exists non-commutative polynomials $(f_i)_{i=1}^{N_1}$, $(g_i)_{i=1}^{N_2}$, $(h_i)_{i=1}^{N_3}$, $(q_i)_{i=1}^{N_0}$, $(r_i)_{i=1}^{N_0}$, such that*

$$\begin{aligned} p(A,B) + \varepsilon I = &\sum_{i=1}^{N_1} f_i(A,B)^* f_i(A,B) + \sum_{i=1}^{N_2} g_i(A,B)^*(I - A^2) g_i(A,B) \\ &+ \sum_{i=1}^{N_3} h_i(A,B)^*(I - B^2) h_i(A,B) + \sum_{i=0}^{N_0} [q_i(A,B), r_i(A,B)]. \end{aligned} \tag{4}$$

The exact form of this result is not very important to us, but one should observe that all the summands on the RHS clearly have non-negative trace whenever A and B are Hermitian contractions.

As one can see, some concessions have to be made to get a working theorem.

1. A small multiple of the identity has to be added, and terms involving $(I - A^2)$ and $(I - B^2)$ considered. This is essentially to make the problem bounded and allow separation in the functional analytic proof of the theorem.

2. Perhaps more crucially, one has to bring von Neumann algebras into the mix. Whether von Neumann algebras could be replaced here by Hermitian matrices was proven by Klep and Schweighofer [KS08a, Theorem 1.6] to be equivalent the Connes embedding conjecture [Con76]. This conjecture was however recently refuted by Ji, Natarajan, Vidick, Wright and Yuen [JNV+21].

We call expressions of the form (4) (tracial) SOS-certificates (sum of squares), and say that a SOS-certificate is *pure* if $\varepsilon = 0$ and no $(I - A^2)$ or $(I - B^2)$ terms are needed. While checking for pure certificates can be turned into a semidefinite program [BKP16], the existence of more general certificates (of the form of 4) cannot be efficiently verified. It is hence important to find tools tailored to specific families of trace inequalities.

0.1.3 Tracial joint spectral measures

My main contribution to the study of trace inequalities is the notion of tracial joint spectral measure.

Theorem 6. *Let n be a positive integer and $A, B \in M_n(\mathbb{C})$ be Hermitian. Then, there exists a positive measure $\mu_{A,B}$ on $\mathbb{R}^2$, that we call the* tracial joint spectral measure *of A and B, such that the following is true:*

Fix any measurable function f on $\mathbb{R}$ with $f(0) = 0$ such that for any $M > 0$,

$$\int_{-M}^{M} \left| \frac{f(t)}{t} \right| \mathrm{d}t < \infty.$$

Define a function $H(f) : \mathbb{R} \to \mathbb{R}$ by

$$H(f)(x) = \int_0^1 \frac{1-t}{t} f(xt) \, \mathrm{d}t.$$

Then, for any $x, y \in \mathbb{R}$, we have

$$\operatorname{tr} H(f)(xA + yB) = \int_{\mathbb{R}^2} f(ax + by) \, \mathrm{d}\mu_{A,B}(a, b). \tag{5}$$

If one sets $f(t) = t^k$ in Theorem 6, $H(f)(t) = t^k/(k(k+1))$, and one recovers Theorem 1.

Tracial joint spectral measures can be understood as a variant of the trace of the Weyl distribution, $\operatorname{tr} \mathcal{W}(A, B)$: $\operatorname{tr} \mathcal{W}_{A,B}$ satisfies identity (5) with $H(f)$ replaced by f. Remarkably, while $\operatorname{tr} \mathcal{W}(A, B)$ is usually not a measure, the subtle difference of $H(f)$ vs f allows one to obtain a positive measure.

In Proposition 5, we will check that the tracial joint spectral measure $\mu_{A,B}$ is necessarily unique away from 0. While the exact form of the measure is usually not important for our applications, $\mu_{A,B}$ turns out to have an explicit expression:

Theorem 7. *Let n, A, B, and $\mu_{A,B}$ be as in Theorem 6. Denote by $\mu_c = \mu_{c,A,B}$ and $\mu_s = \mu_{s,A,B}$ the continuous and singular parts of $\mu_{A,B}$ w.r.t. the Lebesgue measure m_2 on $\mathbb{R}^2$. We assume some linear combination of A and B is invertible. Then, the continuous part μ_c is given by*

$$\frac{\mathrm{d}\mu_c}{\mathrm{d}m_2}(a, b) = \frac{1}{2\pi} \sum_{i=1}^{n} \left| \Im \left(\lambda_i \left(\left(I - \frac{aA + bB}{a^2 + b^2} \right) (bA - aB)^{-1} \right) \right) \right|.$$

Furthermore, if A is invertible and $A^{-1}B$ has n distinct eigenvalues, the singular part μ_s satisfies

$$\mu_s(\varphi) = \sum_{v \in E(A^{-1}B)} \int_0^1 \frac{1-t}{t} \varphi\left(\frac{\langle Av, v\rangle}{\langle v, v\rangle} t, \frac{\langle Bv, v\rangle}{\langle v, v\rangle} t \right) \mathrm{d}t.$$

where $E(C)$ denotes a set of eigenvectors of a matrix $C \in M_n(\mathbb{C})$ and φ is a smooth function with compact support that does not contain 0*.*

Figure 1 illustrates the measure $\mu_{A,B}$ for some choices of A and B.

0.1.4 Derivatives of trace functions and Stahl's theorem

Theorem 2 can be deduced from Theorem 6 by applying identity (5) to the function $f(t) = t_+^{k-1}$. Theorem 2 is well known for $k = 1$ and $k = 2$ [Pet94, Proposition 1], and I first proved the cases $k = 3$ and $k = 4$ in [Hei24]. The general case is intimately connected to the following result of Stahl, formerly the BMV conjecture.

Theorem 8 (Stahl [Sta13])**.** *Let $A, B \in M_n(\mathbb{C})$ be Hermitian with B positive semi-definite. Then the function*

$$t \mapsto \mathrm{tr}(\exp(A - tB)) \tag{6}$$

is the Laplace transform of a positive measure.

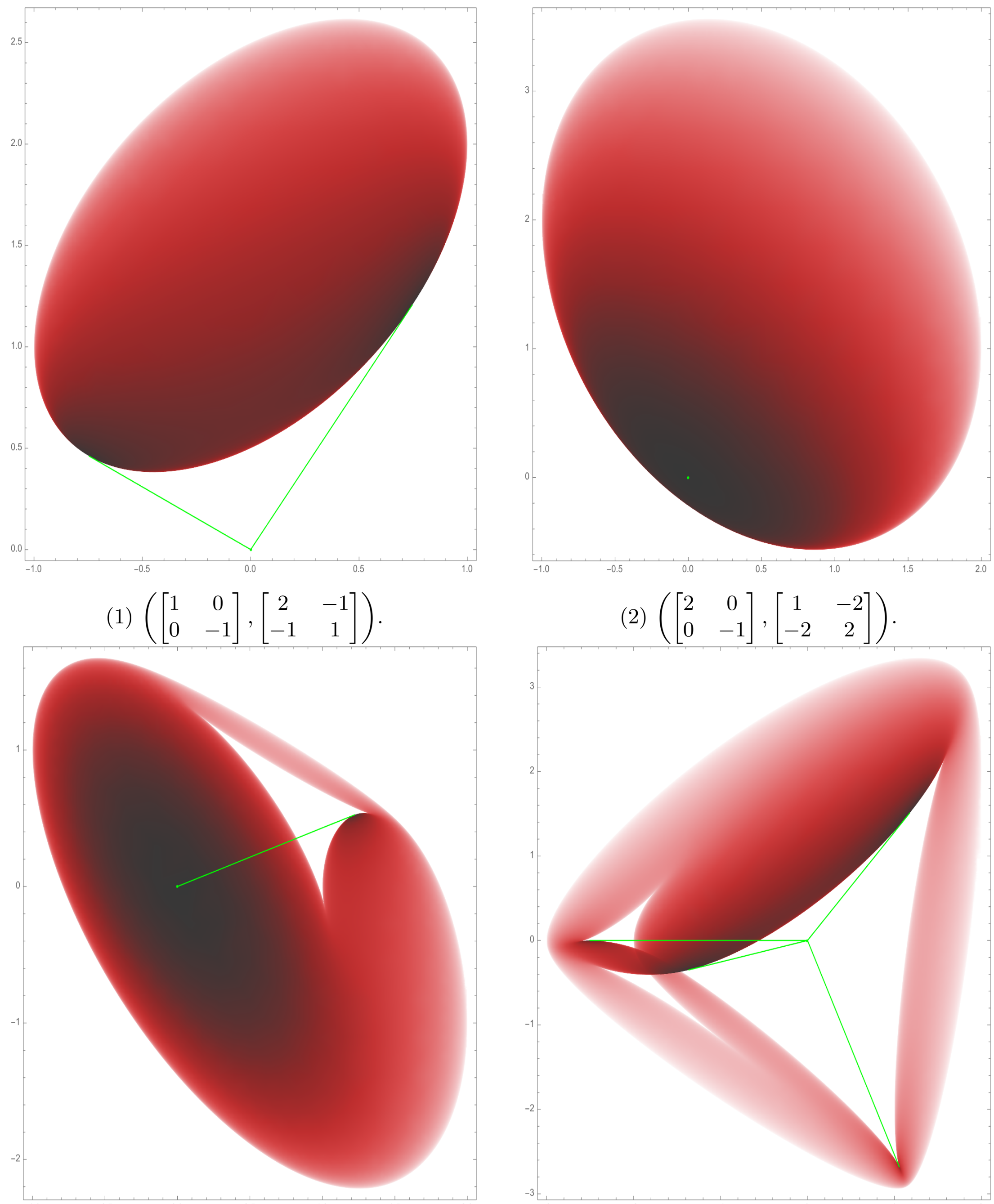
This result was conjecture by Bessis, Moussa and Villani [BMV75] in their study of monotonic approximations of partition functions of quantum statistical systems. While the BMV conjecture garnered much interest during the next three decades (see the survey of Moussa [Mou00]), it was the algebraic formulation of Lieb and Seiringer that inspired a barrage of computational approaches.

Theorem 9 (Lieb–Seiringer [LS04])**.** *Theorem 8 (Stahl's theorem) is true iff for any positive semi-definite $A, B \in M_n(\mathbb{C})$ and positive integer k, all the coefficients of the polynomial*

$$t \mapsto \mathrm{tr}(A + tB)^k \tag{7}$$

are non-negative.

Theorem 9 reduced the BMV conjecture to the task of finding SOS-certificates (4). This approach was carried in a series of works culminating in [KS08b], where non-negativity of the coefficients of



(1) $\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\right)$.

(2) $\left(\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}, \begin{bmatrix}1 & -2\\ -2 & 2\end{bmatrix}\right)$.

(3) $\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1 & 1\\ 1 & -1 & -1\\ 1 & -1 & 1\end{bmatrix}\right)$.

(4) $\left(\begin{bmatrix}-3 & 0 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 2\end{bmatrix}, \begin{bmatrix}0 & 0 & 0 & -1\\ 0 & 0 & 0 & -1\\ 0 & 0 & -2 & -2\\ -1 & -1 & -2 & 2\end{bmatrix}\right)$.

Figure 1: Four illustrations of the measures $\mu_{A,B}$ for the pairs of matrices (A, B) listed below the pictures. The horizontal and vertical axes correspond to a and b respectively. The density of $\mu_{c,A,B}$ is represented with the color running from white (zero) to black (infinity) through red. The green line segments depict the support of the singular part, $\mu_{s,A,B}$.

(7) was proven for $k \leq 13$. Whether the Lieb–Seiringer formulation of the BMV conjecture can be verified in general with SOS-certificates is still an open problem.

Stahl himself doesn't exploit this reformulation of Lieb and Seiringer, but finds a semiexplicit expression for the representing measure which he then proves is positive. It had been observed (see [Mou00]) that should the function (6) correspond to a Laplace transform of a positive measure, say ν, this measure should be supported on the convex hull of the spectrum of B. The measure ν should also have point masses at the spectrum of B, and elsewhere have continuous density w.r.t. the Lebesgue measure. While the point masses are straightforward to deal with, whether the density is non-negative is rather non-trivial. This non-negativity was shown in the case of 2×2 matrices in the original paper of Bessis, Moussa and Villani [BMV75] by finding an explicit formula for the density. For larger matrices, however, such formulas seemed out of reach.

Stahl managed to find a general formula for the density in terms of a contour integral. Correctness of his formula is not terribly difficult to check, but the non-negativity is not clear. To this end, Stahl transforms the contour integral to a Riemann surface, on which he uses symmetry and topological arguments to justify the non-negativity. While Stahl's argument has been rewritten for several expository articles [Ere15, Cli16], I still find it rather mysterious, and quite frankly, magical.

A third approach towards the BMV conjecture was the classic result of Bernstein.

Theorem 10 (Bernstein [SSV09, Theorem 1.4]). *Let $f : (0, \infty) \to \mathbb{R}$ be smooth. Then f is a Laplace transform of a positive measure iff f is completely monotone, i.e. for any positive non-negative integer k,*

$$(-1)^k f^{(k)}(x) \geq 0$$

for $x > 0$.

This reduced the BMV conjecture to an analysis of the derivatives of the function (6). The first two derivatives are relatively easy to analyze [Mou00], but even the case $k = 3$ was open (see discussion in [Cli14, Chapter 4] however) until I proved the cases $k = 3$ and $k = 4$ of Theorem 2 [Hei24].

As we will see in Section 1.4, all the aforementioned reformulations of Stahl's result follow without too much trouble from the theory of tracial joint spectral measures.

0.2 Chapter 2: Planes in Schatten-p and beyond

0.2.1 Schatten-p classes

In his seminal 1937 paper [VN62], John von Neumann defined the concept of a unitarily invariant norm. A norm $\|\cdot\|$ on $M_n(\mathbb{C})$ is unitarily invariant if, for any $A \in M_n(\mathbb{C})$ and unitaries $U, V \in M_n(\mathbb{C})$, we have

$$\|UAV\| = \|A\|.$$

In the same work, von Neumann gave a complete characterization of unitarily invariant norms in terms of symmetric gauge functions.

Definition 1. *A function $\Phi : \mathbb{C}^n \to \mathbb{R}$ is a symmetric gauge function if it is a norm and*

- *(Phase invariance) $\Phi((\omega_i z_i)_{i=1}^n) = \Phi((z_i)_{i=1}^n)$ for any $(\omega_i)_{i=1}^n \in \{w \mid |w| = 1\}^n$, and*
- *(Permutation invariance) $\Phi((z_{\sigma(i)})_{i=1}^n) = \Phi((z_i)_{i=1}^n)$ for any $\sigma \in S_n$.*

Theorem 11. *A norm $\|\cdot\|$ on $M_n(\mathbb{C})$ is unitarily invariant iff there exists a symmetric gauge function Φ such that*

$$\|A\| = \Phi((\sigma_i(A))_{i=1}^n)$$

for any $A \in M_n(\mathbb{C})$.

Here $(\sigma_i(A))_{i=1}^n$ denotes the sequence of singular values of the matrix A.

Theorem 11 created a basis for the theory of cross-norms, norms on tensor products of Hilbert spaces with suitable invariance properties. This theory was developed in a series of works by Robert Schatten and John von Neumann [Sch46, SvN46, SvN48] culminating in the books [Sch50, Sch60] of Schatten. See also [Bha97, Theorem IV.2.1] for a modern treatment of Theorem 11.

For $1 \leq p \leq \infty$, the gauge function $\Phi_p(x) = \|x\|_{\ell_p}$ gives the Schatten-p norm on $M_n(\mathbb{C})$. More generally:

Definition 2. *Let $p \geq 1$. By S_p, we denote the Banach space of compact operators A on a separable complex Hilbert space H such that*

$$\|(\sigma_i(A))_{i=1}^\infty\|_{\ell_p} < \infty,$$

with the norm

$$\|A\|_{S_p} := \|(\sigma_i(A))_{i=1}^{\infty}\|_{\ell_p}.$$

Basic functional calculus implies that we further have

$$\|A\|_{S_p} = (\operatorname{tr}(|A|^p))^{1/p} = \left(\operatorname{tr}(A^*A)^{p/2}\right)^{1/p}.$$

Schatten-p classes S_p can be thought of as non-commutative variants of the classical ℓ_p spaces, and they satisfy the following properties mirroring those of ℓ_p. These can be found in [PX03].

1. If $1 \le p < \infty$, S_q is the dual of S_q with pairing $(A, B) = \operatorname{tr}(AB^*)$. S_p is hence reflexive when $1 < p < \infty$. In finite dimensions, this is a special case of von Neumann's result [VN62], while in infinite dimensions it appears in [Sch60, Chapter V], see also [Sim05, Theorem 3.2].

2. By considering operators diagonal in a fixed orthonormal basis, one sees that ℓ_p is a closed subspace of S_p for $1 \le p \le \infty$.

3. Finite rank operators are dense in S_p for $1 \le p \le \infty$ [McC67, Lemma 5.2].

4. S_p is of type $\min(p, 2)$ and cotype $\max(p, 2)$ when $1 \le p < \infty$ [TJ74].

We have however the following negative results.

1. If $1 \le p < \infty$, $p \ne 2$, S_p is not isomorphic to a subspace of ℓ_p or L_p [McC67].

2. If $p \ne 2$, S_p does not have an unconditional basis [GL74].

0.2.2 Modulus of uniform convexity

As mentioned in the previous section, Schatten-p space is reflexive whenever $1 < p < \infty$. While this follows from the duality $S_p^* = S_q$, one can also see it as a consequence of uniform convexity of S_p for $1 < p < \infty$.

Definition 3. *A normed space X is said to be uniformly convex if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \ge \varepsilon$, then*

$$\left\|\frac{x+y}{2}\right\| \le 1 - \delta.$$

The maximal such δ (depending on ε and X) is denoted by $\delta_X(\varepsilon)$.

Theorem 12 (Milman [Mil38]–Pettis [Pet39])**.** *Any uniformly convex Banach space is reflexive.*

Theorem 13. *For any $1 < p < \infty$, the spaces ℓ_p, L_p and S_p are uniformly convex.*

This result for ℓ_p (or more generally, for L_p) was proven by Clarkson [Cla36]. The main tools are the so-called Clarkson's inequalities for ℓ_p.

Theorem 14 (Clarkson [Cla36])**.** *Let $2 \le p < \infty$ and $f, g \in \ell_p$ or $f, g \in L_p$. Then*

$$\|f\|_p^p + \|g\|_p^p \le \frac{\|f+g\|_p^p + \|f-g\|_p^p}{2}, \tag{8}$$

$$\|f\|_p^p + \|g\|_p^p \le \left(\frac{\|f+g\|_p^q + \|f-g\|_p^q}{2}\right)^{p/q}. \tag{9}$$

For $1 < p \le 2$, the reverse inequalities hold.

Inequality (8), the first Clarkson's inequality, follows from the second (9) by the power mean inequality $((x^r+y^r)/2)^{1/r} \ge (x+y)/2$ for $r \ge 1$, $x, y > 0$. Inequality (8) however admits a somewhat simpler proof. For future illustration, we will briefly sketch the proofs of Clarkson's inequalities.

Sketch of the proofs of Inequalities (8) and (9) for $p \ge 2$. We will start with Inequality (8). Since

$$\|f\|_p^p = \begin{cases} \sum_{i=1}^{\infty} |f(i)|^p & \text{in } \ell_p \\ \int_0^1 |f(t)|^p dt & \text{in } L_p, \end{cases}$$

it is enough to consider the one-dimensional case, i.e. for any $x, y \in \mathbb{R}$ we have

$$|x|^p + |y|^p \le \frac{|x+y|^p + |x-y|^p}{2}.$$

This can be proven directly by scaling y to 1 and analyzing the resulting function in x.

Inequality (9) can also be reduced to a scalar inequality by first observing that for $0 < r \le 1$ and $a, b \ge 0$

$$(|a|^r + |b|^r)^{1/r} = \inf_{\substack{s_1, s_2 > 0 \\ s_1^{r/(r-1)} + s_2^{r/(r-1)} = 1}} (as_1 + bs_2).$$

Indeed, this is the reverse Hölder's inequality. With $r = q/p$, we then have in L_p,

$$
\begin{aligned}
\left(\frac{\|f+g\|_p^q + \|f-g\|_p^q}{2}\right)^{p/q} &= 2^{-p/q} \inf_{\substack{s_1,s_2>0 \\ s_1^{r/(r-1)}+s_2^{r/(r-1)}=1}} \left(s_1\|f+g\|_p^p + s_2\|f-g\|_p^p\right) \\
&\geq 2^{-p/q} \int_0^1 \inf_{\substack{s_1,s_2>0 \\ s_1^{r/(r-1)}+s_2^{r/(r-1)}=1}} \left(s_1|f(t)+g(t)|^p + s_2|f(t)-g(t)|^p\right) dt \\
&= \int_0^1 \left(\frac{|f(t)+g(t)|_p^q + |f(t)-g(t)|_p^q}{2}\right)^{p/q} dt \\
&\geq \int_0^1 |f(t)|_p^p + |g(t)|_p^p dt,
\end{aligned}
$$

where the last inequality is again a pointwise scalar inequality that can be proven without much trouble. □

Here, we see that in the commutative case, inequalities of suitable form can be reduced to scalar inequalities, even if for the second Clarkson's inequality an extra trick was employed. Such simplification is usually not possible for S_p and alternate methods are needed. Clarkson's original method was somewhat different, based on a clever use of Minkowski's inequality to reduce (9) to the scalar case.

The main tool for proving uniform convexity of S_p is the following extension of Clarkson's inequality.

Theorem 15 (Dixmier [Dix53]–McCarthy [McC67]–Klaus). *Let $A, B \in S_p$ for $2 \leq p < \infty$. Then*

$$\|A\|_p^p + \|B\|_p^p \leq \frac{\|A+B\|_p^p + \|A-B\|_p^p}{2} \tag{10}$$

$$\|A\|_p^p + \|B\|_p^p \leq \left(\frac{\|A+B\|_p^q + \|A-B\|_p^q}{2}\right)^{p/q}. \tag{11}$$

For $1 < p \leq 2$, the reverse inequalities hold.

Dixmier proved Inequality (10) for $2 \leq p < \infty$, which is enough to prove uniform convexity for this case. His proof is based on complex interpolation, modelled after a similar argument for commutative Clarkson's inequalities by Boas [Boa40]. The other cases were claimed by McCarthy [McC67], but his proof for (11) is unfortunately erroneous, as observed by Fack and Kosaki [FK86]. Simon [Sim05] sketches an interpolation proof for the remaining cases, attributing this approach to Klaus.

Clarkson's inequalities imply that the moduli of uniform convexity of ℓ_p and S_p satisfy

$$\delta(\varepsilon) \geq \begin{cases} 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p} & \text{for } 2 \leq p < \infty \\ 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^q\right)^{1/q} & \text{for } 1 < p \leq 2. \end{cases}$$

This turns out to be the optimal bound for $p \geq 2$, as can be verified by considering the vectors

$$x = \left(\left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}, \frac{\varepsilon}{2}\right), y = \left(\left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}, -\frac{\varepsilon}{2}\right).$$

The situation for $1 < p < 2$ is more complicated, and Clarkson's inequalities do not give the optimal bound. In 1956 however, Olof Hanner [Han56] proved the following inequalities, now known as Hanner's inequalities, that improve upon those of Clarkson and can be used to determine the modulus of uniform convexity of ℓ_p and L_p in the full range $1 < p < \infty$.

Theorem 16 (Hanner [Han56]). *Let $2 \leq p < \infty$ and $f, g \in \ell_p$ or $f, g \in L_p$. Then*

$$\|f+g\|_p^p + \|f-g\|_p^p \leq (\|f\|_p + \|g\|_p)^p + |\|f\|_p - \|g\|_p|^p. \tag{12}$$

For $1 < p \leq 2$, the reverse inequality holds.

Corollary 1 (Hanner [Han56]). *For $1 < p \leq 2$ the moduli of uniform convexity of ℓ_p and L_p are given by the solution to the equation*

$$2 = \left|1 - \delta_p(\varepsilon) + \frac{\varepsilon}{2}\right|^p + \left|1 - \delta_p(\varepsilon) - \frac{\varepsilon}{2}\right|^p \tag{13}$$

Hanner also established useful asymptotics for δ_p when $\varepsilon \to 0$.

$$\delta_p(\varepsilon) \approx \begin{cases} \frac{1}{p}\left(\frac{\varepsilon}{2}\right)^p & \text{for } 2 \leq p < \infty \\ \frac{p-1}{2}\left(\frac{\varepsilon}{2}\right)^2 & \text{for } 1 < p \leq 2. \end{cases} \tag{14}$$

While the aforementioned generalization of Clarkson's inequality to S_p implies that δ_{S_p} agrees with δ_p for $p \geq 2$, for $1 < p < 2$ a different approach is needed. It was proven by Tomczak-Jaegermann [TJ74] that also for $1 < p < 2$ one obtains quadratic asymptotics $\delta_{S_p}(\varepsilon) \gg_p \varepsilon^2$, but the multiplicative constant obtained was larger than $(p-1)/2$ for p not an even integer.

The main inspiration for this thesis work is then the paper [BCL94] of Ball, Carlen and Lieb, where it was proven that the asymptotics (14) hold also for S_p when $1 < p < 2$. This follows from the following inequality.

Theorem 17. *(Ball–Carlen–Lieb [BCL94]) Let $1 \leq p \leq 2$ and $A, B \in S_p$. Then*

$$\frac{\|A+B\|_{S_p}^2 + \|A-B\|_{S_p}^2}{2} \geq \|A\|_{S_p}^2 + (p-1)\|B\|_{S_p}^2. \tag{15}$$

Ball, Carlen and Lieb also consider an extension of Hanner's inequality for S_p and observe that inequality (15) follows from it.

Theorem 18 (Ball–Carlen–Lieb [BCL94], Heinävaara [Hei23])**.** *Let $2 \leq p < \infty$ and $A, B \in S_p$. Then*

$$\|A+B\|_p^p + \|A-B\|_p^p \leq (\|A\|_p + \|B\|_p)^p + |\|A\|_p - \|B\|_p|^p. \tag{16}$$

For $1 < p \leq 2$, the reverse inequality holds.

Ball, Carlen and Lieb manage however to prove inequality (16) only for $p \geq 4$, and in the dual range $1 < p \leq 4/3$. We can deduce Theorem 18 as a quick corollary of a much more general embedding result, Theorem 3.

0.2.3 Planes in Schatten-p

In [Hei24] and [Hei23], I develop a brand new embedding approach for proving inequalities for S_p.

Theorem 19 (= Theorem 3)**.** *Let $A, B \in S_p$ for some $1 \leq p \leq \infty$. Then there exists $f, g \in L_p$ such that for any $x, y \in \mathbb{R}$,*

$$\|xA + yB\|_{S_p} = \|xf + yg\|_{L_p}.$$

In other words, any two-dimensional real subspace of S_p is linearly isometric to a subspace of L_p.

For $p = 2$ and $p = \infty$ this result is trivial, as S_2 is Hilbert space and L_∞ is universal, i.e. it houses every separable normed space isometrically. For $p = 1$, the result follows from the fact that any two-dimensional normed space is isometric to a subspace of L_1, a classic result of Lindenstrauss [Lin64].

Theorem 19 follows as a quick corollary of Theorem 6. This embedding result, combined with Hanner's theorem [Han56], immediately implies Hanner's inequality for S_p, Theorem 18. More generally, we have the following "reduction to commuting" principle: if an inequality only depends on the S_p-norms of (real) linear combinations of two complex matrices, then it holds as long as it holds for real diagonal matrices. Thus, any property that only depends on two-dimensional subspaces of S_p generalizes directly from L_p to S_p. In particular, S_p has the same modulus of uniform convexity as L_p.

We will see in Section 2.1.1 that the exact form of the embedding allows one to also argue about equality cases of inequalities in S_p, given that the equality cases are known in L_p.

I first proved the cases $p = 3, 4$ in [Hei24] and then the general case in [Hei23]; I will describe proofs of these special cases in Section 2.2.

0.2.4 Beyond two dimensions, and roundness inequalities

In [Hei24], I showed that a natural generalization of Theorem 19 to 3-dimensional subspaces fails.

Theorem 20. *For any $1 \le p < \infty$, $p \ne 2$, there exists a 3-dimensional subspace of S_p which is not linearly isometric to any subspace of L_p.*

In Section 2.3, I will give several examples of inequalities exhibiting this difference, including the so-called roundness inequalities.

After Enflo's work [Enf69a, Enf69b], we say that a Banach space X satisfies the generalized p-roundness inequality if for any positive integer k and $x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k \in X$ one has

$$\sum_{i<j} \|x_i - x_j\|_X^p + \sum_{i<j} \|y_i - y_j\|_X^p \le \sum_{i,j} \|x_i - y_j\|_X^p.$$

Enflo proved that L_p satisfies the generalized p-roundness inequality for $1 \le p \le 2$ and used this to prove that L_p and L_q are not uniformly homeomorphic when $1 \le p \ne q \le 2$.

It turns out that for $p > 2$, even the real line fails to satisfy the generalized p-roundness inequality; instead, one can ask for the least constant $\mathfrak{r}_p(X)$ such that

$$\sum_{i<j} \|x_i - x_j\|_X^p + \sum_{i<j} \|y_i - y_j\|_X^p \le \mathfrak{r}_p(X) \sum_{i,j} \|x_i - y_j\|_X^p.$$

This more general notion was investigated by Naor and Oleszkiewicz in [NO20], where $\mathfrak{r}_p(L_q)$ was bounded for $1 \le p, q < \infty$. In particular, it was proven that if $1 \le p \le q \le 2$, $\mathfrak{r}_p(L_q) = 1$. It follows from a more general result of Naor and Oleszkiewicz that in the non-commutative setting, one has

$$\mathfrak{r}_p(S_q) \le 2^{2-p},$$

whenever $1 \le p \le q \le 2$. They however observe that this bound degenerates for $q = 1$: any Banach space X satisfies $\mathfrak{r}_1(X) \le 2$. Naor and Oleszkiewicz proceed to ask if this can be improved, namely if $\mathfrak{r}_1(S_1) < 2$. We answer this question in the affirmative.

Proposition 2. *There exists a constant $\delta > 0$ such that for any positive integer k and $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_k \in S_1$ one has*

$$\sum_{i<j} \|A_i - A_j\|_{S_1} + \sum_{i<j} \|B_i - B_j\|_{S_1} \le (2-\delta) \sum_{i,j} \|A_i - B_j\|_{S_1}.$$

Naor and Oleszkiewicz also observe that

$$\mathfrak{r}_p(S_1) \ge 2^{p/2+1}$$

and their result readily generalizes to show that

$$\mathfrak{r}_p(S_q) \ge 2^{p(1/q-1/2)+1}.$$

The lower bound is based on a curious set of unitaries $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_k$ having the property that

$$A_iA_j + A_jA_i = 0 = B_iB_j + B_jB_i \text{ for } i \ne j \tag{17}$$

$$A_iB_j = B_jA_i \text{ for } 1 \le i, j \le k. \tag{18}$$

These unitaries arise from the representation theory of Clifford algebras, and were investigated in the context of Schatten-1 norm by Briët, Regev and Saket [BRS17].

Whether this lower bound is the truth for some $1 \le p \le q$ is open, but I have evidence that this might be the case for $p = q = 1$. I'll show in Section 2.4 that this would follow from the following inequality.

Conjecture 1. *Let $k, l \ge 1$ be positive integers and $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_l \in S_1$. Then*

$$\frac{1}{k^2} \sum_{1 \le i < j \le k} \|A_i - A_j\|_{S_1}^2 + \frac{1}{l^2} \sum_{1 \le i < j \le l} \|B_i - B_j\|_{S_1}^2 \le \frac{2}{kl} \sum_{\substack{1 \le i \le k \\ 1 \le j \le l}} \|A_i - B_j\|_{S_1}^2$$

I will show that this inequality is true in the subspaces arising from the aforementioned Clifford algebra construction. I will also sketch an argument showing that the inequality holds when $k = 2$

and A_i's and B_j's are all diagonal.

0.3 Notation and conventions

If not otherwise stated, n will always denote a positive integer that we think of as a dimension, or size of the matrices. Let $A, B \in M_n(\mathbb{C})$ be Hermitian. The multiset of the eigenvalues of A are denoted by $\lambda(A) = (\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$; the ordering is not important for us. The corresponding set of normalized eigenvectors is denoted by $E(A)$. While these vectors are not unique, we make sure to respect this ambiguity.

We will also make use of notions from the theory of matrix pencils (see for instance [Ikr93]). Given A and B as before, we say that the pair/pencil (A, B) is non-degenerate if some linear combination of A and B is invertible, i.e. $\det(bA - aB)$ is not zero for every $(a, b) \in \mathbb{R}^2$. In this case, the determinant has exactly n roots (with multiplicity) in $\mathbb{CP}^1$, which we will call the roots of the pencil (A, B). Real roots, which we interpret as lines in $\mathbb{R}^2$, are called *singular lines.* If a root (a, b) is simple, we may define the corresponding eigenvector as the vector in the kernel of $bA - aB$. The set of such vectors (for simple roots) is denoted by $E(A, B)$. Again, these vectors are not unique, but we will respect this ambiguity.

We say that a pencil (A, B) is definite if some linear combination of A and B is positive definite. A well-known fact about definite pencils is:

Lemma 1 ([LR05, Theorem 10.1])**.** *If (A, B) is definite, all its roots are real.*

We denote the set of (complex valued) Schwartz functions on $\mathbb{R}^d$ by $\mathcal{S}(\mathbb{R}^d)$, and its dual space of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. The Fourier transform of a Schwartz function is defined/normalized with

$$\mathcal{F}(\varphi)(\xi) = \widehat{\varphi}(\xi) = \frac{1}{(2\pi)^{k/2}} \int_{x \in \mathbb{R}^k} \varphi(x) e^{-i\langle x, \xi \rangle} \, \mathrm{d}x.$$

One has $\mathcal{F}(\partial^\alpha \varphi)(\xi) = (i\xi)^\alpha \mathcal{F}(\varphi)(\xi)$ for any multi-index α. The Fourier transform of a tempered distribution T is defined, as usual, using the pairing $(\cdot, \cdot) : \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$, as

$$\left(\widehat{T}, \varphi\right) = (T, \widehat{\varphi}).$$

We make the usual conventions $\ell_p := L_p(\mathbb{N}, \mathbb{R})$ and $L_p := L_p((0, 1), \mathbb{R})$, with the counting and Lebesgue measures, respectively. We say that a norm $\| \cdot \|$ on $\mathbb{R}^k$ is a L_p-norm, if $(\mathbb{R}^k, \| \cdot \|)$ is isometric to a subspace of L_p. L_p-norms are characterized by the following result.

Theorem 21 ([Ney84], [Kol91, Theorem 2])**.** *Norm $\|\cdot\|$ on $\mathbb{R}^k$ is a L_p-norm iff there exists a symmetric measure μ on S^{k-1} (given any inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^k$) such that*

$$\|v\|^p = \int_{S^{k-1}} |\langle v, u\rangle|^p \, \mathrm{d}\mu(u) \tag{19}$$

for any $v \in \mathbb{R}^k$. If p is not an even integer, such a measure is unique.

We shall need basic properties of divided differences. Define

$$[x_0]_f = f(x_0),$$

and recursively for any positive integer k define the divided difference of order k by

$$[x_0, x_1, \ldots, x_k]_f = \frac{[x_0, x_1, \ldots, x_{k-1}]_f - [x_1, x_2, \ldots, x_k]_f}{x_0 - x_k}, \tag{20}$$

when points $x_0, x_1, \ldots, x_k$ are pairwise distinct. If f is C^k, the divided difference of order k has continuous extension to all tuples of $(k+1)$ points. This extension satisfies (20) whenever $x_0 \neq x_k$ and

$$[x_0, x_0, \ldots, x_0]_f = \frac{f^{(k)}(x_0)}{k!},$$

where x_0 appears $(k+1)$ times. For this and lot more, see for instance [dB05].

Chapter 1

Tracial joint spectral measures

In this chapter we discuss tracial joint spectral measures (TJSMs) and their basic properties, with some applications. We begin by proving slight refinements of Theorems 6 (Existence of TJSMs) and 7 (Expression for TJSMs), Theorems 23 and 22.

The defining property of the tracial joint spectral measure of A and B is that for suitable f and $x, y \in \mathbb{R}$, one has

$$\operatorname{tr} H(f)(xA + yB) = \int_{\mathbb{R}^2} f(ax + by)\, \mathrm{d}\mu_{A,B}(a, b),$$

where

$$H(f)(x) = \int_0^1 \frac{1-t}{t} f(xt)\, \mathrm{d}t.$$

It is not difficult to heuristically recover $\mu_{A,B}$ from this identity. Indeed, plugging in $f(t) = e^{it}$ yields

$$\operatorname{tr} H(e^{it})(xA + yB) = \int_{\mathbb{R}^2} e^{i(ax+by)}\, \mathrm{d}\mu_{A,B}(a, b),$$

so one would guess that $\mu_{A,B}$ is the Fourier transform of the function $(x, y) \mapsto \operatorname{tr} H(e^{it})(xA + yB)$. This calculation however doesn't quite make sense since the H-transform cannot be applied to $t \mapsto e^{it}$.

The problem can be fixed by considering $f(t) = e^{it} - 1$ instead. Define a function $g : \mathbb{R} \to \mathbb{C}$ by

$$g(x) = \int_0^1 \frac{e^{ixt} - 1}{t}(1-t)\,\mathrm{d}t = H(t \mapsto e^{it} - 1)(x). \tag{1.1}$$

Observe that g is smooth, $g(0) = 0$, and $g(x) = O(\log|x|)$ at infinity. For any two Hermitian $A, B \in M_n(\mathbb{C})$, define then a continuous function G by

$$\begin{aligned} G := G_{A,B} : \mathbb{R}^2 &\to \mathbb{C} \\ (x, y) &\to \operatorname{tr} g(xA + yB). \end{aligned} \tag{1.2}$$

We will prove that $\mu_{A,B}$ is essentially the Fourier transform of $G_{A,B}$.

Theorem 22. *Let $A, B \in M_n(\mathbb{C})$ be Hermitian. Then there exists a positive measure $\mu_{A,B}$ which agrees with $\widehat{G}_{A,B}$ away from 0, in the sense that if φ is any Schwartz function with compact support not containing 0, then*

$$(\mu_{A,B}, \varphi) = \frac{1}{2\pi}\left(\widehat{G}_{A,B}, \varphi\right).$$

Denote by $\mu_{A,B} = \mu_c + \mu_s$ the Lebesgue decomposition of $\mu_{A,B}$ w.r.t. Lebesgue measure m_2 ($\mu_c \ll m_2$, $\mu_s \perp m_2$). If the pencil (A, B) is non-degenerate, then the continuous part μ_c is given by

$$\rho_{A,B}(a,b) := \frac{\mathrm{d}\mu_c}{\mathrm{d}m_2}(a,b) = \frac{1}{2\pi}\sum_{i=1}^{n}\left|\Im\left(\lambda_i\left(\left(I - \frac{aA + bB}{a^2 + b^2}\right)(bA - aB)^{-1}\right)\right)\right|. \tag{1.3}$$

If the pencil (A, B) is non-degenerate and the real roots of (A, B) are distinct, then the singular part μ_s satisfies

$$\mu_s(\varphi) = \sum_{v \in E(A,B)} \int_0^1 \frac{1-t}{t}\varphi\left(\frac{\langle Av, v\rangle}{\langle v, v\rangle}t, \frac{\langle Bv, v\rangle}{\langle v, v\rangle}t\right)\mathrm{d}t, \tag{1.4}$$

where φ is a Schwartz function with compact support not containing 0.

We note that for any $v \in E(A, B)$ corresponding to a root (a, b), $b\langle Av, v\rangle - a\langle Bv, v\rangle = 0$, and $\langle Av, v\rangle$ and $\langle Bv, v\rangle$ are real. If v further corresponds to a non-real root, then $\langle Av, v\rangle = 0 = \langle Bv, v\rangle$. Thus, the sum in (1.4) can equivalently be taken over all eigenvectors of (A, B) that correspond to real roots.

The points $(\langle Av, v\rangle/\langle v, v\rangle, \langle Bv, v\rangle/\langle v, v\rangle)$ where $v \in E(A, B)$ are called the *singular points*. They lie on the singular lines, as defined in section 0.3.

Remark 1. There is nothing particularly special about the expression

$$C := \left(I - \frac{aA+bB}{a^2+b^2}\right)(bA-aB)^{-1}.$$

Since eigenvalues of C and $C+tI$ have equal imaginary parts, we may replace C by anything of the form

$$(I - f(a,b)A - g(a,b)B)\,(bA-aB)^{-1}$$

where $f(a,b)a + g(a,b)b = 1$. The expression we chose has the desirable property of making sense for every $(a,b) \neq (0,0)$, and works well with the change of variables in the proof.

1.1 Proofs of the main results

Before proving Theorem 22, we will give a mock proof illustrating our strategy. The major unsound steps are indicated by numbered asterisks. We will comment on how to fix them afterwards.

Mock proof of Theorem 22. Our goal is to calculate the Fourier transform of G. By definition (1∗) we have

$$\widehat{G}(a,b) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i(ax+by)} \operatorname{tr} g(xA+yB)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2\pi}\sum_{i=1}^{n}\int_{\mathbb{R}^2} e^{-i(ax+by)} g(\lambda_i(xA+yB))\,\mathrm{d}x\,\mathrm{d}y.$$

Rewriting the integral in polar coordinates, we get

$$\begin{aligned}
&= \frac{1}{2\pi}\sum_{i=1}^{n}\int_0^{\pi}\int_{\mathbb{R}} e^{-ir(a\cos(\theta)+b\sin(\theta))}|r| g(\lambda_i(\cos(\theta)A+\sin(\theta)B)r)\,\mathrm{d}r\,\mathrm{d}\theta\\
&= \frac{1}{2\pi}\sum_{i=1}^{n}\int_0^{\pi}\frac{1}{\lambda_i(\cos(\theta)A+\sin(\theta)B)^2}\left(\int_{\mathbb{R}} e^{-ir\frac{a\cos(\theta)+b\sin(\theta)}{\lambda_i(\cos(\theta)A+\sin(\theta)B)}}|r|g(r)\,\mathrm{d}r\right)\mathrm{d}\theta\\
&= \frac{1}{\sqrt{2\pi}}\sum_{i=1}^{n}\int_0^{\pi}\frac{\mathcal{F}(x\mapsto |x|g(x))\left(\frac{a\cos(\theta)+b\sin(\theta)}{\lambda_i(\cos(\theta)A+\sin(\theta)B)}\right)}{\lambda_i(\cos(\theta)A+\sin(\theta)B)^2}\,\mathrm{d}\theta.
\end{aligned}$$

It turns out that the Fourier transform of $|x|g(x)$ (2∗) equals

$$\xi \mapsto \sqrt{\frac{2}{\pi}}\frac{1}{\xi^2}\log\left|1-\frac{1}{\xi}\right|, \tag{1.5}$$

so plugging this in, and multiplying the eigenvalues to obtain the determinant, allows us to simplify to

$$=\frac{1}{\pi}\sum_{i=1}^{n}\int_0^{\pi}\frac{\log\left|1-\frac{\lambda_i(\cos(\theta)A+\sin(\theta)B)}{a\cos(\theta)+b\sin(\theta)}\right|}{(a\cos(\theta)+b\sin(\theta))^2}\,\mathrm{d}\theta=\frac{1}{\pi}\int_0^{\pi}\frac{\log\left|\det\left(I-\frac{\cos(\theta)A+\sin(\theta)B}{a\cos(\theta)+b\sin(\theta)}\right)\right|}{(a\cos(\theta)+b\sin(\theta))^2}\,\mathrm{d}\theta.$$

Making a change of variable

$$t=\frac{1}{a^2+b^2}\frac{a\cos(\theta)+b\sin(\theta)}{b\cos(\theta)-a\sin(\theta)}$$

transforms (3∗) the integral to

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\log\left|\det\left(I-\frac{aA+bB}{a^2+b^2}+t(bA-aB)\right)\right|\mathrm{d}t.$$

Finally, we factorize the determinant and use the fact (4∗) that

$$\int_{-\infty}^{\infty}\log|a+bt|\,\mathrm{d}t=\pi\left|\Im\left(\frac{a}{b}\right)\right| \tag{1.6}$$

to prove the identity (1.3). △

(1∗) The function $G_{A,B}$ is not integrable, so we will instead calculate its Fourier transform as a distribution, testing against a Schwartz function φ.

(2∗) The function $|x|g(x)$ does not have a Fourier transform in the usual sense, and in any case (1.5) is not correct. In Lemma 2, we calculate the correct Fourier transform as a tempered distribution, which is similar to (1.5) but contains some corrections terms.

(3∗) The integrals at hand are not integrable. Instead, we apply a cutoff, split the integral to three parts, and apply a change of variables to each part.

(4∗) The identity (1.6) is not true, but instead we employ a similar looking identity from Lemma 3.

Additionally, this mock proof doesn't see the singular part. It is hidden (together with the terms making the expressions converge) in the correction terms of the Fourier transform of $|x|g(x)$. These terms bring complications, as we will need to understand the behaviour of the eigenvalues of C (as in Remark 1) near the singular lines. These eigenvalue estimates are done in Lemma 5.

Proof of Theorem 22. Our goal is to calculate the Fourier transform of $G=G_{A,B}$. Fix a Schwartz function φ. We can rewrite our integral in polar coordinates,

$$(\widehat{G},\varphi)=(G,\widehat{\varphi})=\int_{\mathbb{R}^2}\operatorname{tr} g(xA+yB)\widehat{\varphi}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$=\int_0^\pi\sum_{i=1}^n\left(\int_{\mathbb{R}}|r|g(\lambda_i(\cos(\theta)A+\sin(\theta)B)r)\widehat{\varphi}(r\cos(\theta),r\sin(\theta))\,\mathrm{d}r\right)\mathrm{d}\theta.$$

Let $\lambda=\lambda_i(\cos(\theta)A+\sin(\theta)B)$. The inner integral vanishes when $\lambda=0$, and when $\lambda\neq 0$ it equals

$$\int_{\mathbb{R}}|r|g(\lambda r)\widehat{\varphi}(r\cos(\theta),r\sin(\theta))\,\mathrm{d}x=\frac{1}{\lambda^2}\int_{\mathbb{R}}|r|g(r)\widehat{\varphi}\left(\frac{r\cos(\theta)}{\lambda},\frac{r\sin(\theta)}{\lambda}\right)\mathrm{d}r. \tag{1.7}$$

Let $\overline{v}_\theta=(\cos(\theta),\sin(\theta))$, $\overline{u}_\theta=(-\sin(\theta),\cos(\theta))$, and $\varphi(\overline{w})=\varphi(\overline{w}_1,\overline{w}_2)$, for any $\overline{w}\in\mathbb{R}^2$. By direct calculation, one sees that the term in the integrand

$$r\mapsto\widehat{\varphi}\left(\frac{r\cos(\theta)}{\lambda},\frac{r\sin(\theta)}{\lambda}\right)$$

is the Fourier transform of the function mapping x to

$$\frac{\lambda^2}{\sqrt{2\pi}}\int_{\mathbb{R}}\varphi(\lambda x\cos(\theta)-\lambda y\sin(\theta),\lambda x\sin(\theta)+\lambda y\cos(\theta))\,\mathrm{d}y=\frac{\lambda^2}{\sqrt{2\pi}}\int_{\mathbb{R}}\varphi(\lambda x\overline{v}_\theta+\lambda y\overline{u}_\theta)\,\mathrm{d}y=:\frac{\lambda^2}{\sqrt{2\pi}}\phi_{\theta,\lambda}.$$

Observe also that $\phi_{\theta,\lambda}(x)=\phi_{\theta,1}(\lambda x)/|\lambda|=:\phi_\theta(\lambda x)/|\lambda|$. Consequently, the integral in (1.7) simplifies to

$$\frac{1}{\lambda^2}\int_{\mathbb{R}}|r|g(r)\widehat{\varphi}\left(\frac{r}{\lambda}\overline{v}_\theta\right)\mathrm{d}r=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}|r|g(r)\widehat{\phi_{\theta,\lambda}}(r)\,\mathrm{d}r=\frac{1}{\sqrt{2\pi}}\left(\widehat{|r|g(r)},\phi_{\theta,\lambda}\right),$$

where by $\widehat{|r|g(r)}$ we mean the Fourier transform of the tempered distribution $r\mapsto|r|g(r)$.

Lemma 2. *For any Schwartz function ϕ on $\mathbb{R}$, one has*

$$(\widehat{|x|g(x)},\phi)=\sqrt{\frac{2}{\pi}}\int_{\mathbb{R}}(\phi(x)-\phi(0)-\phi'(0)x)\frac{1}{x^2}\log\left|1-\frac{1}{x}\right|\mathrm{d}x. \tag{1.8}$$

Proof. If $\phi(x)=x^2\psi(x)$ for some Schwartz function ψ, we have

$$(\widehat{|x|g(x)},\phi)=(-\partial_x^2\widehat{|x|g(x)},\psi)=\left(\widehat{\frac{1-e^{ix}}{|x|}},\psi\right)=\sqrt{\frac{2}{\pi}}\left(\log\left|1-\frac{1}{x}\right|,\psi\right)=\sqrt{\frac{2}{\pi}}\left(\frac{1}{x^2}\log\left|1-\frac{1}{x}\right|,\phi\right),$$

as desired. For the third equality, see [GS16, p. 361].

The result is thus true up to a multiple of δ_0 and δ_0'. To take care of them, consider the function

$$\phi_{\varepsilon,\alpha,\beta}(x) = (\alpha + \beta x) e^{-\frac{1}{2}\varepsilon^2 x^2}.$$

It is enough to check that for any fixed $\alpha, \beta \in \mathbb{R}$, when $\varepsilon \to 0^+$, both the left- and right-hand side of (1.8) tend to 0. For the right-hand side, this follows from the dominated convergence theorem. For the left-hand side, note that

$$\begin{aligned}(\widehat{|x|g(x)}, \phi_{\varepsilon,\alpha,\beta}) = (|x|g(x), \widehat{\phi_{\varepsilon,\alpha,\beta}}) &= \int_{\mathbb{R}} |x|g(x) \left(\frac{\alpha}{\varepsilon} + i\frac{\beta}{\varepsilon^3}x\right) e^{-\frac{1}{2\varepsilon^2}x^2} \,\mathrm{d}x \\ &= \int_{\mathbb{R}} |x|g(\varepsilon x)\, (\varepsilon\alpha + i\beta x)\, e^{-\frac{1}{2}x^2} \,\mathrm{d}x,\end{aligned}$$

which tends to 0 as $\varepsilon \to 0^+$, since g is continuous and $g(0) = 0$. □

Using Lemma 2, we can write

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \left(\widehat{|r|g(r)}, \phi_{\theta,\lambda}\right) &= \frac{1}{\pi}\int_{\mathbb{R}} (\phi_{\theta,\lambda}(r) - \phi_{\theta,\lambda}(0) - \phi_{\theta,\lambda}'(0) r) \frac{1}{r^2} \log\left|1 - \frac{1}{r}\right| \mathrm{d}r \\ &= \frac{1}{\pi}\int_{\mathbb{R}} (\phi_\theta(x) - \phi_\theta(0) - \phi_\theta'(0)x)\frac{1}{x^2}\log\left|1 - \frac{\lambda}{x}\right| \mathrm{d}x.\end{aligned}$$

Recall that λ was an eigenvalue of $\cos(\theta)A + \sin(\theta)B$. We can now sum the above expression over all the eigenvalues of $\cos(\theta)A + \sin(\theta)B$, and integrate over θ. The eigenvalues multiply to form the determinant, and we obtain

$$(\widehat{G}, \varphi) = \frac{1}{\pi}\int_0^\pi \int_{\mathbb{R}} (\phi_\theta(x) - \phi_\theta(0) - \phi_\theta'(0)x) \frac{\log\left|\det\left(I - \frac{\cos(\theta)A + \sin(\theta)B}{x}\right)\right|}{x^2} \,\mathrm{d}x\,\mathrm{d}\theta.$$

We would like to split this integral to the three parts corresponding to $\phi_\theta(x)$, $\phi_\theta(0)$, and $\phi_\theta'(0)$. While the resulting parts don't converge, we can remedy this with a cutoff:

$$\begin{aligned}(\widehat{G}, \varphi) = \lim_{\varepsilon\to 0^+} \Bigg(&\frac{1}{\pi}\int_0^\pi \int_{|x|>\varepsilon} \phi_\theta(x) \frac{\log\left|\det\left(I - \frac{\cos(\theta)A + \sin(\theta)B}{x}\right)\right|}{x^2} \,\mathrm{d}x\,\mathrm{d}\theta \\ &- \frac{1}{\pi}\int_0^\pi \int_{|x|>\varepsilon} \phi_\theta(0) \frac{\log\left|\det\left(I - \frac{\cos(\theta)A + \sin(\theta)B}{x}\right)\right|}{x^2} \,\mathrm{d}x\,\mathrm{d}\theta \\ &- \frac{1}{\pi}\int_0^\pi \int_{|x|>\varepsilon} \phi_\theta'(0) \frac{\log\left|\det\left(I - \frac{\cos(\theta)A + \sin(\theta)B}{x}\right)\right|}{x} \,\mathrm{d}x\,\mathrm{d}\theta \Bigg)\end{aligned}$$

We will now analyze the three integrals inside the limit for $\varepsilon > 0$; these integrals are absolutely integrable.

(i) **The $\phi_\theta(x)$-term:** By definition of ϕ_θ,

$$\frac{1}{\pi}\int_0^\pi \int_{|x|>\varepsilon} \phi_\theta(x) \frac{\log\left|\det\left(I - \frac{\cos(\theta)A + \sin(\theta)B}{x}\right)\right|}{x^2} \,\mathrm{d}x\,\mathrm{d}\theta$$
$$= \frac{1}{\pi}\int_0^\pi \int_{|x|>\varepsilon}\int_{\mathbb{R}} \varphi(\lambda x\overline{v}_\theta + \lambda y\overline{u}_\theta) \frac{\log\left|\det\left(I - \frac{\cos(\theta)A + \sin(\theta)B}{x}\right)\right|}{x^2} \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\theta$$

We make the change of variables

$$(a, b, t) = \left(x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta), \frac{y}{x(x^2 + y^2)}\right),$$

and get

$$\int_{t^2(a^2+b^2)^2 < (a^2+b^2)/\varepsilon^2 - 1} \log\left|\det\left(I - \frac{aA + bB}{a^2 + b^2} - t(bA - aB)\right)\right| \varphi(a, b)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}t.$$

(ii) **The $\phi_\theta(0)$-term:** With a change of variables similar to the previous case,

$$(a, b, t) = \left(-\sin(\theta)y, \cos(\theta)y, \frac{1}{xy}\right),$$

we simplify to

$$\int_{t^2(a^2+b^2)^2 < (a^2+b^2)/\varepsilon^2} \log\left|\det\left(I - t(bA - aB)\right)\right| \varphi(a, b)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}t.$$

(iii) **The $\phi'_\theta(0)$-term:** With the same change of variables as in the $\phi_\theta(0)$ case, one can simplify to

$$\int_{t^2(a^2+b^2)^2 < (a^2+b^2)/\varepsilon^2} \log\left|\det\left(I - t(bA - aB)\right)\right| \frac{\frac{d}{dh}\varphi(a + hb, b - ha)\Big|_{h=0}}{(a^2 + b^2)t}\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}t.$$

At this point, we have proven that

$$\left(\widehat{G},\varphi\right) = \lim_{\varepsilon\to 0^+}\left(\int_{t^2(a^2+b^2)^2<(a^2+b^2)/\varepsilon^2-1}\log\left|\det\left(I-\frac{aA+bB}{a^2+b^2}-t(bA-aB)\right)\right|\varphi(a,b)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}t\right.$$
$$-\int_{t^2(a^2+b^2)^2<(a^2+b^2)/\varepsilon^2}\log|\det(I-t(bA-aB))|\,\varphi(a,b)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}t$$
$$\left.-\int_{t^2(a^2+b^2)^2<(a^2+b^2)/\varepsilon^2}\log|\det(I-t(bA-aB))|\,\frac{\frac{d}{dh}\varphi(a+hb,b-ha)\Big|_{h=0}}{(a^2+b^2)t}\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}t\right).$$

Write

$$C_1(a,b) := \left(I-\frac{aA+bB}{a^2+b^2}\right)(bA-aB)^{-1},\ \text{ and } C_2(a,b) := (bA-aB)^{-1}. \tag{1.9}$$

To prove identity (1.3), we recall our additional assumption that (A,B) is non-degenerate. We now also need to assume that φ has compact support not containing 0. We can then rewrite our expression as

$$\frac{1}{\pi}\lim_{\varepsilon\to 0^+}\left(\int_{(a^2+b^2)/\varepsilon^2-1<t^2(a^2+b^2)^2<(a^2+b^2)/\varepsilon^2}\log|\det(C_1(a,b)-t)|\,\varphi(a,b)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}t\right.$$
$$+\int_{t^2(a^2+b^2)<1/\varepsilon^2}\log\left|\frac{\det(C_1(a,b)-tI)}{\det(C_2(a,b)-tI)}\right|\varphi(a,b)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}t$$
$$-\int_{(a^2+b^2)/\varepsilon^2-1<t^2(a^2+b^2)^2<(a^2+b^2)/\varepsilon^2}\log|\det(C_2(a,b))|\,\varphi(a,b)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}t$$
$$\left.-\int_{t^2(a^2+b^2)<1/\varepsilon^2}\log\left|\det\left(I-tC_2(a,b)^{-1}\right)\right|\frac{\frac{d}{dh}\varphi(a+hb,b-ha)\Big|_{h=0}}{(a^2+b^2)t}\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}t\right).$$

We claim that the first term here tends to 0 as $\varepsilon\to 0^+$. To that end, we integrate first over t and then a and b. Observe that for small ε and fixed a and b, the term $\log|\det(C_1(a,b)-t)|$ is a sum of functions $\log|t-c|$, where c is an eigenvalue of $C_1(a,b)$. The integral in t is over an interval of length $O(\varepsilon)$ with distance $O(1/\varepsilon)$ from 0. One checks that such an integral is $O(\varepsilon\log(1+|c|)+\varepsilon\log(1/\varepsilon))$, i.e.

$$\left|\int_{(a^2+b^2)/\varepsilon^2-1<t^2(a^2+b^2)^2<(a^2+b^2)/\varepsilon^2}\log|\det(C_1(a,b)-t)|\,\varphi(a,b)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}t\right|$$
$$= O\left(\int_{\mathbb{R}^2}(\varepsilon\sum_{i=1}^{n}\log(1+|\lambda_i(C_1(a,b))|)+\varepsilon\log(1/\varepsilon))|\varphi(a,b)|\,\mathrm{d}a\,\mathrm{d}b\right).$$

Since the eigenvalues of $C_1(a,b)$ explode at most polynomially near the singular lines of $C_1(a,b)$, these integrals converge to zero. By a similar argument, one sees that the third term converges to zero.

So, we know that if φ has compact support not containing 0, then

$$\begin{aligned}\left(\widehat{G}, \varphi\right) =& \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \left(\int_{t^2(a^2+b^2)<1/\varepsilon^2} \log \left| \frac{\det\left(C_1(a,b) - tI\right)}{\det(C_2(a,b) - tI)} \right| \varphi(a,b)\, \mathrm{d}a\, \mathrm{d}b\, \mathrm{d}t \right. \\ & \left. - \int_{t^2(a^2+b^2)<1/\varepsilon^2} \log \left| \det\left(I - tC_2(a,b)^{-1}\right) \right| \frac{\frac{d}{dh}\varphi(a+hb, b-ha)\Big|_{h=0}}{(a^2+b^2)t}\, \mathrm{d}a\, \mathrm{d}b\, \mathrm{d}t \right).\end{aligned}$$

We will now integrate out t using the following two computational lemmas.

Lemma 3. *For any $\lambda \in \mathbb{C}$, consider the integral*

$$I_1(\lambda, M) := \int_{|t|<M} \log \left| 1 - \frac{\lambda}{t} \right| \mathrm{d}t.$$

Then, we have

$$\begin{aligned} & I_1(\lambda, M) \\ =& M \log \left| 1 - \frac{\lambda^2}{M^2} \right| + \Re(\lambda) \log \left| \frac{\lambda + M}{\lambda - M} \right| + \Im(\lambda) \left(\arctan\left(\frac{M - \Re(\lambda)}{\Im(\lambda)} \right) - \arctan\left(\frac{-M - \Re(\lambda)}{\Im(\lambda)} \right) \right) \end{aligned} \tag{1.10}$$

$$=: \pi |\Im(\lambda)| + E_1(\lambda, M),$$

where

$$E_1(\lambda, M) = O\left(M \log\left(1 + \frac{\lambda^2}{M^2} \right) \right) + |\Im(\lambda)| o(1);$$

the $o(1)$ term tends to zero with $|\lambda|/M$.

Proof. The first identity is straightforward, if somewhat tedious to verify. The limit of the expression is $\pi|\Im(\lambda)|$, so it remains to prove the error term estimate.

By scaling, we may assume that $M = 1$. Since logarithm is locally integrable, it is enough to consider the cases with $|\lambda| \ll 1$ and $|\lambda| \gg 1$. The first two terms of (1.10) can be estimated via Taylor expansion, while the third term is straightforward. □

Lemma 4. *For any $\lambda \in \mathbb{R}$, consider the integral*

$$I_2(\lambda, M) := \int_{|t|<M} \log \left| 1 - \frac{t}{\lambda} \right| \frac{\mathrm{d}t}{t}.$$

Then,

$$I_2(\lambda, M) = -\frac{\pi^2}{2} \operatorname{sign}(\lambda) + E_2(\lambda, M),$$

where the $E_2(\lambda, M)$ is bounded and tends to zero with $|\lambda|/M$.

Proof. By scaling, we may assume that $\lambda = 1$. We have

$$\begin{aligned} \lim_{M\to\infty} \int_{|t|<M} \log|1-t| \frac{\mathrm{d}t}{t} &= \int_0^\infty \log \left| \frac{1-t}{1+t} \right| \frac{\mathrm{d}t}{t} = \int_0^1 \log \left(\frac{1-t}{1+t} \right) \frac{\mathrm{d}t}{t} + \int_1^\infty \log \left(\frac{t-1}{t+1} \right) \frac{\mathrm{d}t}{t} \\ &= 2 \int_0^1 \log \left(\frac{1-t}{1+t} \right) \frac{\mathrm{d}t}{t} = -2 \operatorname{Li}_2(1) + 2 \operatorname{Li}_2(-1) = -\frac{\pi^2}{2}. \end{aligned}$$

Here, Li_2 stands for the dilogarithm function, defined as

$$\operatorname{Li}_2(z) = \int_0^z \frac{\log(1-t)}{t} \, \mathrm{d}t,$$

whose properties and special values are well-documented (see [Zag07]). The error term estimates are straightforward. □

Having integrated out t, we are then left with

$$
\begin{aligned}
&\left(\widehat{G}, \varphi\right) \\
&= \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \left(\int_{\mathbb{R}^2} \left[\sum_{i=1}^{n} I_1 \left(\lambda_i \left(C_1(a,b)\right), \frac{1}{\varepsilon\sqrt{a^2+b^2}} \right) - \sum_{i=1}^{n} I_1 \left(\lambda_i \left(C_2(a,b)\right), \frac{1}{\varepsilon\sqrt{a^2+b^2}} \right) \right] \varphi(a,b) \, \mathrm{d}a \, \mathrm{d}b \right. \\
&\left. - \int_{\mathbb{R}^2} \left[\sum_{i=1}^{n} I_2 \left(\lambda_i \left(C_2(a,b)\right), \frac{1}{\varepsilon\sqrt{a^2+b^2}} \right) \right] \frac{\frac{d}{dh}\varphi(a+hb, b-ha)\Big|_{h=0}}{(a^2+b^2)} \, \mathrm{d}a \, \mathrm{d}b \right) \\
&= \int_{\mathbb{R}^2} \sum_{i=1}^{n} \left|\Im(\lambda_i(C_1(a,b)))\right| \varphi(a,b) \, \mathrm{d}a \, \mathrm{d}b \qquad (1.11) \\
&+ \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \left(\int_{\mathbb{R}^2} \left[\sum_{i=1}^{n} E_1 \left(\lambda_i \left(C_1(a,b)\right), \frac{1}{\varepsilon\sqrt{a^2+b^2}} \right) - \sum_{i=1}^{n} E_1 \left(\lambda_i \left(C_2(a,b)\right), \frac{1}{\varepsilon\sqrt{a^2+b^2}} \right) \right] \varphi(a,b) \, \mathrm{d}a \, \mathrm{d}b \right. \qquad (1.12) \\
&\left. - \int_{\mathbb{R}^2} \left[\sum_{i=1}^{n} \left(-\frac{\pi^2}{2} \operatorname{sign}(\lambda_i(C_2(a,b))) + E_2 \left(\lambda_i \left(C_2(a,b)\right), \frac{1}{\varepsilon\sqrt{a^2+b^2}} \right) \right) \right] \frac{\frac{d}{dh}\varphi(a+hb, b-ha)\Big|_{h=0}}{(a^2+b^2)} \, \mathrm{d}a \, \mathrm{d}b \right). \qquad (1.13)
\end{aligned}
$$

We are finally ready to isolate the continuous part. Indeed, we will assume that φ has compact support disjoint from the singular lines. In this case, the eigenvalues of $C_1(a,b)$ and $C_2(a,b)$ are bounded uniformly on the support of φ, so the error terms E_1 and E_2 tend uniformly to zero. Additionally, the sign term is locally constant, so integrating it by parts against the derivative term along circular arcs yields 0. We are left with the first term, (1.11), which is the desired continuous part (after dividing by 2π).

It remains to work out the singular part. We recall the additional assumption that (A,B) does not have repeated real roots. If (A,B) does not have real roots, there is no singular part. Otherwise, we can assume that $(0,1) \in \mathbb{RP}^1$ is one of the roots. This means that B is singular, and the a-axis is a singular line; it suffices to consider φ for which the support hits only this single singular line. In the following lemma, we analyze the behaviour of the eigenvalues of $C_1(a,b)$ and $C_2(a,b)$ near this singular line.

Lemma 5. *Assume that the pencil (A,B) has a simple root $(0,1)$ with unit eigenvector v. Then $\langle Av, v\rangle \neq 0$, and for $a \neq 0$, the matrices $C_1(a,b)$ and $C_2(a,b)$ (as defined in (1.9)) have big eigen-*

values, with asymptotics as follows,

$$\lambda_{\text{big}}\left(C_1(a,b)\right) = \frac{a - \langle Av, v\rangle}{ab\langle Av, v\rangle}(1 + O(b^{1/n})),$$
$$\lambda_{\text{big}}\left(C_2(a,b)\right) = \frac{1}{b\langle Av, v\rangle}(1 + O(b)).$$

The O-terms are uniform for $(a,b) \in K \times (-\delta, \delta)$, where K is compact and does not contain 0 *and δ is sufficiently small. All the other eigenvalues of $C_1(a,b)$ and $C_2(a,b)$ are $O(b^{1/n-1})$ and $O(1)$ respectively, with the same uniformity properties.*

Proof. By our assumption, $\det(xA + B)$ has a single zero at 0. By expanding this determinant in an eigenbasis for B, we can see that $\langle Av, v\rangle \neq 0$.

Observe that

$$C_2(a,b) = \frac{1}{b\langle Av, v\rangle} vv^* + O(1),$$

where the error term is Hermitian with uniformly bounded entries. This follows at once from Cramer's rule, applied in an eigenbasis for B. Consequently, for $C_1(a,b)$ we have

$$C_1(a,b) = \frac{1}{b\langle Av, v\rangle}\left(I - \frac{aA + bB}{a^2 + b^2}\right) vv^* + O(1),$$

where the error term again has uniformly bounded entries but not necessarily Hermitian. The main terms have the desired eigenvalues, and the error estimates follow from well-known eigenvalue perturbation bounds for general (C_1) and Hermitian (C_2) matrices; see [Kat66]. □

These eigenvalue estimates imply that (1.11) is indeed integrable.

We will evaluate (1.12) and (1.13) using the eigenvalue expansions, starting with the former. Making the change of variable $b = c\varepsilon$, and observing that $E_1(s,t) = E_1(\varepsilon s, \varepsilon t)/\varepsilon$, we can rewrite (1.12) as

$$\frac{1}{\pi}\int_{\mathbb{R}^2}\sum_{i=1}^{n}\left[E_1\left(\varepsilon\lambda_i\left(C_1(a,\varepsilon c)\right), \frac{1}{\sqrt{a^2+\varepsilon^2c^2}}\right) - E_1\left(\varepsilon\lambda_i\left(C_2(a,\varepsilon c)\right), \frac{1}{\sqrt{a^2+\varepsilon^2c^2}}\right)\right]\varphi(a,\varepsilon c)\,\mathrm{d}a\,\mathrm{d}c.$$

By Lemma (5), the integrand converges pointwise, and we obtain

$$\frac{1}{\pi}\int_{\mathbb{R}^2}\left[I_1\left(\frac{a - \langle Av, v\rangle}{ca\langle Av, v\rangle}, \frac{1}{|a|}\right) - I_1\left(\frac{1}{c\langle Av, v\rangle}, \frac{1}{|a|}\right)\right]\varphi(a,0)\,\mathrm{d}a\,\mathrm{d}c.$$

To justify taking the limit inside, it suffices to note that by Lemma 5, for $i \in [n]$, we have $\varepsilon|\lambda_i\left(C_1(a, \varepsilon c)\right)| = O(1/|c|)$, with imaginary part $O(|c|^{1/n-1}\varepsilon^{1/n})$. Apply the estimates of Lemma (3) to get an integrable majorant $c \mapsto C(\log(1+1/c^2) + |c|^{1/n-1})$ for some $C > 0$.

We may now use Lemma 3 to evaluate the I_1-terms to get

$$\frac{1}{\pi}\int_{\mathbb{R}^2}\left[\frac{a-\langle Av,v\rangle}{c|a|\langle Av,v\rangle}\log\left|\frac{\frac{a-\langle Av,v\rangle}{\langle Av,v\rangle}+c}{\frac{a-\langle Av,v\rangle}{\langle Av,v\rangle}-c}\right| + \log\left|1-\frac{1}{c^2}\frac{(a-\langle Av,v\rangle)^2}{\langle Av,v\rangle^2}\right|\right]\varphi(a,0)\,\mathrm{d}a\,\mathrm{d}c$$
$$-\frac{1}{\pi}\int_{\mathbb{R}^2}\left[\frac{a}{c|a|\langle Av,v\rangle}\log\left|\frac{\frac{a}{\langle Av,v\rangle}+c}{\frac{a}{\langle Av,v\rangle}-c}\right| + \log\left|1-\frac{1}{c^2}\frac{a^2}{\langle Av,v\rangle^2}\right|\right]\varphi(a,0)\,\mathrm{d}a\,\mathrm{d}c.$$

Finally, use Lemma 3 and 4 to calculate the integral in c, ending up with

$$\pi\int_{\mathbb{R}}\left(\left|\frac{1}{a}-\frac{1}{\langle Av,v\rangle}\right| - \frac{1}{|\langle Av,v\rangle|}\right)\varphi(a,0)\,\mathrm{d}a.$$

We will now turn to (1.13). Since the E_2-term is bounded and converges to zero, it vanishes in the limit, and we are left with

$$\frac{\pi}{2}\int_{\mathbb{R}^2}\left[\sum_{i=1}^{n}\operatorname{sign}(\lambda_i(C_2(a,b)))\right]\frac{\frac{d}{dh}\varphi(a+hb,b-ha)\Big|_{h=0}}{(a^2+b^2)}\,\mathrm{d}a\,\mathrm{d}b$$
$$=\frac{\pi}{2}\int_{\mathbb{R}^2}\operatorname{sign}(\langle Av,v\rangle b)\frac{\frac{d}{dh}\varphi(a+hb,b-ha)\Big|_{h=0}}{(a^2+b^2)}\,\mathrm{d}a\,\mathrm{d}b.$$

Here, the equality follows from the fact that only the big eigenvalue can change its sign in the support of φ, and its sign is determined by Lemma 5. This integral can be further simplified by integration along the $\sqrt{a^2+b^2}$-radius arcs, with say the change of variables $(a,b) = (r\cos(\theta), r\sin(\theta))$. This results in the integral

$$\pi\int_{\mathbb{R}}\frac{\operatorname{sign}(\langle Av,v\rangle)}{a}\varphi(a,0)\,\mathrm{d}a.$$

Putting the terms together, we can see that the singular part is given by

$$\pi\int_{\mathbb{R}}\left(\left|\frac{1}{a}-\frac{1}{\langle Av,v\rangle}\right| - \frac{1}{|\langle Av,v\rangle|} + \frac{\operatorname{sign}(\langle Av,v\rangle)}{a}\right)\varphi(a,0)\,\mathrm{d}a$$
$$= 2\pi\int_0^1\frac{1-t}{t}\varphi(\langle Av,v\rangle t,0)\,\mathrm{d}t = 2\pi\int_0^1\frac{1-t}{t}\varphi(\langle Av,v\rangle t,\langle Bv,v\rangle)\,\mathrm{d}t,$$

as desired.

It remains to get rid of the extra assumptions for the existence of $\mu_{A,B}$. This can be done with approximation: one can find a sequence of pairs (A_m, B_m) converging to (A, B), such that 1) pencils (A_m, B_m) are non-degenerate, and 2) all roots of (A_m, B_m) are pairwise distinct. These conditions are Zariski open, so are satisfied by small generic perturbations. Then, $(\widehat{G_{A_m,B_m}}, \varphi) = (G_{A_m,B_m}, \widehat{\varphi}) \to (G_{A,B}, \widehat{\varphi}) = (\widehat{G_{A,B}}, \varphi)$ for any φ as before. So, $\widehat{G_{A,B}}$ is a weak limit of positive measures and hence a positive measure itself. □

Theorem 23. *Let $\mu_{A,B}$ be as in Theorem 22. Fix any measurable function f with $f(0) = 0$ such that for any $M > 0$,*

$$\int_{-M}^{M} \left| \frac{f(t)}{t} \right| \mathrm{d}t < \infty.$$

Define a function $H(f) : \mathbb{R} \to \mathbb{R}$ by

$$H(f)(x) = \int_0^1 \frac{1-t}{t} f(xt) \, \mathrm{d}t.$$

Then, for any $x, y \in \mathbb{R}$, one has

$$\operatorname{tr} H(f)(Ax + By) = \int_{\mathbb{R}^2} f(ax + by) \, \mathrm{d}\mu_{A,B}(a, b). \tag{1.14}$$

Proof. Let us start by considering a Schwartz function f with compact support not containing 0. By a change of variables (see Proposition 3, (1)), we may assume that $(x, y) = (1, 0)$. Define $\varphi_\varepsilon(a, b) = f(a)e^{-1/2\varepsilon^2 b^2}$. By the defining property of the measure $\mu_{A,B}$ from Theorem 22,

$$\begin{aligned}
\int_{\mathbb{R}^2} f(a) \, \mathrm{d}\mu_{A,B}(a, b) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^2} \varphi_\varepsilon(a, b) \, \mathrm{d}\mu_{A,B}(a, b) &= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^2} \widehat{\varphi}_\varepsilon(a, b) G_{A,B}(a, b) \, \mathrm{d}a \, \mathrm{d}b \\
&= \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \widehat{f}(a) e^{-\frac{1}{2\varepsilon^2} b^2} \operatorname{tr} g(aA + bB) \, \mathrm{d}a \, \mathrm{d}b = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^2} \widehat{f}(a) e^{-\frac{1}{2} b^2} \operatorname{tr} g(aA + c\varepsilon B) \, \mathrm{d}a \, \mathrm{d}c \\
&= \int_{\mathbb{R}^2} \widehat{f}(a) e^{-\frac{1}{2} b^2} \operatorname{tr} g(aA) \, \mathrm{d}a \, \mathrm{d}c = \sqrt{2\pi} \sum_{i=1}^n \int_{\mathbb{R}} \widehat{f}(a) g(a\lambda_i(A)) \, \mathrm{d}a \\
&= \sqrt{2\pi} \sum_{i=1}^n \int_{\mathbb{R}} \mathcal{F}(x \mapsto f(\lambda_i(A)x))(a) g(a) \, \mathrm{d}a = \sqrt{2\pi} \sum_{i=1}^n \left(\widehat{g}, x \mapsto f(\lambda_i(A)x)\right).
\end{aligned}$$

It therefore suffices to check that

$$\int_0^1 \frac{1-t}{t} f(t) \, \mathrm{d}t = \frac{1}{\sqrt{2\pi}} \left(\hat{g}, f\right).$$

Writing $f(t) = t^2 h(t)$, we are left to verify that

$$\int_0^1 t(1-t)h(t)\,\mathrm{d}t = -\frac{1}{\sqrt{2\pi}}\left(\widehat{\partial^2 g}, h\right) = -\frac{1}{\sqrt{2\pi}}\left(\mathcal{F}\left(x \mapsto \frac{(x+2i)e^{ix}}{x^3} + \frac{x-2i}{x^3}\right), h\right).$$

But this is straightforward to check by calculating the inverse Fourier transform of $t(1-t)\chi_{[0,1]}(t)$.

A general f can be dealt with approximation. Start by assuming that f is bounded and compactly supported with the support not containing 0. One can then find a sequence $(f_m)_{m=1}^{\infty}$ of Schwartz functions with the same bound converging pointwise a.e. to f. Dominated convergence theorem then implies that both sides of (1.14) converge when $m \to \infty$, so the identity (1.14) is also true for such an f. A general non-negative f can be now dealt with monotone convergence theorem, and to finish, decompose f to positive and negative parts. □

Remark 2. If $h := H(f)$, one may calculate

$$\begin{aligned} h'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 f(xt)\frac{1-t}{t}\,\mathrm{d}t = \int_0^1 f'(xt)(1-t)\,\mathrm{d}t = \frac{1}{x}\int_0^1 f(xt)\,\mathrm{d}t \\ h^{(2)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x}\int_0^1 f(xt)\,\mathrm{d}t = -\frac{1}{x^2}\int_0^1 f(xt) + \frac{1}{x}\int_0^1 f'(xt)t\,\mathrm{d}t = -\frac{2}{x^2}\int_0^1 f(xt) + \frac{1}{x^2}f(x), \end{aligned}$$

so that $2xh'(x) + x^2 h^{(2)}(x) = f(x)$. This allows one to recover f from $H(f)$, at least if $H(f)$ is regular enough, say twice continuously differentiable.

1.2 Basic properties

Proposition 3. *1. (Basis change) Let $V : \mathbb{R}^2 \to \mathbb{R}^2$ be linear and invertible with $V(a,b) = (v_{1,1}a + v_{1,2}b, v_{2,1}a + v_{2,2}b)$ for $a, b \in \mathbb{R}$. Define $(A', B') = V(A,B) = (v_{1,1}A + v_{1,2}B, v_{2,1}A + v_{2,2}B)$. Then, $\mu_{A',B'}$ is given by the pushforward measure $V_*(\mu_{A,B})$.*

2. (Invariance) The measure $\mu_{A,B}$ only depends on the homogeneous polynomial, the so called Kippenhahn polynomial,

$$p_{A,B}(x,y,z) = \det(zI + xA + yB),$$

in the sense that if $p_{A,B} = p_{A',B'}$ for a different pair (A',B'), then $\mu_{A,B} = \mu_{A',B'}$.

3. *(Block matrices) Assume that (A,B) is block diagonal, i.e. in some basis we have*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \text{ and } B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

for some $(A_1, B_1) \in M_{n_1}(\mathbb{C}) \times M_{n_1}(\mathbb{C})$ and $(A_2, B_2) \in M_{n_2}(\mathbb{C}) \times M_{n_2}(\mathbb{C})$ (with $n_1 + n_2 = n$, $n_1, n_2 > 0$). Then,

$$\mu_{A,B} = \mu_{A_1,B_1} + \mu_{A_2,B_2}. \tag{1.15}$$

4. *If $p_{A,B}$ is reducible, then for some $n_1, n_2 > 0$ with $n_1 + n_2 = n$, there exists $(A_1, B_1) \in M_{n_1}(\mathbb{C}) \times M_{n_1}(\mathbb{C})$ and $(A_2, B_2) \in M_{n_2}(\mathbb{C}) \times M_{n_2}(\mathbb{C})$ such that $p_{A,B} = p_{A_1,B_1} p_{A_2,B_2}$ and hence (1.15) holds.*

Proof. 1. It follows by manipulating (1.14) that $V_*(\mu_{A,B})$ satisfies the defining identity (1.14) for $\mu_{A',B'}$. Hence, we will be done by uniqueness of the measure, Proposition 5.

Alternatively, this follows from the explicit formulas (1.3) and (1.4). Indeed, Remark 1 implies that

$$\rho_{A',B'} = \frac{1}{|\det(V)|} \rho_{A,B} \circ V^{-1},$$

which is exactly the density of the pushforward of the continuous part of $\mu_{A,B}$. The singular points are respected by the pushforward, and hence the singular part in its entirety.

2. This is clear since the left-hand side of (1.14) only depends on the eigenvalues (with multiplicities) of linear combinations of A and B; and these are the same for (A, B) and (A', B') if $p_{A,B} = p_{A',B'}$.

3. This follows from $\operatorname{tr} f(xA + yB) = \operatorname{tr} f(xA_1 + yB_1) + \operatorname{tr} f(xA_2 + yB_2)$, and uniqueness of the measure.

4. The polynomial $p_{A,B}$ and hence all its factors are hyperbolic in the sense of Gårding [Går59]. The existence of (A_1, B_1) and (A_2, B_2) then follows from the Helton–Vinnikov theorem [HV07].

□

Property (4) of Proposition 3 implies that degenerate pencils can be reduced to the non-degenerate case by factoring the polynomial $p_{A,B}$.

Proposition 4. *Let μ_c be the continuous part of the measure $\mu_{A,B}$ for some Hermitian matrices A and B. Then the following things hold true:*

1. *For $(a,b) \in \mathbb{R}^2$ we have $\rho_{A,B}(a,b) > 0$ iff $(A - aI, B - bI)$ has non-real root.*

2. *If the pencil $(A - aI, B - bI)$ is definite, then $\rho_{A,B}(a,b) = 0$. In particular, μ_c is supported on the joint numerical range of (A,B):*

$$\operatorname{supp}(\mu_c) \subset W(A,B) \subset [\min(\lambda(A)), \max(\lambda(A))] \times [\min(\lambda(B)), \max(\lambda(B))].$$

3. *$\mu_c \equiv 0$ iff A and B commute.*

Proof. 1. Roots of the pencil $(A - aI, B - bI)$ are images of the eigenvalues of $C_1(a,b)$ (see (1.9)) under the map

$$\lambda \mapsto \frac{b\lambda - a}{a\lambda + b}\frac{1}{a^2 + b^2},$$

2. The first claim follows from the previous part and the Lemma 1. For the second claim, observe that if $\alpha(A - aI) + \beta(B - \beta)$ is positive definite, (α, β) is exactly the functional separating (a,b) from the convex set $W(A,B)$.

3. Assume that $\mu_c \equiv 0$ so that

$$\operatorname{tr} H(f)(Ax + By) = \sum_{v \in E(A,B)} H(f)\left(\frac{\langle Av, v\rangle}{\langle v, v\rangle}x + \frac{\langle Bv, v\rangle}{\langle v, v\rangle}y\right).$$

Assuming w.l.o.g. that the largest eigenvalue of A equals 1, we will apply this identity with $(x,y) = (1,0)$ and the function $f = \chi_{x > 1-\varepsilon}$ for $0 < \varepsilon < 1$. Note that $H(f)$ vanishes on $(-\infty, 1-\varepsilon]$ and is positive on $(1-\varepsilon, \infty)$. If $\langle Av, v\rangle < \langle v, v\rangle$ for every $v \in E(A,B)$, then we may choose ε so small that LHS is positive while RHS is zero, a contraction. If on the other hand $\langle Av, v\rangle = \langle v, v\rangle$, then v maximizes the Rayleigh quotient, and is hence an eigenvector of A. Since it is also a root of the pencil (A,B), it must be eigenvector of B. Consequently A and B have a common eigenvector and we are done by induction.

□

Proposition 5. *There is at most one measure μ with $\mu(\{0\}) = 0$ satisfying the condition of Theorem 23.*

Proof. While μ is not a finite measure, the measure $\tilde{\mu}$ defined by

$$\tilde{\mu}(f) = \int_{\mathbb{R}^2} (a^2 + b^2) f(a,b) \,\mathrm{d}\mu(a,b)$$

is. Applying (1.14) for different polynomials and $x, y \in \mathbb{R}$, one can solve all moments of $\tilde{\mu}$. As $\tilde{\mu}$ is a compactly supported measure by Proposition 4, the moments uniquely determine it (via expansion of its characteristic function), and hence also μ. $\square$

1.3 Applications to polynomials

Applying Theorem 6 to integer powers $f(t) = t^k$ for $k > 0$ yields

$$\frac{1}{k(k+1)} \operatorname{tr}(xA + yB)^k = \int_{\mathbb{R}^2} (ax + by)^k \,\mathrm{d}\mu_{A,B}(a,b). \tag{1.16}$$

Write

$$\operatorname{tr}(xA + yB)^k = \sum_{l=0}^{k} \binom{k}{l} s_{l,k-l}(A,B) x^l y^{k-l}.$$

The coefficient $s_{k,l}(A,B)$ is equal to the average of traces of words with k A's and l B's. If $\max(k,l) \leq 1$, $s_{k,l}(A,B) = \operatorname{tr}(A^k B^l)$, while for instance $s_{2,2}(A,B) = (2\operatorname{tr}(A^2B^2) + \operatorname{tr}(ABAB))/3$ and $s_{3,2}(A,B) = (\operatorname{tr}(A^3B^2) + \operatorname{tr}(A^2BAB))/2$.

Equating coefficients in (1.16) one finds expressions for mixed moments:

$$\int_{\mathbb{R}^2} a^k b^l \,\mathrm{d}\mu_{A,B}(a,b) = \frac{s_{k,l}(A,B)}{(k+l)(k+l+1)}$$

for $k, l > 0$. This allows one to deduce a trace inequality for any non-negative bivariate polynomial vanishing at origin.

Corollary 2. *If $p(a,b) = \sum_{k,l\geq 0} c_{k,l} a^k b^l$ is non-negative with $c_{0,0} = 0$, then for any Hermitian $A, B \in M_n(\mathbb{C})$*

$$0 \leq \int_{\mathbb{R}^2} p(a,b) \,\mathrm{d}\mu_{A,B}(a,b) = \sum_{k,l\geq 0} c_{k,l} \frac{s_{k,l}(A,B)}{(k+l)(k+l+1)}.$$

In particular, if p is homogeneous and non-negative with $p(a,b) = \sum_{l=0}^{k} c_l a^l b^{k-l}$, then also

$$\sum_{l=0}^{k} c_l s_{l,k-l}(A,B) \geq 0$$

for any Hermitian $A, B \in M_n(\mathbb{C})$.

Example 3. Considering $p(a,b) = (a^2 + b^2 - a)^2$ in Corollary 2 yields

$$\operatorname{tr}(A^2) - (\operatorname{tr}(A^3) + \operatorname{tr}(AB^2)) + \frac{3\operatorname{tr}(A^4) + 4\operatorname{tr}(A^2B^2) + 2\operatorname{tr}(ABAB) + 3\operatorname{tr}(B^4)}{10} \geq 0.$$

This inequality doesn't have any non-trivial equality cases. Indeed, if

$$\int_{\mathbb{R}^2} (a^2 + b^2 - a)^2 \, \mathrm{d}\mu_{A,B}(a,b) = 0,$$

measures $\mu_{A,B}$ would need to be supported on a circle $\{(a,b) \mid a^2 + b^2 - a\}$ and this is only possible if $A = 0 = B$.

More generally, one could ask for the best constant c in the inequality

$$c\operatorname{tr}(A^2) - (\operatorname{tr}(A^3) + \operatorname{tr}(AB^2)) + \frac{3\operatorname{tr}(A^4) + 4\operatorname{tr}(A^2B^2) + 2\operatorname{tr}(ABAB) + 3\operatorname{tr}(B^4)}{10} \geq 0. \tag{1.17}$$

As we observed, $c \leq 1$. If $c \geq 5/4$, one may rewrite this inequality as

$$\left(c - \frac{5}{4}\right)\operatorname{tr}(A^2) + \frac{1}{10}\operatorname{tr}(AB+BA)^2 + \frac{5}{4}\operatorname{tr}\left(\frac{2}{5}(A^2+B^2) - A\right)^2 + \frac{1}{10}\operatorname{tr}(A^2 - B^2)^2 \geq 0.$$

Proposition 6. *If $c < \frac{5}{4}$, the inequality (1.17) doesn't admit a pure SOS-certificate.*

For basics of working with SOS-certificates, see for instance [BKP16, Section 1.3].

Proof. Existence of a pure SOS-certificate is equivalent to the existence of a positive semidefinite 5×5 matrix M such that

$$\operatorname{tr} MG = c\operatorname{tr}(A^2) - (\operatorname{tr}(A^3) + \operatorname{tr}(AB^2)) + \frac{3\operatorname{tr}(A^4) + 4\operatorname{tr}(A^2B^2) + 2\operatorname{tr}(ABAB) + 3\operatorname{tr}(B^4)}{10} =: S,$$

where G denotes the tracial Gram matrix of (A, A^2, B^2, AB, BA), i.e.

$$G = \begin{bmatrix} \operatorname{tr}(A^2) & \operatorname{tr}(A^3) & \operatorname{tr}(AB^2) & \operatorname{tr}(A^2B) & \operatorname{tr}(A^2B) \\ \operatorname{tr}(A^3) & \operatorname{tr}(A^4) & \operatorname{tr}(A^2B^2) & \operatorname{tr}(A^3B) & \operatorname{tr}(A^3B) \\ \operatorname{tr}(AB^2) & \operatorname{tr}(A^2B^2) & \operatorname{tr}(B^4) & \operatorname{tr}(AB^3) & \operatorname{tr}(AB^3) \\ \operatorname{tr}(A^2B) & \operatorname{tr}(A^3B) & \operatorname{tr}(AB^3) & \operatorname{tr}(A^2B^2) & \operatorname{tr}(ABAB) \\ \operatorname{tr}(A^2B) & \operatorname{tr}(A^3B) & \operatorname{tr}(AB^3) & \operatorname{tr}(ABAB) & \operatorname{tr}(A^2B^2) \end{bmatrix}.$$

Observe now that the entries $G_{i,j}$ with $i \leq 3 < j$ have odd degree in B, and such terms don't exist in S. We may hence restrict our attention to two block matrices and note that

$$\begin{aligned} S = &\operatorname{tr}\left(\begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix} \begin{bmatrix} \operatorname{tr}(A^2) & \operatorname{tr}(A^3) & \operatorname{tr}(AB^2) \\ \operatorname{tr}(A^3 & \operatorname{tr}(A^4) & \operatorname{tr}(A^2B^2) \\ \operatorname{tr}(AB^2) & \operatorname{tr}(A^2B^2) & \operatorname{tr}(B^4) \end{bmatrix} \right) \\ &+ \operatorname{tr}\left(\begin{bmatrix} M_{4,4} & M_{4,5} \\ M_{5,4} & M_{5,5} \end{bmatrix} \begin{bmatrix} \operatorname{tr}(A^2B^2) & \operatorname{tr}(ABAB) \\ \operatorname{tr}(ABAB) & \operatorname{tr}(A^2B^2) \end{bmatrix} \right). \end{aligned}$$

Since the term $\operatorname{tr}(ABAB)$ appears only at the entries $G_{4,5}$ and $G_{5,4}$ of G, we must have $M_{4,5}+M_{5,4} = 1/5$. Positivity of M then implies that $M_{4,4} + M_{5,5} \geq 1/5$. Subtracting these terms from both sides of the equation reveals that

$$\begin{aligned} 0 \leq &\operatorname{tr}\left(\begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix} \begin{bmatrix} \operatorname{tr}(A^2) & \operatorname{tr}(A^3) & \operatorname{tr}(AB^2) \\ \operatorname{tr}(A^3 & \operatorname{tr}(A^4) & \operatorname{tr}(A^2B^2) \\ \operatorname{tr}(AB^2) & \operatorname{tr}(A^2B^2) & \operatorname{tr}(B^4) \end{bmatrix} \right) \\ =&S - (M_{4,4} + M_{5,5}) \operatorname{tr}(A^2B^2) - (M_{4,5} + M_{5,4}) \operatorname{tr}(ABAB) \\ \leq & c \operatorname{tr}(A^2) - (\operatorname{tr}(A^3) + \operatorname{tr}(AB^2)) + \frac{3\operatorname{tr}(A^4) + 2\operatorname{tr}(A^2B^2) + 3\operatorname{tr}(B^4)}{10}. \end{aligned}$$

Pluggin in $A, B = 5/4$ one deduces that $c \geq 5/4$. □

Picking

$$A = \frac{3}{4}\begin{bmatrix} 1+\sqrt{\frac{2}{3}} & 0 \\ 0 & 1-\sqrt{\frac{2}{3}} \end{bmatrix}, \quad B = \frac{3}{4}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

shows that $c \geq \frac{9}{10}$, but we don't know if this bound is optimal. If A and B commute, $c = \frac{5}{6}$ is the optimal bound since the expression rewrites to

$$\frac{1}{30}\operatorname{tr}(3(A^2+B^2)-5A)^2.$$

1.4 Derivatives of trace functions and Stahl's theorem

If one applies Theorem 6 to the functions $f(t)=t_+^{k-1}$ for a positive integer k, one quickly arrives at Theorem 2.

Proof of Theorem 2. Smoothness follows from a classical result of Rellich, see [Kat66, VII, Theorem 3.9]. Furthermore, by [Bul71, Corollary 8], it is enough to proof that if f is of the form

$$t \mapsto p_{k-1}(t)+\sum_{i=1}^{M} m_i(t-c_i)_+^{k-1} \tag{1.18}$$

where p_{k-1} is a polynomial of degree at most $k-1$, $(c_i)_{i=1}^M \in \mathbb{R}^M$, and $(m_i)_{i=1}^M \in \mathbb{R}_+^M$, then $\operatorname{tr} f(tA+B)$ is a pointwise limit of functions of the same form. The desired conclusion is clear for the polynomial part, and for the remaining terms we can assume that $M=1$, $c_1=0$, and $m_1=1$. Also, the case $k=1$ is classical [Pet94, Proposition 1], so we may assume that $k\geq 2$.

Applying Theorem 6 for $f(t)=t_+^{k-1}$ (so that $H(f)(t)=f(t)/(k(k-1))$) and $(x,y)=(t,1)$, one gets

$$\operatorname{tr}(tA+B)_+^{k-1}=k(k-1)\int_{\mathbb{R}^2}(at+b)_+^{k-1}\,\mathrm{d}\mu_{A,B}(a,b).$$

If k is an even integer, the integrand $(at+b)_+^{k-1}$ is of the form (1.18) for any $a,b\in\mathbb{R}$, and we are done by the positivity of $\mu_{A,B}$. If k is odd, we further need that the support of $\mu_{A,B}$ is contained in the half plane $\{(a,b)\in\mathbb{R}^2 \mid a\geq 0\}$, which follows from Proposition 4. $\square$

As explained in the introduction, Theorem 2 is closely related to Stahl's theorem, which admits the following reformulations.

Theorem 24 (Reformulations of Stahl's theorem)**.** *Let $A,B\in M_n(\mathbb{C})$ be positive semidefinite. Then*

1. *the function*

$$t\mapsto \operatorname{tr}(\exp(A-tB))$$

is Laplace transform of a positive measure,

2. *for positive integer k all the coefficients of the polynomial*

$$t \mapsto \mathrm{tr}(A + tB)^k$$

are non-negative, and

3. *the function*

$$t \mapsto \mathrm{tr}(\exp(A - tB))$$

is completely monotone.

Proof. 1. By approximation (see for instance [Cli14, Section 2.1]) we may assume that all the eigenvalues of B are pairwise distinct. Apply identity (1.14) for $(x, y) = (1, 0)$ and positive definite A. Since $A^{-1}B \sim A^{-1/2}BA^{-1/2}$, all the roots of (A, B) are real and the singular part hence has n terms; we assume for now that all the roots of (A, B) are pairwise distinct. We hence have

$$\mathrm{tr}\, H(f)(A) = \sum_{v \in E(A,B)} H(f)\left(\frac{\langle Av, v\rangle}{\langle v, v\rangle}\right) + \int_{\mathbb{R}^2} f(a)\rho_{A,B}(a, b)\,\mathrm{d}m_2(a, b). \tag{1.19}$$

By Theorem 7 and Remark 1 the density $\rho_{A,B}$ satisfies

$$\rho_{A,B}(a, b) = \frac{1}{2\pi}\sum_{i=1}^{n}\left|\Im\left(\lambda_i\left(\left(I - \frac{A}{a}\right)(bA - aB)^{-1}\right)\right)\right|.$$

Writing $A' = A^{-1}, B' = A^{-1/2}BA^{-1/2}$ this rewrites to

$$\rho_{A,B}(a, b) = \frac{1}{2\pi}\sum_{i=1}^{n}\left|\Im\left(\lambda_i\left(\left(A' - \frac{I}{a}\right)(bI - aB')^{-1}\right)\right)\right|.$$

Note also that

$$\left\{\frac{\langle Av, v\rangle}{\langle v, v\rangle}\right\}_{v \in E(A,B)} = \left\{\frac{\langle A^{1/2}v, A^{1/2}v\rangle}{\langle A'A^{1/2}v, A^{1/2}v\rangle}\right\}_{v \in E(A^{1/2}, A^{-1/2}B)} = \left\{\frac{\langle w, w\rangle}{\langle A'w, w\rangle}\right\}_{w \in E(B')},$$

and (A, B) having n distinct roots means exactly that B' has n distinct eigenvalues. We can hence simplify (1.19) to

$$\begin{aligned}\operatorname{tr} H(f)(A'^{-1}) = &\sum_{v\in E(B')} H(f)\left(\frac{\langle v,v\rangle}{\langle A'v,v\rangle}\right)\\ &+\frac{1}{2\pi}\int_{\mathbb{R}^2} f(a)\sum_{i=1}^{n}\left|\Im\left(\lambda_i\left(\left(A'-\frac{I}{a}\right)(bI-aB')^{-1}\right)\right)\right| \mathrm{d}m_2(a,b).\end{aligned}$$

The idea is now to choose f so that $H(f)(1/x)=\exp(-x)$. Remark 2 allows us to solve for f and we obtain

$$f(t)=\begin{cases} t^{-2}e^{-1/t} & t>0\\ 0 & t\le 0\end{cases}.$$

Simple integration yields that $H(f)(t)=t^2 f(t)$ and we have

$$\begin{aligned}\operatorname{tr}\exp(-A') = &\sum_{v\in E(B')} \exp\left(-\langle A'v,v\rangle/\langle v,v\rangle\right)\\ &+\frac{1}{2\pi}\int_{\mathbb{R}^2} a^{-2}e^{-1/a}\sum_{i=1}^{n}\left|\Im\left(\lambda_i\left(\left(A'-\frac{I}{a}\right)(bI-aB')^{-1}\right)\right)\right| \mathrm{d}m_2(a,b)\\ =&\sum_{v\in E(B')} \exp\left(-\langle A'v,v\rangle/\left\langle v,v\right\rangle\right)\\ &+\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-a'}\sum_{i=1}^{n}\left|\Im\left(\lambda_i\left((A'-a'I)(b'I-B')^{-1}\right)\right)\right| \mathrm{d}m_2(a',b')\end{aligned}$$

where we performed the change of variables $(a',b')=(1/a,b/a)$. While this formula a priori only holds when A' is positive definite, multiplying both sides by a constant and translating A' and a' appropriately reveals that such restriction is not necessary.

Substitute $(A',B')=(Bt-A,B)$. Again, a simple change of variables $(a',b')=(bt-a,b)$ yields

$$\begin{aligned}\operatorname{tr}\exp(A-tB) = &\sum_{v\in E(B)} \exp\left(\langle Av,v\rangle/\langle v,v\rangle - t\langle Bv,v\rangle/\langle v,v\rangle\right)\\ &+\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{a-bt}\sum_{i=1}^{n}\left|\Im\left(\lambda_i\left((aI-A)(bI-B)^{-1}\right)\right)\right| \mathrm{d}m_2(a,b).\end{aligned}$$

The measure we seek is therefore

$$\sum_{v \in E(B)} e^{\frac{\langle Av,v \rangle}{\langle v,v \rangle}} \delta_{\frac{\langle Bv,v \rangle}{\langle v,v \rangle}} + \nu_c$$

where

$$\frac{\mathrm{d}\nu_c}{\mathrm{d}m_1}(b) = \frac{1}{2\pi} \int_{\mathbb{R}} e^a \left| \Im \left(\lambda_i \left((aI - A)(bI - B)^{-1} \right) \right) \right| \mathrm{d}a.$$

2. Applying Theorem 6 to $f(t) = t^k$ one sees that

$$\mathrm{tr}(xA + yB)^k = k(k+1) \int_{\mathbb{R}^2} (xa + yb)^k \, \mathrm{d}\mu_{A,B}(a,b).$$

   The claim now follows once one observes that by Proposition 4 the measure $\mu_{A,B}$ is supported on the first quadrant $\{(a,b) \mid a, b \geq 0\}$.

3. This readily follows from Theorem 2 applied to the exponential function.

□

Chapter 2

Chapter 2: Planes in Schatten-p and beyond

In this chapter we study isometric questions in Schatten-p classes. We start by proving Theorem 25, a slight refinement of Theorem 3/Theorem 19.

2.1 The embedding result

Theorem 25. *Let $A, B \in S_p$ for some $0 < p < \infty$. Then there exists $f, g \in L_p$ such that for any $x, y \in \mathbb{R}$*

$$\|xA + yB\|_{S_p} = \|xf + yg\|_{L_p}.$$

In other words, any two-dimensional real subspace of S_p is linearly isometric to a subspace of L_p.

It turns that Theorem 25 follows as a quick corollary from Theorem 23. Indeed, if in Theorem 23 we set $f(t) = |t|^p$ for $p > 0$, then $H(f)(t) = |t|^p/(p(p+1))$, and the identity we obtain is

$$\operatorname{tr}|xA + yB|^p = p(p+1)\int_{\mathbb{R}^2} |ax + by|^p \,\mathrm{d}\mu_{A,B}(a,b).$$

Note that this is giving us an embedding of the span of A and B in S_p to $L_p(\mu_{A,B})$, mapping A to $(a,b) \to a$ and B to $(a,b) \to b$. This embedding is (proportional to) an isometry for every $p > 0$ simultaneously.

Passing from the case of Hermitian matrices A, B to general two dimensional subspaces of S_p is standard.

Proof of Theorem 25. We have seen that the result holds as a consequence of Theorem 6 if $A, B \in M_n(\mathbb{C})$ are Hermitian. General complex matrices $A, B \in M_n(\mathbb{C})$ can be reduced to this special case by considering the following Hermitian block matrices

$$A' = \frac{1}{2^{1/p}} \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}, \qquad B' = \frac{1}{2^{1/p}} \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}, \tag{2.1}$$

and noting that $\|xA' + yB'\|_{S_p} = \|xA + yB\|_{S_p}$ for any $x, y \in \mathbb{R}$.

For general $A, B \in S_p$, by approximating A and B with finite rank operators and applying the finite dimensional result, one sees that for any $k \in \mathbb{N}$ there exists a measure μ_k and a $(1 + 1/k)$-distortion embedding $\mathrm{span}(A, B)_{S_p} \to L_p(\mu_k)$. Take an ultraproduct of these maps with respect to a non-principal ultrafilter $\mathcal{U}$ to get an isometric embedding of $\mathrm{span}(A, B)_{S_p}$ to the ultraproduct $(\prod_k L_p(\mu_k))_{\mathcal{U}}$, which is known to be isometric to $L_p(\mu)$ for some measure μ; see [DCK72] for the case $p \geq 1$ and [Nao98] for the case $0 < p < 1$. □

2.1.1 Equality cases

It turns out that the tracial joint spectral measure can be in many cases used to determine equality cases in various inequalities.

Theorem 26. *Let $p > 0$ and ν a finite symmetric signed measure on S^1 such that for any $A, B \in S_p$ one has*

$$\int_{S^1} \|t_1 A + t_2 B\|_{S_p}^p d\nu(t_1, t_2) \geq 0. \tag{2.2}$$

Assume also that two Hermitian matrices A, B not both zero exist such that (2.2) holds with equality. Then at least one of the following is true:

- *A and B commute.*
- *The set*

$$\left\{ (a, b) \in \mathbb{R}^2 \mid \int_{S^1} |t_1 a + t_2 b|^p d\nu(t_1, t_2) = 0 \right\}$$

  contains an open set.

Proof. Take such a Hermitian A and B and consider $\mu_{A,B}$. We then have

$$\int_{S^1} \|t_1 A + t_2 B\|_{S_p}^p d\nu(t_1,t_2) = p(p+1)\int_{\mathbb{R}^2}\int_{S^1}\sum_{i=1}^n |t_1 a + t_2 b|^p d\nu(t_1,t_2) d\mu(a,b).$$

Since the integrand is non-negative by the assumption, it has to vanish on the support of $\mu_{A,B}$. If A and B don't commute, by Proposition 4, the support of $\mu_{A,B}$ contains an open set, so the second condition is true. □

By analysing dilations (2.1) one could formulate similar statements about more general elements $A, B \in S_p$.

While Hanner's inequality Theorem 18 doesn't fit in the framework of the Theorem 26, the knowledge of the situation in the commutative case can be still used to deduce equality case in the non-commutative one.

Theorem 27. *Let $p > 2$. Assume that A, B are Hermitian matrices for which inequality (16) holds with equality. Then A and B commute.*

Proof. As in the proof of the embedding result, consider the functions $f_{A,B} = (a,b) \to a \in L_p(\mu_{A,B})$ and $g_{A,B} = (a,b) \to b \in L_p(\mu_{A,B})$. Since

$$\|xA + yB\|_{S_p}^p = p(p+1)\|x f_{A,B} + y g_{A,B}\|_{L_p}^p$$

the pair $(f_{A,B}, g_{A,B})$ satisfies inequality (12) with equality. It was however proven by Hanner [Han56] that equality holds in (12) only if $|f(t)|$ and $|g(t)|$ are proportional. This means that the support of $\mu_{A,B}$ has to lie on two lines, which by Proposition 4 implies that A and B commute. □

2.2 Embeddings via polynomial certificates

In my earlier work [Hei24] I proved Theorem 25 for $p = 3$ and $p = 4$. In these special cases the embedding result admits considerably simpler proofs which I will now describe.

2.2.1 The case $p = 4$

For even integers subspace structure of L_p is somewhat special: for any $f, g \in L_p$ the function

$$t \mapsto \|tf + g\|_{L_p}^p$$

is a polynomial of degree p. This turns the question of embedding to L_p essentially to a moment problem. Indeed, by Theorem 21 we are to find a measure μ on S^1 such that for any $x, y \in \mathbb{R}$,

$$\operatorname{tr}(xA + yB)^p = \int_{S^1} |xv_1 + yv_2|^p \, \mathrm{d}\mu(v_1, v_2).$$

Both sides of this equation are homogeneous polynomials of degree p and we may compare their coefficients to obtain constraints for the moments $\mu(v_1^k v_2^{p-k})$, $0 \le k \le p$. Isometric embeddability to L_p for even integer p is then equivalent to the solvability of this moment problem.

The same relationship works in any dimension, but for 2-dimensional spaces the situation is somewhat simpler: moment problem is one-dimensional and we can use Fourier analysis to connect this problem to a well known variant of Bochner's theorem.

Theorem 28. *If $A, B \in M_n(\mathbb{C})$ are Hermitian matrices, there exists a measure μ on S^1 such that for any $x, y \in \mathbb{R}$,*

$$\operatorname{tr}(xA + yB)^4 = \int_{S^1} |xv_1 + yv_2|^4 \, \mathrm{d}\mu(v_1, v_2). \tag{2.3}$$

Proof. Expanding both sides of (2.3) leads to an equality between two degree 4 homogeneous polynomials. On the LHS the coefficients of this polynomial can be written as sum of traces, while on the RHS they can be written in terms of Fourier coefficients of μ by writing

$$(v_1, v_2) = \left(\frac{\exp(it) + \exp(-it)}{2}, \frac{\exp(it) - \exp(-it)}{2i} \right).$$

Equating the coefficients leads to the following equations for the Fourier coefficients:

$$\begin{aligned}
\hat{\mu}(0) &= \operatorname{tr}(A^4) + \frac{2}{3}\left(2\operatorname{tr}(A^2B^2) + \operatorname{tr}(ABAB)\right) + \operatorname{tr}(B^4) \\
\hat{\mu}(2) &= \operatorname{tr}(A^4) - \operatorname{tr}(B^4) + 2i\left(\operatorname{tr}(A^3B) + \operatorname{tr}(AB^3)\right) \\
\hat{\mu}(4) &= \operatorname{tr}(A^4) + \operatorname{tr}(B^4) - 2\left(2\operatorname{tr}(A^2B^2) + \operatorname{tr}(ABAB)\right) + 4i\left(\operatorname{tr}(A^3B) - \operatorname{tr}(AB^3)\right).
\end{aligned}$$

It then suffices to determine if a measure with such Fourier coefficients exists. Since only the even coefficients are constrained, we may consider even measures ($\mu(f) = \mu(f(\cdot + \pi))$). If ν denotes the respective measure on the half circle $[0, \pi]$, we have $\hat{\nu}(k) = \hat{\mu}(2k)$ for every $k \in \mathbb{Z}$.

Existence of a positive measure ν with given moments $\hat{\nu}(0), \hat{\nu}(1), \hat{\nu}(2)$ is equivalent to (see for instance [BW11, Theorem 1.3.6])

$$\begin{bmatrix} \hat{\nu}(0) & \hat{\nu}(1) & \hat{\nu}(2) \\ \overline{\hat{\nu}(1)} & \hat{\nu}(0) & \hat{\nu}(1) \\ \overline{\hat{\nu}(2)} & \overline{\hat{\nu}(1)} & \hat{\nu}(0) \end{bmatrix} \geq 0,$$

i.e. $\nu(p) \geq 0$ for any trigonometric polynomial p of degree 2 non-negative on S^1. By [BW11, Theorem 1.1.7] it is enough to check such polynomials with roots on the unit circle, namely that

$$\begin{aligned} 0 \leq &\nu(\theta \mapsto (e^{i\theta} - e^{i\theta_1})(e^{i\theta} - e^{i\theta_2})(e^{-i\theta} - e^{-i\theta_1})(e^{-i\theta} - e^{-i\theta_1})) \\ =&(4 + e^{i(\theta_1-\theta_2)} + e^{i(\theta_2-\theta_1)})\hat{\nu}(0) \\ &- 2(e^{-i\theta_1} + e^{-i\theta_2})\hat{\nu}(1) - 2(e^{i\theta_1} + e^{i\theta_2})\overline{\hat{\nu}(1)} \\ &+ e^{-i(\theta_1+\theta_2)}\hat{\nu}(2) + e^{i(\theta_1+\theta_2)}\overline{\hat{\nu}(2)} \end{aligned}$$

for any $\theta_1, \theta_2 \in \mathbb{R}$. But since this can be expressed as

$$\begin{aligned} &(2 - e^{i\theta_1} - e^{-i\theta_1})(2 - e^{i\theta_2} - e^{-i\theta_2}) \operatorname{tr}(A^4) \\ +&4i \left(e^{i\theta_1} + e^{i\theta_2} - e^{i(\theta_1+\theta_2)} - e^{-i\theta_1} - e^{-i\theta_2} + e^{-i(\theta_1+\theta_2)}\right) \operatorname{tr}(A^3B) \\ +&\left(8 + 2(e^{i(\theta_1-\theta_2)} + e^{i(\theta_2-\theta_1)}) - 6(e^{i(\theta_1+\theta_2)} + e^{-i(\theta_2+\theta_1)})\right) \frac{2\operatorname{tr}(A^2B^2) + \operatorname{tr}(ABAB)}{3} \\ +&4i \left(e^{i\theta_1} + e^{i\theta_2} + e^{i(\theta_1+\theta_2)} - e^{-i\theta_1} - e^{-i\theta_2} - e^{-i(\theta_1+\theta_2)}\right) \operatorname{tr}(AB^3) \\ +&(2 + e^{i\theta_1} + e^{-i\theta_1})(2 + e^{i\theta_2} + e^{-i\theta_2}) \operatorname{tr}(A^4) \\ =&\|(e^{i\theta_1} - 1)(e^{i\theta_2} - 1)A^2 - i(e^{i(\theta_1+\theta_2)} - 1)(AB + BA) - (e^{i\theta_1} + 1)(e^{i\theta_2} + 1)B^2\|_{S_2}^2 \\ +&\frac{|e^{i\theta_1} - e^{i\theta_2}|^2}{3}\|AB - BA\|_{S_2}^2 \\ \geq&0, \end{aligned}$$

we are done. □

For $p = 6$ a similar argument is considerably more difficult but still possible. The main difference is that a new identity is needed to prove non-negativity of $\nu(p)$ for trigonometric polynomial p of degree 3 non-negative on S^1. Again, it suffices to check polynomials with roots at $e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}$. For such a polynomial, non-negativity is exhibited by the following identity:

$$
\begin{aligned}
&\nu(\theta \to |e^{i\theta} - e^{i\theta_1}|^2 |e^{i\theta} - e^{i\theta_2}|^2 |e^{i\theta} - e^{i\theta_3}|^2) \qquad (2.4)\\
=& \left\| c_{3,0}A^3 + c_{2,1}(BA^2 + ABA + A^2B) + c_{1,2}(BA^2 + ABA + A^2B) + c_{0,3}B^3 \right\|_{S_2}^2 \\
+& \frac{|e^{i\theta_2} - e^{i\theta_3}|^2}{360} \left(\|[((e^{i\theta_1}-1)A + i(e^{i\theta_1}+1)B), [A,B]]\|_{S_2}^2 + 15\|[((e^{i\theta_1}-1)A - i(e^{i\theta_1}+1)B), [A,B]]\|_{S_2}^2 \right) \\
+& \frac{|e^{i\theta_3} - e^{i\theta_1}|^2}{360} \left(\|[((e^{i\theta_2}-1)A + i(e^{i\theta_2}+1)B), [A,B]]\|_{S_2}^2 + 15\|[((e^{i\theta_2}-1)A - i(e^{i\theta_2}+1)B), [A,B]]\|_{S_2}^2 \right) \\
+& \frac{|e^{i\theta_1} - e^{i\theta_2}|^2}{360} \left(\|[((e^{i\theta_3}-1)A + i(e^{i\theta_3}+1)B), [A,B]]\|_{S_2}^2 + 15\|[((e^{i\theta_3}-1)A - i(e^{i\theta_3}+1)B), [A,B]]\|_{S_2}^2 \right),
\end{aligned}
$$

where

$$
\begin{aligned}
c_{3,0} &= (e^{i\theta_1} - 1)(e^{i\theta_2} - 1)(e^{i\theta_3} - 1) \\
c_{2,1} &= -i\frac{3e^{i(\theta_1+\theta_2+\theta_3)} + 3 - e^{i\theta_1} - e^{i\theta_2} - e^{i\theta_3} - e^{i(\theta_1+\theta_2)} - e^{i(\theta_2+\theta_3)} - e^{i(\theta_3+\theta_1)}}{3} \\
c_{1,2} &= -\frac{3e^{i(\theta_1+\theta_2+\theta_3)} - 3 - e^{i\theta_1} - e^{i\theta_2} - e^{i\theta_3} + e^{i(\theta_1+\theta_2)} + e^{i(\theta_2+\theta_3)} + e^{i(\theta_3+\theta_1)}}{3} \\
c_{0,3} &= i(e^{i\theta_1} + 1)(e^{i\theta_2} + 1)(e^{i\theta_3} + 1).
\end{aligned}
$$

Remark 3. Identity (2.4) was discovered with heavy use of computer algebra. I first guessed the form of the leading term based on the commutative case. I then analyzed the remainder in various special cases with semidefinite programming tools until patterns started emerging and I could guess a general formula.

For $p = 8$ I have not been able to find a similar expression, and have some numerical evidence that it might not exist. In particular, one can check that

$$
\begin{aligned}
&\nu(\theta \mapsto |(e^{2i\theta} + 1)^2|^2) \\
=& s_{8,0}(A,B) - 4s_{6,2}(A,B) + 6s_{4,4}(A,B) - 4s_{2,6}(A,B) + s_{0,8}(A,B), \qquad (2.5)
\end{aligned}
$$

where, as in the section 1.3, the coefficients $s_{k,l}(A,B)$ are defined with

$$
\operatorname{tr}(tA + B)^k = \sum_{l=0}^{k} s_{l,k}(A,B) \binom{k}{l} t^l.
$$

Question 1. *Can the expression (2.5) be written as a sum of squares?*

2.2.2 The case $p = 3$

For odd integers p determining embeddability to L_p also simplifies, but for a rather different reason. It was shown by Koldobsky [Kol92] that embeddability of two-dimensional normed spaces to L_p is essentially determined by non-negativity of a fractional derivative of an auxiliary function. For odd integers this fractional derivative is just the usual derivative. More explicitly, one can make use of the following result.

Lemma 6. *Let $\|\cdot\|$ be a norm in $\mathbb{R}^2$ such that for any $v, w \in \mathbb{R}^2$ the 4th distributional derivative of*

$$t \mapsto \|tv + w\|^3$$

is a finite positive Borel measure μ, for which $\int |x|^3 d\mu(x) < \infty$. Then $\|\cdot\|$ is linearly isometrically embeddable into L_3.

The converse is also true [Kol92, Theorem 4].

Proof of Lemma 6. It follows directly Theorem 5 of [Kol92] that for any $v, w \in \mathbb{R}^2$ there exists a function $h_{v,w} : \mathbb{R}^2 \to \mathbb{R}$ and a symmetric positive measure $\mu_{v,w}$ on S^1 such that for any $x, y \in \mathbb{R}$

$$\|xv + yw\|^3 = \int_{S^1} |\langle xv + yw, u\rangle|^3 d\mu_{v,w}(u) + h_{v,w}(x, y). \tag{2.6}$$

Additionally, there exists some $m_{v,w} \in \mathbb{N}$ such that $h_{v,w}(x, y)$ is polynomial of degree less than $m_{v,w}$ in y for every fixed x. By checking the asymptotics at $\pm\infty$ one sees that $m_{v,w}$ is at most 3. By using homogeneity one sees that $h_{v,w}(x, y) = |x|^3 p_{v,w}(y/x)$ for some polynomial $p_{v,w}$ of degree at most 3 whenever $x \neq 0$.

Swapping v and w one similarly obtains $\mu_{w,v}$ and $p_{w,v}$. Subtract the resulting representations to get the equation

$$\int_{S^1} |\langle xv + yw, u\rangle|^3 d(\mu_{v,w} - \mu_{w,v})(u) = |y|^3 p_{w,v}(x/y) - |x|^3 p_{v,w}(y/x) =: q(x, y). \tag{2.7}$$

We claim that $\mu_{v,w} - \mu_{w,v} =: \nu$ is supported on $\{u \in S^1 \mid \langle v, u\rangle = 0\} \cup \{u \in S^1 \mid \langle w, u\rangle = 0\} = v^\perp \cup w^\perp$, and that $p_{v,w}$ and $p_{w,v}$ are both constant.

To that end, note that for fixed non-zero $y_0 \in \mathbb{R}$, the restrictions of $q(\cdot, y_0)$ to intervals $(-\infty, 0)$ and $(0, \infty)$ coincide with polynomials, though these polynomials need not be the same. Integration by parts then implies that for any smooth φ with compact support not containing 0 one has

$$\begin{aligned}0 &= \int_{\mathbb{R}} \frac{\partial^4}{\partial x^4} q(x, y_0)\varphi(x)dx = \int_{\mathbb{R}} q(x, y_0)\varphi^{(4)}(x)dx = \int_{\mathbb{R}}\int_{S^1} |\langle xv + y_0 w, u\rangle|^3 \varphi^{(4)}(x) d\nu(u) dx \\ &= -6\int_{S^1}\int_{\mathbb{R}} \langle v, u\rangle^3 \operatorname{sign}(\langle xv + y_0 w, u\rangle)\varphi'(x)dx d\nu(u) = 12\int_{S^1} |\langle v, u\rangle|^3 \varphi\left(-y_0 \frac{\langle w, u\rangle}{\langle v, u\rangle}\right) d\nu(u).\end{aligned}$$

Since this holds for any φ as before, it follows that the signed measure ν is supported on $v^{\perp} \cup w^{\perp}$. This means that the LHS of (2.7) is a linear combination of $|x|^3$ and $|y|^3$, and consequently for some $\alpha, \beta \in \mathbb{R}$ we have

$$\begin{aligned}\beta|y|^3 - \alpha|x|^3 &= |y|^3 p_{w,v}(x/y) - |x|^3 p_{v,w}(y/x) \\ \Leftrightarrow \operatorname{sign}(x)(x^3(p_{v,w}(y/x) - \alpha))) &= \operatorname{sign}(y)(y^3(p_{w,v}(x/y) - \beta))\end{aligned}$$

whenever $x, y \neq 0$. Comparing the bivariate polynomials $x^3(p_{v,w}(y/x) - \alpha)$ and $y^3(p_{w,v}(x/y) - \beta)$ one then sees that in fact $p_{v,w} \equiv \alpha$ and $p_{w,v} \equiv \beta$.

We have thus proven that in (2.6), $h_{v,w}(x, y)$ is a multiple of $|x|^3$, and we may hence absorb it into the measure $\mu_{v,w}$ to obtain measure $\tilde{\mu}_{v,w}$ for which

$$\|xv + yw\|^3 = \int_{S^1} |\langle xv + yw, u\rangle|^3 d\tilde{\mu}_{v,w}(u)$$

for any $x, y \in \mathbb{R}$. While $\tilde{\mu}_{v,w}$ is not a priori positive, it is positive in $w^{\perp}$. Since by Theorem 21 the measure $\tilde{\mu}_{v,w}$ however is independent of v and w, it has to be a positive everywhere. □

Theorem 29. *Let $A, B \in M_n(\mathbb{C})$ be Hermitian. Then there exists $f, g \in L_3$ such that for any $x, y \in \mathbb{R}$*

$$\|xA + yB\|_{S_3} = \|xf + yg\|_{L_3}.$$

Given Lemma 6, proving Theorem 29 reduces to analyzing the fourth derivative of

$$t \mapsto \operatorname{tr}|tA + B|^3$$

for Hermitian A and B. The moment bounds are mostly technicalities and the points of non-smoothness also cause no trouble, but checking non-negativity elsewhere is quite non-trivial.

Lemma 7. *Let $A, B \in M_n(\mathbb{C})$ be Hermitian. Then*

$$\left(\operatorname{tr}|A + tB|^3\right)^{(4)} = O(1/|t|^5)$$

when $|t| \to \infty$.

Proof. Recall that since A and B are Hermitian, by the result of Rellich [RB69] the eigenvalues of $B + \varepsilon A$ are n analytic functions of ε, and hence for $i \in [n]$ we can write

$$\lambda_i(A + tB) = t\lambda_i(B) + \sum_{j=0}^{\infty} c_j t^{-j}$$

with $c_j \in \mathbb{R}$ for $j \geq 0$ and the series converging for large enough t. Now, either $\lambda_i(A + tB)$ is identically zero, or for some integer $d \leq 1$ one has

$$|\lambda_i(A + tB)|^3 = |t|^{3d}(1 + O(1/t)), \tag{2.8}$$

where the multiplier is analytic outside a compact set. Differentiating termwise one sees that the 4th derivative of (2.8) is $O(|t|^{-5})$, and summing over i then yields the claim. □

Theorem 30. *For Hermitian $A, B \in M_n(\mathbb{C})$, with B invertible, the function*

$$T : t \mapsto \operatorname{tr}|A + tB|^3$$

is analytic outside finitely many points where $A + tB$ is singular. Outside these points T has non-negative 4th derivative. At the singular points T behaves like

$$C|t - c|^3 + D|t - c|^3(t - c) + f(t),$$

where $C \geq 0$ and f is C^4 near c. Consequently, the 4th derivative is a non-negative multiple of δ measure at the singular points and therefore altogether a non-negative measure.

Proof. The second claim follows straightforwardly from the analyticity of the eigenvalues. Indeed, assume that say 0 is a singular point and let $\lambda_i(A+tB)$ denote the analytic branches of eigenvalues of $A + tB$ near 0 for $i \in [n]$. If $\lambda_i(A) \neq 0$, $|\lambda_i(A + tB)|^3$ is analytic near 0. If on the other hand $\lambda_i(A) = 0$, we can write $\lambda_i(A + tB) = t^k\nu_i(t)$ for some analytic ν_i with $\nu_i(0) \neq 0$ and k a positive integer (recall that B is invertible). Now

$$|\lambda_i(t)|^3 = |t|^{3k}|\nu_i(t)|^3.$$

Note however that the second term is analytic near 0. If $k > 1$ the whole function is C^4 near 0. If $k = 1$, we may expand $|\nu_i|^3$ at 0 to get required expansion for a single eigenvalue. Repeating this for all the eigenvalues yields the claim. Since $\det(At + B)$ is a non-zero polynomial, it can only have finitely many zeroes, and there can consequently only be finitely many points where T is not analytic.

Let us then move to the heart of the matter, identifying the analytic part.

Lemma 8. *(cf. [Hia10, Theorem 2.3.1.]) Let $A, B \in M_n(\mathbb{C})$ be Hermitian and f be analytic near the eigenvalues of A. Write $F(t) = \operatorname{tr}(f(A + tB))$. Then*

$$\frac{F^{(k)}(0)}{k!} = \frac{1}{k}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_k=1}^{n}[\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k}]_{f'} B_{i_1,i_2}B_{i_2,i_3}\cdots B_{i_k,i_1}. \tag{2.9}$$

Here $B_{i,j}$ is the matrix of B in the eigenbasis of A, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A.

Proof. The LHS clearly only depends on the value of f and its first k derivatives at the eigenvalues of A. Hence it is sufficient to prove such an identity for a special class of functions, say for polynomials. For the function $(\cdot)^m$ the LHS is simply

$$\begin{aligned}
\frac{(\operatorname{tr}(A + tB)^m)^{(k)}(0)}{k!} &= \sum_{j_0, j_1, \ldots, j_k \geq 0, \sum j_i = m-k} \operatorname{tr}(A^{j_0}BA^{j_1}B\cdots BA^{j_k}) \\
&= \sum_{j_1, \ldots, j_k \geq 0, \sum j_i = m-k} (j_k + 1)\operatorname{tr}(BA^{j_1}B\cdots BA^{j_k}) \\
&= \frac{m}{k}\sum_{j_1, \ldots, j_k \geq 0, \sum j_i = m-k} \operatorname{tr}(A^{j_1}B\cdots BA^{j_k}B) \\
&= \frac{m}{k}\sum_{j_1, \ldots, j_k \geq 0, \sum j_i = m-k}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_k=1}^{n}\lambda_{i_1}^{j_1}\lambda_{i_2}^{j_2}\cdots\lambda_{i_k}^{j_k}B_{i_1i_2}B_{i_2i_3}\cdots B_{i_ki_1}.
\end{aligned}$$

But since

$$\sum_{j_1, \ldots, j_k \geq 0, \sum j_i = m-k} \lambda_{i_1}^{j_1}\lambda_{i_2}^{j_2}\cdots\lambda_{i_k}^{j_k} = [\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k}]_{x^{m-1}},$$

we are done. $\square$

Let us now focus on the main expression (2.9) with $k = 4$ and $f = |\cdot|^3$. Write $g = f'/3 = (\cdot)|\cdot|$ We have the following simple identities.

Lemma 9. *Let $a_1, a_2, a_3, a_4 < 0 < b_1, b_2, b_3, b_4$. Then the following identities hold:*

$$\begin{aligned}
[a_1, a_2, a_3, a_4]_g =& 0 && (2.10)\\
[a_1, a_2, a_3, b_1]_g =& \frac{2b_1^2}{(b_1 - a_1)(b_1 - a_2)(b_1 - a_3)} && (2.11)\\
[a_1, a_2, b_1, b_2]_g =& \frac{2(b_1 + b_2)a_1a_2 - 2(a_1 + a_2)b_1b_2}{(b_1 - a_1)(b_1 - a_2)(b_2 - a_1)(b_2 - a_2)} && (2.12)\\
=& \frac{-2a_1b_1}{(b_1 - a_1)(b_1 - a_2)(b_2 - a_1)} + \frac{-2a_2b_2}{(b_2 - a_2)(b_2 - a_1)(b_1 - a_2)} &&\\
[a_1, b_1, b_2, b_3]_g =& \frac{2a_1^2}{(b_1 - a_1)(b_2 - a_1)(b_3 - a_1)} && (2.13)\\
[b_1, b_2, b_3, b_4]_g =& 0. && (2.14)
\end{aligned}$$

Proof. Straightforward to check from the definitions. □

Let us then complete the proof of Theorem 30. Assume that the matrix A has n_1 negative $(\lambda_1, \lambda_2, \ldots, \lambda_{n_1})$ and n_2 positive $(\lambda_{n_1+1}, \ldots, \lambda_n)$ eigenvalues $(n_1+n_2 = n)$. To prove that (2.9) is non-negative for our f, we first group the sum in several parts. The sum consists of n^4 summands; assign each summand a sign pattern depending on whether i_1, i_2, i_3 and i_4 are at most or greater than n_1. If say $i_1, i_3 \le n_1 < i_2, i_4$, assign the corresponding term the pattern $(-, +, -, +)$. Write then $S_{-,+,-,+}$ for the sum of all the terms with the pattern $(-, +, -, +)$. Note that since $[x_0, x_1, \ldots, x_k]_f = [x_{\sigma(0)}, x_{\sigma(1)}, \ldots, x_{\sigma(k)}]_f$ for any permutation σ, many of these sums are equal, and they can be in fact split into following groups:

(a) $S_{-,-,-,-}$

(b) $S_{+,-,-,-}$, $S_{-,+,-,-}$, $S_{-,-,+,-}$, $S_{-,-,-,+}$

(c) $S_{+,+,-,-}$, $S_{-,+,+,-}$, $S_{-,-,+,+}$, $S_{+,-,-,+}$

(d) $S_{+,-,+,-}$, $S_{-,+,-,+}$

(e) $S_{-,+,+,+}$, $S_{+,-,+,+}$, $S_{+,+,-,+}$, $S_{+,+,+,-}$

(f) $S_{+,+,+,+}$.

Note also that by (2.10) and (2.14), $S_{-,-,-,-} = 0 = S_{+,+,+,+}$. Moreover, by (2.11), (2.12) and (2.13), we have

$$S_{+,-,-,-} = \sum_{i_1=n_1+1}^{n} \sum_{i_2,i_3,i_4=1}^{n_1} \frac{2\lambda_{i_1}^2 B_{i_1i_2}B_{i_2i_3}B_{i_3i_4}B_{i_4i_1}}{(\lambda_{i_1}-\lambda_{i_2})(\lambda_{i_1}-\lambda_{i_3})(\lambda_{i_1}-\lambda_{i_4})}$$

$$S_{+,+,-,-} = \sum_{i_1,i_2=n_1+1}^{n} \sum_{i_3,i_4=1}^{n_1} \frac{-2\lambda_{i_1}\lambda_{i_3} B_{i_1i_2}B_{i_2i_3}B_{i_3i_4}B_{i_4i_1}}{(\lambda_{i_1}-\lambda_{i_3})(\lambda_{i_2}-\lambda_{i_3})(\lambda_{i_1}-\lambda_{i_4})} + \sum_{i_1,i_2=n_1+1}^{n} \sum_{i_3,i_4=1}^{n_1} \frac{-2\lambda_{i_2}\lambda_{i_4} B_{i_1i_2}B_{i_2i_3}B_{i_3i_4}B_{i_4i_1}}{(\lambda_{i_2}-\lambda_{i_4})(\lambda_{i_1}-\lambda_{i_4})(\lambda_{i_2}-\lambda_{i_3})}$$

$$S_{+,-,+,-} = \sum_{i_1,i_3=n_1+1}^{n} \sum_{i_2,i_4=1}^{n_1} \frac{-2\lambda_{i_1}\lambda_{i_3}(\lambda_{i_2}+\lambda_{i_4}) B_{i_1i_2}B_{i_2i_3}B_{i_3i_4}B_{i_4i_1}}{(\lambda_{i_1}-\lambda_{i_2})(\lambda_{i_3}-\lambda_{i_2})(\lambda_{i_1}-\lambda_{i_4})(\lambda_{i_3}-\lambda_{i_4})} + \sum_{i_1,i_3=n_1+1}^{n} \sum_{i_2,i_4=1}^{n_1} \frac{2\lambda_{i_2}\lambda_{i_4}(\lambda_{i_1}+\lambda_{i_3}) B_{i_1i_2}B_{i_2i_3}B_{i_3i_4}B_{i_4i_1}}{(\lambda_{i_1}-\lambda_{i_2})(\lambda_{i_3}-\lambda_{i_2})(\lambda_{i_1}-\lambda_{i_4})(\lambda_{i_3}-\lambda_{i_4})}$$

$$S_{-,+,+,+} = \sum_{i_1=1}^{n_1} \sum_{i_2,i_3,i_4=n_1+1}^{n} \frac{2\lambda_{i_1}^2 B_{i_1i_2}B_{i_2i_3}B_{i_3i_4}B_{i_4i_1}}{(\lambda_{i_2}-\lambda_{i_1})(\lambda_{i_3}-\lambda_{i_1})(\lambda_{i_4}-\lambda_{i_1})}.$$

Our goal is to prove that

$$(2.9) = 4S_{+,-,-,-} + 4S_{+,+,-,-} + 2S_{+,-,+,-} + 4S_{-,+,+,+} \geq 0.$$

We shall in fact prove that

$$S_{+,-,-,-} + S_{+,+,-,-} + S_{-,+,+,+} \geq 0 \tag{2.15}$$

and

$$S_{+,-,+,-} \geq 0. \tag{2.16}$$

The inequalities (2.15) and (2.16) correspond to terms where the (cyclic) sign sequences change signs 2 and 4 times, respectively. To prove (2.15), first note that (simply relabel the indices, factor, and use the Hermitian property $B_{j,i} = \overline{B_{i,j}}$)

$$S_{+,-,-,-} = \sum_{i>n_1\geq j} \frac{2\lambda_i^2}{\lambda_i-\lambda_j}\left|\sum_{l=1}^{n_1} \frac{B_{i,l}B_{l,j}}{\lambda_i-\lambda_l}\right|^2$$

$$S_{-,+,+,+} = \sum_{i>n_1\geq j} \frac{2\lambda_j^2}{\lambda_i-\lambda_j}\left|\sum_{l=n_1+1}^{n} \frac{B_{i,l}B_{l,j}}{\lambda_l-\lambda_j}\right|^2.$$

In a similar vein,

$$S_{+,+,-,-} = \sum_{i>n_1\geq j} \frac{-2\lambda_i\lambda_j}{\lambda_i-\lambda_j}\left(\sum_{l=n_1+1}^{n} \frac{B_{i,l}B_{l,j}}{\lambda_l-\lambda_j}\right)\overline{\left(\sum_{l=1}^{n_1}\frac{B_{i,l}B_{l,j}}{\lambda_i-\lambda_l}\right)}$$
$$+\sum_{i>n_1\geq j} \frac{-2\lambda_i\lambda_j}{\lambda_i-\lambda_j}\left(\sum_{l=1}^{n_1} \frac{B_{i,l}B_{l,j}}{\lambda_i-\lambda_l}\right)\overline{\left(\sum_{l=n_1+1}^{n}\frac{B_{i,l}B_{l,j}}{\lambda_l-\lambda_j}\right)}.$$

But this just means that

$$S_{+,-,-,-}+S_{+,+,-,-}+S_{-,+,+,+} = 2\sum_{i>n_1\geq j}\frac{1}{\lambda_i-\lambda_j}\left|\sum_{l=1}^{n_1}\frac{\lambda_i B_{i,l}B_{l,j}}{\lambda_i-\lambda_l}+\sum_{l=n_1+1}^{n}\frac{\lambda_j B_{i,l}B_{l,j}}{\lambda_j-\lambda_l}\right|^2 \geq 0. \tag{2.17}$$

Quite similarly

$$S_{+,-,+,-} = 2\sum_{i_1,i_2=1}^{n_1}(-\lambda_{i_1}-\lambda_{i_2})\left|\sum_{l=n_1+1}^{n}\frac{\lambda_l B_{i_1,l}B_{l,i_2}}{(\lambda_{i_1}-\lambda_l)(\lambda_{i_2}-\lambda_l)}\right|^2 \tag{2.18}$$
$$+2\sum_{i_1,i_2=n_1+1}^{n}(\lambda_{i_1}+\lambda_{i_2})\left|\sum_{l=1}^{n_1}\frac{\lambda_l B_{i_1,l}B_{l,i_2}}{(\lambda_{i_1}-\lambda_l)(\lambda_{i_2}-\lambda_l)}\right|^2 \geq 0.$$

The proof is thus complete. □

Corollary 3. *For any Hermitian $n\times n$ matrices A and B the function*

$$T : t \mapsto \operatorname{tr}|A+tB|^3$$

has non-negative 4th distributional derivative.

Proof. If B is invertible, the claim follows from Theorem 30. In the general case, approximate B by sequence of invertible matrices $(B_k)_{k=1}^{\infty}$. If B_k converges to B, $T_k = t\mapsto \operatorname{tr}|A+tB_k|^3$ converges pointwise to T. Since functions with non-negative 4th distributional derivative are closed under pointwise limits, we are done. □

Proof of Theorem 29. This follows at once from Corollary 3 and Lemmas 6 and 7. □

It would be interesting to determine if similar proof can be given for other odd integers. For $p=1$ this is straightforward. Indeed, one ends up considering the second derivative of the function

$$T : t\mapsto \operatorname{tr}|tA+B|.$$

It is not difficult to check that at the points where T is smooth ($tA + B$ is non-singular), the second derivative of T equals

$$4 \sum_{i>n_1\geq j} \frac{1}{\lambda_i - \lambda_j} |B_{i,j}|^2,$$

which is clearly non-negative. Here, as in the proof of Theorem 30 $\lambda_1, \lambda_2, \ldots, \lambda_{n_1} < 0 < \lambda_{n_1+1}, \ldots, \lambda_n$.

For larger integers the key difficult is finding such certificates of positivity. Order $p+1$ derivative of

$$t \mapsto |tA + B|^p$$

can be always written as a polynomial of degree $p+1$ in the entries B in the eigenbasis of A, where the coefficients are rational functions of the respective eigenvalues. By the converse of Lemma 6 (for general p) we know that this polynomial is non-negative in the real variables

$$(B_{i,i})_{i=1}^n \times (\Re(B_{i,j}))_{1\leq i<j\leq n} \times (\Im(B_{i,j}))_{1\leq i<j\leq n} \in \mathbb{R}^{n^2},$$

so the question remains: is this polynomial also a sum of squares (even for fixed λ_i's)? In the case $p = 3$ the necessary identities were discovered mostly with laborious trial and error process starting with small n. For larger p, I have attempted to find representations with the aid of semidefinite programming and computer algebra systems. Despite some effort, I haven't been able to find such a sum of squares representation even for $p = 5$ and $n = 3$.

Question 2. *For which odd integers p can one prove the non-negativity of the $(p+1)$:st derivative of the function*

$$t \mapsto |tA + B|^p$$

by a sum of squares representation as described above?

Since every non-negative polynomial is sum of squares of rational functions, it would also be interesting to find an explicit certificates by sums of squares of rational functions of entries of B.

2.3 3-dimensional subspaces of S_p

In this section we show that Theorem 25 doesn't have analogue for more than two matrices/operators.

Theorem 31. *For any $1 \leq p < \infty$, $p \neq 2$ there exists a 3-dimensional subspace of S_p which is not linearly isometric to any subspace of L_p. In fact one may consider the space of real symmetric 2×2 matrices.*

To illustrate the main idea of the following proof, we will first sketch the argument for $p = 1$. It is straightforward to check that for $x, y \in \mathbb{R}$ one has

$$\left\| x \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + y \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + I \right\|_{S_1} = 2\max(1, \sqrt{x^2 + y^2}).$$

We are therefore looking for a measure ν on S^2 such that

$$2\max(1, \sqrt{x^2 + y^2}) = \int_{S^2} |t_1 x + t_2 y + t_3| \, \mathrm{d}\nu(t_1, t_2, t_3) \tag{2.19}$$

for any $x, y \in \mathbb{R}$.

The impossiblity of the existence of measure ν then follows from the following three observations:

1. Denote by S the set of open radial segments not intersecting the unit circle. The LHS of (2.19) is affine on any such segment.

2. If $(t_1, t_2, t_3) \in \operatorname{supp}(\nu)$, the RHS of (2.19) is not affine on any open segment transversally intersecting $\{(x, y) \in \mathbb{R}^2 \mid t_1 x + t_2 y + t_3 = 0\}$.

3. Any line on the plane intersects some segment in S transversally.

For $p \neq 1$, the idea of affine segments doesn't quite work. One can use the uniqueness of representing measure ν instead, Theorem 21.

Proof of Theorem 31. Assume first that p is not an even integer. Towards a contradiction, assume that there exists a measure μ

$$\left\| \begin{bmatrix} z + x & y \\ y & z - x \end{bmatrix} \right\|_{S_p}^p = \left| z + \sqrt{x^2 + y^2} \right|^p + \left| z - \sqrt{x^2 + y^2} \right|^p = \int_{S^2} |x t_1 + y t_2 + z t_3|^p d\mu(t_1, t_2, t_3).$$

Setting $(x, y) = (r\cos(\theta), r\sin(\theta))$ we see that

$$|z + r|^p + |z - r|^p = \int_{S^2} |r(\cos(\theta) t_1 + \sin(\theta) t_2) + z t_3|^p d\mu(t_1, t_2, t_3). \tag{2.20}$$

Note however that by the uniqueness of the symmetric representing measure for 2-dimensional L_p-spaces, Theorem 21, this equality implies that for any θ, the point $(\cos(\theta)t_1 + \sin(\theta)t_2, t_3) \in \mathbb{R}^2$ has to lie in $\{(x, x) \mid x \in \mathbb{R}\} \cup \{(x, -x) \mid x \in \mathbb{R}\}$, for every (t_1, t_2, t_3) in the support of μ. Applying this for $\theta = 0, \pi/4, \pi/2$ we see that

$$\mathrm{supp}(\mu) \subset \{(x,y,z) \mid |x| = |z|\} \cap \{(x,y,z) \mid |y| = |z|\} \cap \{(x,y,z) \mid |x+y| = \sqrt{2}|z|\} = \{(0,0,0)\},$$

which is a contradiction.

Consider then an even integer $p \geq 4$. Again, it is enough to refute the existence of measure μ on S^2 such that (2.20) holds. The same argument as before doesn't work now since for even integers the uniqueness result fails. We will instead show that the moment problem doesn't have a solution by calculating a couple of the first moments.

Expanding both sides around $r = 0$ and averaging over $\theta \in [0, 2\pi]$ we see that

$$\begin{aligned} \mu(t_3^p) &= 2 \\ \mu(t_3^{p-2}(1 - t_3^2))) &= 4 \\ \mu(t_3^{p-4}(1 - t_3^2)^2) &= \frac{16}{3}. \end{aligned}$$

But now $\mu((2t_3^2 - 1)^2 t_3^{p-4}) = -2/3$, which is impossible. □

This result has an important consequence in terms of equality cases: there is no version of Theorem 26 for more than two matrices. In other words: there exists inequalities in S_p all equality cases of which are truly non-commutative. While the existence of such inequalities of quite explicit nature follows by the work of Krivine [Kri65] from Theorem 31, one can give quite simple and explicit examples.

Proposition 7. *Let $p > 2$. Then there exists a constant $c_p < 2^{p-2}$ such that for any $f, g, h \in L_p$ one has*

$$\mathbb{E}_t \|f + \cos(t)g + \sin(t)h\|_{L_p}^p \leq c_p \left(\|f\|_{L_p}^p + \mathbb{E}_t \|\cos(t)g + \sin(t)h\|_{L_p}^p \right), \tag{2.21}$$

where t is uniform on $[0, 2\pi]$. Conversely, if $0 < p < 2$, there exists a constant $c_p > 2^{p-2}$ such that Inequality (2.21) holds in reverse.

Proof. Consider first $p > 2$. It is clearly enough to check the claim for scalars $f, g, h \in \mathbb{R}$. Since

$\cos(t)g+\sin(t)h$ has the same distribution as $\cos(t)\sqrt{g^2+h^2}$, one is reduced to consider the inequality

$$\mathbb{E}_t|1+\cos(t)x|^p \leq c_p\left(1+\mathbb{E}_t|\cos(t)x|^p\right). \tag{2.22}$$

Applying Clarkson's inequality to numbers 1 and $\cos(t)x$ one sees that

$$\frac{|1+\cos(t)x|^p+|1-\cos(t)x|^p}{2} \leq 2^{p-2}\left(1+|\cos(t)x|^p\right)$$

for any $0 \leq t \leq 2\pi$. Taking the expectation over t yields (2.22) with constant $c_p = 2^{p-2}$. Since Clarkson's inequality is satisfied with equality only for finitely many t (i.e. whenever $|\cos(t)x| = 1$), the inequality with $c_p = 2^{p-2}$ is strict for every x. Analyzing (2.22) at infinity then implies that one can improve the constant c_p slightly below 2^{p-2}.

Argument for $0 < p < 2$ is similar. □

Proposition 8. *Let $p \geq 2$. Then for any $A, B, C \in S_p$*

$$\mathbb{E}_t\|A+\cos(t)B+\sin(t)C\|_{S_p}^p \leq 2^{p-2}\left(\|A\|_{S_p}^p+\mathbb{E}_t\|\cos(t)B+\sin(t)C\|_{S_p}^p\right), \tag{2.23}$$

where the constant 2^{p-2} is the best possible. The reverse inequality is true for $0 < p < 2$ with the constant 2^{p-2} again the best possible.

Proof. The inequalities follow from Clarkson's inequality as in the proof of Proposition 7. If

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

(2.23) holds as equality. □

Remark 4. The choice of the uniform measure in Proposition 7 is by no means the only possible: any symmetric measure on the circle, support of which doesn't lie on two lines symmetric about the origin, would work as an counterexample.

Inequality 2.21 is very much inspired by the proof of Theorem 31. The key aspect exploited in the proof is that for any unit vector (x, y), the matrix

$$xA + yB = \begin{bmatrix} x & y \\ y & -x \end{bmatrix}$$

has the same eigenvalues. This means that if one finds an inequality of two matrices, with I and A as the equality case, then also I and $\cos(t)A+\sin(t)B$ should be an equality case for $t \in [0, 2\pi]$. Such behaviour cannot occur in commutative world as for real numbers $\cos(t)x + \sin(t)y$ will fluctuate between $-\sqrt{x^2+y^2}$ and $\sqrt{x^2+y^2}$.

2.4 Roundness inequalities

In propositions 7 and 8 we considered an inequality demonstrating difference between L_p and S_p. This inequality was somewhat artificially constructed but such inequalities also arise naturally from geometric applications. As an example, we will consider generalized roundness inequalities of Enflo.

Definition 4. *For a Banach space X, its generalized p-roundness constant is the least constant $\mathfrak{r}_p(X)$ for which for any positive integer k and $x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k \in X$ one has*

$$\sum_{i<j} \|x_i - x_j\|_X^p + \sum_{i<j} \|y_i - y_j\|_X^p \le \mathfrak{r}_p(X) \sum_{i,j} \|x_i - y_j\|_X^p.$$

For any Banach space X one has $\mathfrak{r}_p(X) \le 2^p$, as follows from the inequality

$$\|x-y\|_X^p \le 2^{p-1} \left(\|x-z\|_X^p + \|z-y\|_X^p \right)$$

averaged over tuples $(x, y, z) = (x_i, x_j, y_k), (y_i, y_j, x_k)$. Naor and Oleszkiewicz [NO20] asked whether this trivial bound can be improved for $p = 1$ and $X = S_1$. We prove that this is indeed the case.

Proposition 9. *There exists constant $\delta > 0$ such that for any positive integer k and $A_1, A_2, \dots, A_k, B_1, B_2, \dots, B_k \in S_1$ one has*

$$\sum_{i<j} \|A_i - A_j\|_{S_1} + \sum_{i<j} \|B_i - B_j\|_{S_1} \le (2-\delta) \sum_{i,j} \|A_i - B_j\|_{S_1}. \tag{2.24}$$

Proof of this proposition is based on the following rigidity result, which itself is a quick consequence of a result of Regev and Vidick [RV20].

Lemma 10. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that the following is true. If $A_1, A_2, B_1, B_2 \in S_1$ are such that $\|A_i - B_j\|_{S_1} \le 1$, $1 \le i, j \le 2$ and $\|A_1 - A_2\|_{S_1}, \|B_1 - B_2\|_{S_1} \ge 2 - \delta$, then $\|A_1 + A_2 - B_1 - B_2\|_{S_1} < \varepsilon$.*

Lemma 11 ([RV20, Proposition 4])**.** *Let $A, B, C \in M_n(\mathbb{C})$ such that C is positive definite with trace*

1,

$$\begin{aligned}\|A\|_{S_1} + \|C - A\|_{S_1} &\leq 1 + \delta \\ \|B\|_{S_1} + \|C - B\|_{S_1} &\leq 1 + \delta \\ \|A - B\|_{S_1} &\geq 1 - \delta.\end{aligned}$$

Then there exists orthogonal projections P, Q with $P + Q = I$ such that

$$\|A - C^{1/2}PC^{1/2}\|_{S_1}, \|B - C^{1/2}QC^{1/2}\|_{S_1} = O(\delta^{1/8}).$$

Proof of Lemma 10. By approximation we may assume that we are working in $M_n(\mathbb{C})$ for some $n \geq 1$, and by translation and unitary invariance that further $B_2 = 0$ and B_1 is positive definite. Lemma 11 then implies that there exists orthogonal projections P and Q in $M_n(\mathbb{C})$ such that $P + Q = I$ and

$$\|A_1 - B_1^{1/2}PB_1^{1/2}\|_{S_1}, \|A_2 - B_1^{1/2}QB_1^{1/2}\|_{S_1} = O(\delta^{1/8}).$$

We hence have $\|A_1 + A_2 - B_1 - B_2\|_{S_1} = O(\delta^{1/8})$ and can take $\delta = c\varepsilon^8$ for small enough $c > 0$. □

Proof of Proposition 9. We start by observing that it suffices to show the existence of $\delta > 0$ such that for any $A_1, A_2, A_3, B_1, B_2, B_3 \in S_1$ one has

$$\frac{1}{3}\sum_{1\leq i<j\leq 3}\|A_i - A_j\|_{S_1} + \frac{1}{3}\sum_{1\leq i<j\leq 3}\|B_i - B_j\|_{S_1} \leq \frac{2}{9}(1-\delta)\sum_{i=1}^{3}\sum_{j=1}^{3}\|A_i - B_j\|_{S_1} \tag{2.25}$$

Indeed, by replacing $(A_1, A_2, A_3) \to (A_i, A_j, A_k)$ and $(B_1, B_2, B_3) \to (B_{i'}, B_{j'}, B_{k'})$ and averaging over all pairwise disjoint tuples (i, j, k) and (i', j', k') of $[k]$ uniformly and independently then yields (2.24) with $2 - \delta$ replaced by $2(1-\delta)(1-1/k)$.

It also suffices to consider the case where $\|A_i - A_j\|_{S_1} = \|B_{i'} - B_{j'}\|_{S_1}$ for $i \neq j$, $i' \neq j'$ and $\|A_i - B_j\|_{S_1} = 1$ for $1 \leq i, j \leq k$. Indeed; if we can find operators on H violating (2.25), the operators

$$\begin{aligned}A'_i &= \bigoplus_{\sigma,\sigma'\in\mathrm{Sym}(k)\times\mathrm{Sym}(k)} A_{\sigma(i)} \oplus B_{\sigma'(i)} \\ B'_i &= \bigoplus_{\sigma,\sigma'\in\mathrm{Sym}(k)\times\mathrm{Sym}(k)} B_{\sigma(i)} \oplus A_{\sigma'(i)}\end{aligned}$$

acting on $H^{2|\operatorname{Sym}(k)|^2}$ also violate (2.25) and, properly normalized, satisfy the distance conditions. The violation then simplifies to $\|A_i - A_j\|_{S_1}, \|B_{i'} - B_{j'}\|_{S_1} \geq 2(1-\delta)$ which for small enough δ by Lemma 10 implies that $\|A_i + A_j - B_{i'} - B_{j'}\| < \varepsilon$. This however forces all A_i's and B_j's close to each other, contradicting our distance conditions whenever $\varepsilon < 1/2$. □

Naor and Oleszkiewicz observe that $\mathfrak{r}_1(S_1) \geq \sqrt{2}$. This is based on a Clifford algebra construction of Briët, Regev and Saket [BRS17] that allows one to embed $(M_{N,2}(\mathbb{R}), \|\cdot\|_{S_\infty})$ into S_1.

Proposition 10. *For any $N \in \mathbb{N}_+$ the space $(M_{N,2}(\mathbb{R}), \|\cdot\|_{S_\infty})$, real $2 \times N$ matrices with the operator norm, can be isometrically embedded into $(M_{2^N}(\mathbb{C}), S_1)$.*

Proof. By [BRS17, Lemma 5.2] there exists a linear map $T : \mathbb{C}^N \to M_{2^N}(\mathbb{C})$ for which for any $a \in \mathbb{C}^N$

$$\|C(a)\|_{S_1} = \frac{1}{2}\sqrt{\|a\|_2^2 + 2\Lambda(a)} + \frac{1}{2}\sqrt{\|a\|_2^2 - 2\Lambda(a)},$$

where

$$\Lambda(a) = \sqrt{\|\Re(a)\|_2^2 \|\Im(a)\|_2^2 - |\langle \Re(a), \Im(a)\rangle|^2}.$$

Write $(v, w) = (\Re(a), \Im(a))$ and consider $A = (v, w) \in M_{N,2}(\mathbb{R})$. It suffices to check that $\|A\|_{S_\infty} = \|C(a)\|_{S_1}$. To that end, observe that

$$A^* A = \begin{bmatrix} \|v\|_2^2 & \langle v, w\rangle \\ \langle w, v\rangle & \|w\|_2^2 \end{bmatrix}.$$

One calculates that $\sigma_1(A)\sigma_2(A) = \sqrt{\det(A^*A)} = \Lambda(a)$ and $\sigma_1(A)^2 + \sigma_2(A)^2 = \|v\|_2^2 + \|w\|_2^2 = \|a\|_2^2$, and thus

$$\|A\|_{S_\infty} = \frac{\sigma_1(A) + \sigma_2(A)}{2} + \frac{|\sigma_1(A) - \sigma_2(A)|}{2} = \frac{1}{2}\sqrt{\|a\|_2^2 + 2\Lambda(a)} + \frac{1}{2}\sqrt{\|a\|_2^2 - 2\Lambda(a)} = \|C(a)\|_{S_1},$$

as desired. □

To prove that $\mathfrak{r}_1(S_1) \geq \sqrt{2}$ one may then proceed as follows: pick $2k$ orthonormal vectors in $\mathbb{R}^{2k}$, $e_1, e_2, \ldots, e_{2k}$ and let $A_i = (e_i, 0)$ and $B_i = (0, e_{i+k})$ for $i \in [k]$. It is then straightforward to check that $\|A_i - A_j\|_{S_\infty} = \sqrt{2} = \|A_i - A_j\|_{S_\infty}$ for $i \neq j \in [k]$, while $\|A_i - B_j\|_{S_\infty} = 1$ for $i, j \in [k]$.

Plugging these matrices in the main definition yields

$$k(k-1)\sqrt{2} \leq \mathfrak{r}_1(S_1)k^2$$

which implies the claim.

For fixed k one may slightly improve this lower bound by having $(e_1, e_2, \ldots, e_k)$ and $(e_{k+1}, e_{k+2}, \ldots, e_{2k})$ as the vertices of two regular $(k-1)$-simplices living in orthogonal spaces. A natural question arises:

Question 3. *Does $\mathfrak{r}_1(S_1)$ equal $\sqrt{2}$?*

While we haven't been able to answer this question, we introduce the following conjectural inequality that would that would imply that the answer is yes.

Conjecture 2. *Let $k, l \geq 1$ be positive integers and $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_l \in S_1$. Then*

$$\frac{1}{k^2} \sum_{1 \leq i < j \leq k} \|A_i - A_j\|_{S_1}^2 + \frac{1}{l^2} \sum_{1 \leq i < j \leq l} \|B_i - B_j\|_{S_1}^2 \leq \frac{2}{kl} \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq l}} \|A_i - B_j\|_{S_1}^2 \tag{2.26}$$

If Conjecture 2 is true, then $\mathfrak{r}_1(S_1) = \sqrt{2}$. Indeed, the averaging argument employed in the proof of Proposition 9 implies that we may assume that all the summands on LHS of (2.24) are equal, and so are the summands on the RHS. If say $\|A_1 - A_2\| = t$ and $\|A_1 - B_1\| = 1$, then (2.26) implies that

$$\left(\frac{k-1}{2k} + \frac{l-1}{2l} \right) t^2 \leq 2,$$

which implies that $t \leq \sqrt{2}$. Plugging this back in (2.24) implies that $\mathfrak{r}_1(S_1) = \sqrt{2}$.

Noteworthy and desirable aspect of the inequality (2.26) is that it has rich set of equality cases coming from the aforementioned Clifford algebra construction. It is possible to check Conjecture 2 for such matrices.

Proposition 11. *Let $k, l \geq 1$ be positive integers and $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_l \in M_{N \times 2}(\mathbb{C})$. Then*

$$\frac{1}{k^2} \sum_{1 \leq i < j \leq k} \|A_i - A_j\|_{S_\infty}^2 + \frac{1}{l^2} \sum_{1 \leq i < j \leq l} \|B_i - B_j\|_{S_\infty}^2 \leq \frac{2}{kl} \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq l}} \|A_i - B_j\|_{S_\infty}^2. \tag{2.27}$$

Equality holds in 2.27 iff there exists orthonormal vectors $e_1, e_2 \in \mathbb{C}^2$ and $(k+l)$ vectors all with *equal norm* $v_1, v_2, \ldots, v_k \in \mathbb{C}^N$ *and* $w_1, w_2, \ldots, w_l \in \mathbb{C}^N$ *such that after a translation*

$$\sum_{i=1}^{k} v_i = 0 = \sum_{j=1}^{l} w_j$$

$$\langle v_i, w_j \rangle = 0 \text{ for } 1 \le i \le k, 1 \le j \le l$$

$$A_i = v_i \otimes e_1 \in \mathbb{C}^N \otimes \mathbb{C}^2 = M_{N\times 2}(\mathbb{C}) \text{ for } i \in [k]$$

$$B_j = w_j \otimes e_2 \in \mathbb{C}^N \otimes \mathbb{C}^2 = M_{N\times 2}(\mathbb{C}) \text{ for } j \in [l]$$

Proof. Since for $C \in M_{N\times 2}(\mathbb{C})$

$$\|C\|_{S_\infty} \le \|C\|_{S_2} \le \sqrt{2}\|C\|_{S_\infty}, \tag{2.28}$$

to prove the inequality it is enough to check that

$$\frac{1}{k^2} \sum_{1\le i<j\le k} \|A_i - A_j\|_{S_2}^2 + \frac{1}{l^2} \sum_{1\le i<j\le l} \|B_i - B_j\|_{S_2}^2 \le \frac{1}{kl} \sum_{\substack{1\le i\le k\\ 1\le j\le l}} \|A_i - B_j\|_{S_2}^2,$$

but this rewrites to

$$0 \le \left\| \frac{\sum_{i=1}^{k} A_i}{k} - \frac{\sum_{i=1}^{l} B_j}{l} \right\|_{S_2}^2. \tag{2.29}$$

From the equality cases in 2.28 one deduces that $\operatorname{rank}(A_i - A_j), \operatorname{rank}(B_i - B_j) \le 1$ whenever $i \ne j$; and that $A_i - B_j$ is an isometry for any i, j. The first condition easily implies that

$$A_i = A_0 + v_i \otimes e_1 \text{ for } i \in [k]$$

$$B_j = B_0 + w_j \otimes e_2 \text{ for } j \in [l]$$

for some $A_0, B_0 \in \mathbb{C}^N \otimes \mathbb{C}^2$, $e_1, e_2 \in \mathbb{C}^2$ and $v_1, v_2, \ldots, v_k \in \mathbb{C}^N$ and $w_1, w_2, \ldots, w_l \in \mathbb{C}^N$. We further translate v_i's and w_j's so that $\sum v_i = \sum w_j$ after which equation (2.29) implies we can assume $A_0 = 0 = B_0$.

Direct calculation shows that $A_i - B_j$ is then isometry iff for all $i \in [k], j \in [l]$ we have $\|v_i\|_2 = \|w_j\|_2$ and $\langle v_i, w_j\rangle/(\|v_i\|_2\|w_j\|_2) = \langle e_1, e_2\rangle$. Since v_i's and w_j's sum to zero however, this can only be the case if $\langle e_1, e_2\rangle = 0$, and $\langle v_i, w_j\rangle = 0$ for all $i \in [k], j \in [l]$. □

Note that while the usual roundness inequality for L_1 can be reduced to scalars and then checked with relative ease with the aid of the special conic structure of L_1, the same is not true for the quadratic roundness inequality (2.26). Indeed, I don't know if Conjecture 2 is true even for L_1. For L_1 I have obtained a partial result (stated as a conjecture), which I'll now give a rough proof sketch for.

Conjecture 3. *Let k be positive integer and $f_1, f_2, \ldots, f_k, g_1, g_2 \in L_1$. Then*

$$\frac{1}{k^2} \sum_{1 \le i < j \le k} \|f_i - f_j\|_{L_1}^2 + \frac{1}{4}\|g_1 - g_2\|_{L_1}^2 \le \frac{1}{k} \sum_{1 \le i \le k} \left(\|f_i - g_1\|_{L_1}^2 + \|f_i - g_2\|_{L_1}^2 \right) \tag{2.30}$$

Proof idea for Conjecture 3. The main idea is to write the difference of RHS and LHS as a sum of various terms that are clearly non-negative. Interpreting $f_1, \ldots, f_k, g_1, g_2$ as $k+2$ points in L_1, and rewriting $(f_1, f_2, \ldots, f_k, g_1, g_2) = (h_1, h_2, \ldots, h_{k+2})$, consider the vector of distances $(\|h_i - h_j\|_{L_1})_{1 \le i < j \le k+2}$. It turns out that such distance vectors from a polyhedral cone, so called cut cone CUT_{k+2}. While the extreme rays of this cone are very simple, its dual cone is very complicated. In fact it is NP-complete problem to determine whether a given rational vector of length $\binom{k+2}{2}$ arises from distances in ℓ_1. This result along with wealth of information about cut cones can be found in the book of Deza and Laurent [DL10].

While all inequalities constraining the distance vectors are hard to describe, many families are known. It is for instance known and quite easy to show that if $b_1, b_2, \ldots, b_{k+2}$ are integers summing to 1, one has

$$\sum_{1 \le i < j \le k+2} b_i b_j \|h_i - h_j\|_{L_1} \le 0$$

If $(b_1, b_2, \ldots, b_{k+2})$ is an reordering of $(1, 1, -1, 0, 0, \ldots, 0)$, this simplifies to the triangle inequality, but for other tuples rich family additional inequalities, so called hypermetric inequalities, is obtained. Taking products of such inequalities one obtains family of quadratic inequalities that can used to certify 2.30. For fixed k, denote by $Q_1^{(k+2)}, Q_2^{(k+2)}, \ldots, Q_{N_{k+2}}^{(k+2)}$ a generating list of extremal inequalities spanning the dual cut cone of order $k+2$.

In addition to the inequalities arising from the dual cut cone, one can also consider sums of squares of linear combinations of distance. This is the same as specifying a positive semidefinite $\binom{k+2}{2} \times \binom{k+2}{2}$ matrix M indexed by pairs $(i,j), 1 \le i < j \le k+2$, and considering expressions of the form

$$\sum_{\substack{1\le i<j\le k+2\\ 1\le i'<j'\le k+2}} M_{(i,j),(i',j')}\|h_i-h_j\|_{L_1}\|h_{i'}-h_{j'}\|_{L_1}.$$

All in all, to prove inequality (2.30), it is enough to find a positive definitive matrix M and non-negative scalars $t_{i',j'}$ for $1\le i'\le j'\le N_{k+2}$ such that

$$\begin{aligned} RHS-LHS = &\sum_{\substack{1\le i<j\le k+2\\ 1\le i'<j'\le k+2}} M_{(i,j),(i',j')}\|h_i-h_j\|_{L_1}\|h_{i'}-h_{j'}\|_{L_1}\\ &+\sum_{1\le i'\le j'\le N_{k+2}} t_{i',j'}Q^{(k+2)}_{i'}((\|h_i-h_j\|_{L_1})_{1\le i<j\le k+2})Q^{(k+2)}_{j'}((\|h_i-h_j\|_{L_1})_{1\le i<j\le k+2}).\end{aligned}$$

While this problem is still intractable as $Q_i^{(k+2)}$'s cannot be enumerated for large k, it turns out to be possible to find such a representation by only allowing hypergeometric inequalities. Exploiting some inherent symmetries of the problem, the task of finding the representation boils down to a semidefinite programming problem which can solved efficiently for small k. The emerging structure of the certificates can be then used to write a guess for the general form of the certificate, and this guess is straightforward if somewhat tedious to formally verify. □

This approach could in theory work for any fixed $k,l\ge 2$, but it was for $l=2$ when special pattern (for varying k) emerged for the matrix M; and it happened to be enough to consider cone inequalities of simpler hypergeometric form.

It is worth noting that even if this approach could generalize to deal with the full L_1-case, for S_1 the cone structure is lost, and the approach is doomed to fail. Indeed, assume we are able to express the quadratic inequality (2.26) as sum of squares, and terms of the form

$$\left(\sum_{1\le i<j\le k+l} t_{i,j}\|C_i-C_j\|_{S_1}\right)\left(\sum_{1\le i<j\le k+l} s_{i,j}\|C_i-C_j\|_{S_1}\right) \tag{2.31}$$

where $(A_1,A_2,\dots,A_k,B_1,B_2,\dots,B_k)=(C_1,C_2,C_3,\dots,C_{k+l})$, $t_{i,j},s_{i,j}\in\mathbb{R}$, and

$$\sum_{1\le i<j\le k+l} t_{i,j}\|C_i-C_j\|_{S_1}\ge 0 \tag{2.32}$$

$$\sum_{1\le i<j\le k+l} s_{i,j}\|C_i-C_j\|_{S_1}\ge 0 \tag{2.33}$$

for any $(C_i)_{i=1}^{k+l} \in S_1^{k+l}$. Since this inequality is not true for every norm, expressions of the form (2.31) are needed. Choose $k = 5 = l$ and consider an equality case which in the notation of the proof of Proposition 11 corresponds to vectors $(v_1, v_2, v_3, v_4, v_5, w_1, w_2, w_3, w_4, w_5)$. The inequalities (2.32) and (2.33) now read

$$\sum_{i,j=1}^{5} t_{i,j+5} + \sum_{1\leq i<j\leq 5} t_{i,j}\|v_i - v_j\|_2 + \sum_{1\leq i<j\leq 5} t_{i+5,j+5}\|w_i - w_j\|_2 \geq 0$$
$$\sum_{i,j=1}^{5} s_{i,j+5} + \sum_{1\leq i<j\leq 5} s_{i,j}\|v_i - v_j\|_2 + \sum_{1\leq i<j\leq 5} s_{i+5,j+5}\|w_i - w_j\|_2 \geq 0$$

Since the corresponding matrices C_i constitute an equality case in Proposition 11, at least one of the inequalities has to however hold as an equality. By varying v_i's and w_i's, it is not hard to see that either $t_{i,j} = 0 = t_{i+5,j+5}$ for $i, j \in [5]$ and $\sum_{i,j} t_{i,j} = 0$, or similarly for s. It therefore suffices to analyse inequalities of the form

$$\sum_{i,j=1}^{5} t_{i,j}\|A_i - B_j\|_{S_1} \geq 0$$

where $\sum_{i,j} t_{i,j} = 0$. It is however easy to see this implies $t_{i,j} = 0$ for $i, j \in [5]$ by picking A_i's and B_j's to be either 0 or 1 combining the resulting inequalities with the condition $\sum_{i,j} t_{i,j} = 0$.

In conclusion, new ideas are needed to tackle conjectures like Conjecture 2.

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