Tracial joint spectral measures

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Backstory 1/3

Definition (Schatten-p spaces)

For $p \geq 1$ and a compact operator A, define the S_p-norm with

$$\|A\|_{S_p} = \left(\sum_{i=1}^{\infty} \sigma_i(A)^p \right)^{1/p} = (\operatorname{tr} |A|^p)^{1/p} = \left(\operatorname{tr}(A^* A)^{p/2} \right)^{1/p}.$$

S_2 is the Hilbert–Schmidt norm.
S_∞ is the operator norm.
S_1 is the trace/nuclear norm.

Backstory 2/3

If $A, B \in M_n(\mathbb{C})$ are Hermitian, the triangle inequality in S_p reads

$$\left(\sum_{i=1}^{n} |\lambda_i(A+B)|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |\lambda_i(A)|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |\lambda_i(B)|^p\right)^{1/p}.$$

How is this proven?

1. Von Neumann's trace inequality:

$$\operatorname{tr}(AB) \leq \sum_{i=1}^{n} |\lambda_i(A)||\lambda_i(B)|.$$

2. Majorization: eigenvalues of $A+B$ are spread out the most when A and B commute, $(\lambda_i(A+B))_{i=1}^n \prec (\lambda_i(A)+\lambda_i(B))_{i=1}^n$.
3. Complex interpolation.

Backstory 3/3

Theorem (Hanner, 1955)

If $p \geq 2$ and $f, g \in L_p(0,1)$, then

$$\|f+g\|_{L_p}^p + \|f-g\|_{L_p}^p \leq (\|f\|_{L_p} + \|g\|_{L_p})^p + |\|f\|_{L_p} - \|g\|_{L_p}|^p.$$

A question of Ball, Carlen and Lieb (1994)

Does Hanner's inequality generalize to S_p? Namely, for $p \geq 2$, is the following true for $A, B \in M_n(\mathbb{C})$?

$$\|A+B\|_{S_p}^p + \|A-B\|_{S_p}^p \leq (\|A\|_{S_p} + \|B\|_{S_p})^p + |\|A\|_{S_p} - \|B\|_{S_p}|^p$$

Ball, Carlen and Lieb proved that this is true for $p \geq 4$.

Embedding conjecture

Conjecture (H, 2022)

For $p \geq 1$ and $A, B \in M_n(\mathbb{C})$, there exists functions $f, g \in L_p(0,1)$ such that for any $x, y \in \mathbb{R}$,

$$\|xA + yB\|_{S_p} = \|xf + yg\|_{L_p}.$$

Embedding result!

Theorem (H, 2023)

For $p > 0$ and $A, B \in M_n(\mathbb{C})$, there exists functions $f, g \in L_p(0,1)$ such that for any $x, y \in \mathbb{R}$,

$$\|xA + yB\|_{S_p} = \|xf + yg\|_{L_p}.$$

Subspaces of L_p, 1/5

Characterization of subspaces of L_p

$\mathbb{R}^k$ with norm $\|\cdot\|$ is isometric to a subspace of L_p iff there exists a (necessarily unique) measure μ_p on S^{k-1} such that for any $(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$,

$$\|(x_1, x_2, \ldots, x_k)\|^p = \int_{S^{k-1}} |x_1 t_1 + \ldots + x_k t_k|^p \, \mathrm{d}\mu_p(t_1, \ldots, t_k).$$

Measure μ_p can be explicitly calculated for (Hermitian) 2×2 matrices ($\|(x_1, x_2)\| := \|x_1 A + x_2 B\|_{S_p}$), but for bigger matrices, this seems hopeless.

Subspaces of L_p, 2/5

Simultaneous embedding theorem to L_p?

For $A, B \in M_n(\mathbb{C})$, does there exist a measure μ on S^1 such that for any $(x_1, x_2) \in \mathbb{R}^2$,

$$\|x_1 A + x_2 B\|_{S_p}^p = \int_{S^1} |x_1 t_1 + x_2 t_2|^p \, \mathrm{d}\mu(t_1, t_2)$$

for every $p > 0$?

No! μ_p is unique, and in general $\mu_p \neq \mu_q$ for $p \neq q$.

Subspaces of L_p, 3/5

Better simultaneous embedding to L_p?

For $A, B \in M_n(\mathbb{C})$, does there exist a measure μ on $\mathbb{R}^2$ such that for any $(x_1, x_2) \in \mathbb{R}^2$,

$$\|x_1 A + x_2 B\|_{S_p}^p = \int_{\mathbb{R}^2} |x_1 t_1 + x_2 t_2|^p \, \mathrm{d}\mu(t_1, t_2)$$

for every $p > 0$?

No... μ is usually not a measure, but a distribution.

Subspaces of L_p, 4/5

Betterer simultaneous embedding to L_p?

Does there exists a scaling function $c : \mathbb{R}_+ \to \mathbb{R}_+$ with the following property: for $A, B \in M_n(\mathbb{C})$, there exists a measure μ on $\mathbb{R}^2$ such that for any $(x_1, x_2) \in \mathbb{R}^2$,

$$\|x_1 A + x_2 B\|_{S_p}^p = c(p) \int_{\mathbb{R}^2} |x_1 t_1 + x_2 t_2|^p \, \mathrm{d}\mu(t_1, t_2)$$

for every $p > 0$?

What should $c(p)$ be?

Subspaces of L_p, 5/5

Simultaneous embedding to L_p (H, 2023)

For $A, B \in M_n(\mathbb{C})$, there exists a measure μ on $\mathbb{R}^2$ such that for any $(x_1, x_2) \in \mathbb{R}^2$ and $p > 0$,

$$\|x_1 A + x_2 B\|_{S_p}^p = p(p+1) \int_{\mathbb{R}^2} |x_1 t_1 + x_2 t_2|^p \, \mathrm{d}\mu(t_1, t_2).$$

Tracial joint spectral measure

Theorem (H, 2023)

For **Hermitian** $A, B \in M_n(\mathbb{C})$, there exists a unique measure $\mu_{A,B}$ on $\mathbb{R}^2 \setminus \{0\}$ such that for any $x, y \in \mathbb{R}^2$ and $k \in \mathbb{N}_+$,

$$\operatorname{tr}(xA + yB)^k = k(k+1) \int_{\mathbb{R}^2} (ax + by)^k \, \mathrm{d}\mu_{A,B}(a, b).$$

This $\mu_{A,B}$ is the **tracial joint spectral measure** of A and B.

Tracial joint spectral measure

Theorem (H, 2023)

For Hermitian $A, B \in M_n(\mathbb{C})$, there exists a unique measure $\mu_{A,B}$ on $\mathbb{R}^2 \setminus \{0\}$ such that for any $x, y \in \mathbb{R}^2$ and any $f : \mathbb{R} \to \mathbb{R}$,

$$\operatorname{tr} H(f)(xA + yB) = \int_{\mathbb{R}^2} f(ax + by)\, \mathrm{d}\mu_{A,B}(a, b),$$

where

$$H(f)(x) = \int_0^1 f(xt) \frac{1-t}{t}\, \mathrm{d}t.$$

$H(t^k) = t^k/(k(k+1))$.
$H(|t|^p) = |t|^p/(p(p+1))$.

Formula for tracial joint spectral measure

Theorem (H, 2023)

Decompose $\mu_{A,B} = \mu_c + \mu_s$ w.r.t. the Lebesgue measure ($\mu_c \ll m_2$, $\mu_s \perp m_2$). Then

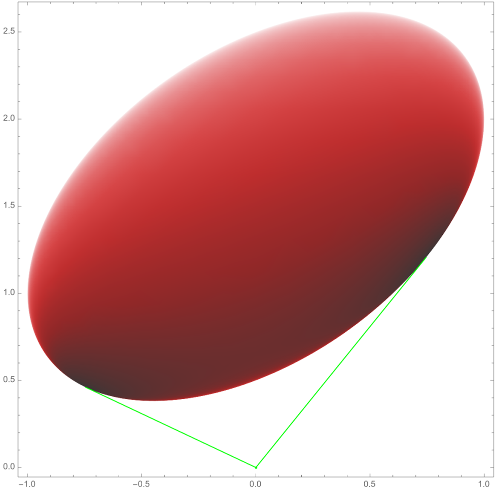
$$\frac{\mathrm{d}\mu_c}{\mathrm{d}m_2}(a,b) = \frac{1}{2\pi}\sum_{i=1}^{n}\left|\operatorname{Im}\left(\lambda_i\left(\left(I - \frac{aA+bB}{a^2+b^2}\right)(bA-aB)^{-1}\right)\right)\right|,$$

and for $\varphi \in C_c(\mathbb{R}^2 \setminus \{0\})$,

$$\int_{\mathbb{R}^2}\varphi(a,b)\,\mathrm{d}\mu_s(a,b) = \sum_{i=1}^{k}\int_0^1 \varphi\left(\frac{\langle Av_i, v_i\rangle}{\langle v_i, v_i\rangle}t, \frac{\langle Bv_i, v_i\rangle}{\langle v_i, v_i\rangle}t\right)\frac{1-t}{t}\,\mathrm{d}t.$$

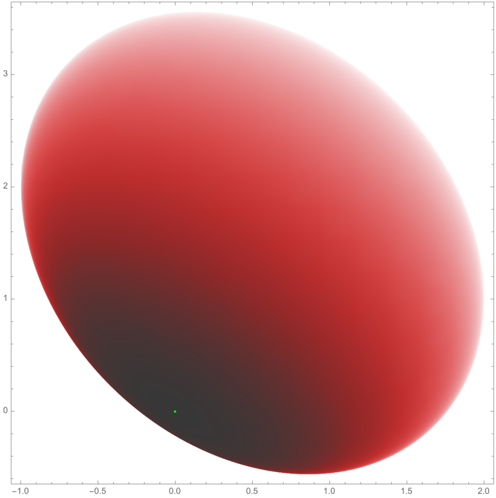
where $\{v_1, v_2, \ldots, v_k\}$ are eigenvectors of $A^{-1}B$ corresponding to the real eigenvalues.

2×2 example 1



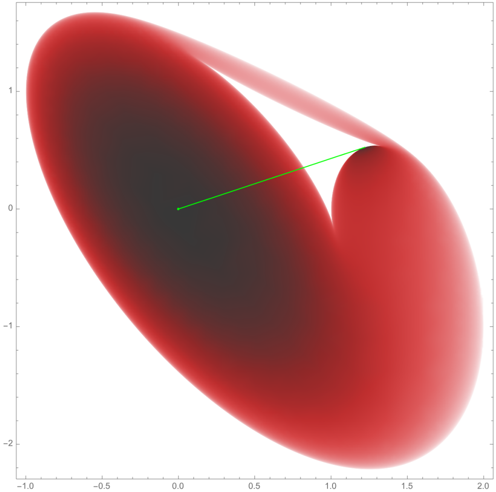
$$(A, B) = \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \right).$$

2×2 example 2



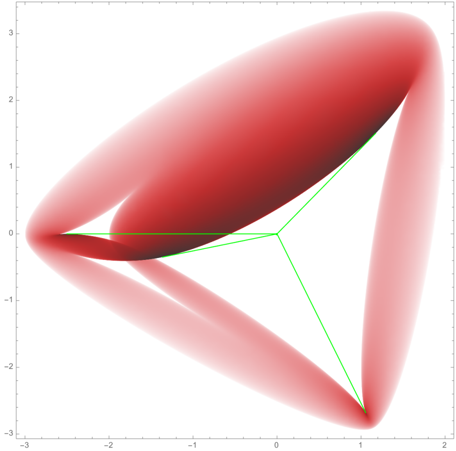
$$(A, B) = \left(\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ -2 & 2 \end{bmatrix} \right).$$

3×3 example



$$(A, B) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \right).$$

4×4 example



$$(A, B) = \left(\begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -2 & -2 \\ -1 & -1 & -2 & 2 \end{bmatrix} \right).$$

Basic properties of tracial joint spectral measures

1. The continuous part μ_c is supported on (a subset of) the joint numerical range

$$\mathcal{W}(A) = \{(\langle Av, v\rangle, \langle Bv, v\rangle) \mid v \in S^{n-1}\} \subset \mathbb{R}^2.$$

2. Singular part is supported on tangents from the origin to the boundary curve of the continuous part, *Kippenhahn curve*.
3.

$$\text{If} \qquad A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix},$$

$$\text{then} \qquad \mu_{A,B} = \mu_{A_1,B_1} + \mu_{A_2,B_2}.$$

4. The continuous part μ_c vanishes iff A and B commute.

The main application

Theorem (H, 2023)

For $p > 0$ and $A, B \in M_n(\mathbb{C})$, there exists functions $f, g \in L_p(0,1)$ such that for any $x, y \in \mathbb{R}$,

$$\|xA + yB\|_{S_p} = \|xf + yg\|_{L_p}.$$

Proof.

Tracial joint spectral measure of A and B applied to the function $t \mapsto |t|^p$ implies that for $x, y \in \mathbb{R}$,

$$\frac{\|xA + yB\|_{S_p}^p}{p(p+1)} = \frac{\operatorname{tr}|xA + yB|^p}{p(p+1)} = \int_{\mathbb{R}^2} |ax + by|^p \, \mathrm{d}\mu_{A,B}(a,b).$$

This means that we should choose $f, g \in L_p(\mu_{A,B})$ with $f = (a,b) \mapsto a$ and $g = (a,b) \mapsto b$. □

Further applications 1/2

Theorem (H, 2023)

If $f : \mathbb{R} \to \mathbb{R}$ has non-negative k:th derivative, then for any Hermitian $A, B \in M_n(\mathbb{C})$ with $A \geq 0$, so does

$$t \mapsto \operatorname{tr} f(tA + B).$$

Proof.

Apply tracial joint spectral measure to $f(t) = t_+^{k-1}$. □

Applying this result to $f(t) = \exp(t)$ recovers a result of Stahl (formerly the BMV conjecture).

Theorem (Stahl, 2011)

Function $t \mapsto \operatorname{tr} \exp(B - tA)$ *is a Laplace transform of a positive measure for Hermitian* $A, B \in M_n(\mathbb{C})$ *with* $A \geq 0$.

Further applications 2/2

Any non-negative bivariate polynomial p with $p(0,0) = 0$ gives rise to an (often non-trivial) inequality.

Example

If $p(a,b) = (a^2 + b^2 - a)^2$,

$$0 \leq 6 \int p(a,b)\, \mathrm{d}\mu_{A,B}(a,b) = \mathrm{tr}(A^2) - \mathrm{tr}(A^3) - \mathrm{tr}(AB^2) + \frac{3\,\mathrm{tr}(A^4) + 4\,\mathrm{tr}(A^2B^2) + 2\,\mathrm{tr}(ABAB) + 3\,\mathrm{tr}(B^4)}{10}.$$

Limitations 1/2

Tracial joint spectral measures don't generalize to triplets of matrices.

Theorem (H, 2022)

If $0 < p < \infty$, $p \neq 2$, the 3-dimensional space of 2×2 real symmetric matrices is **not** *isometric to a subspace of $L_p(0,1)$.*

Limitations 2/2

The proof only works for matrices, and while a compactness argument can deal with compact operators on a Hilbert space:

Question

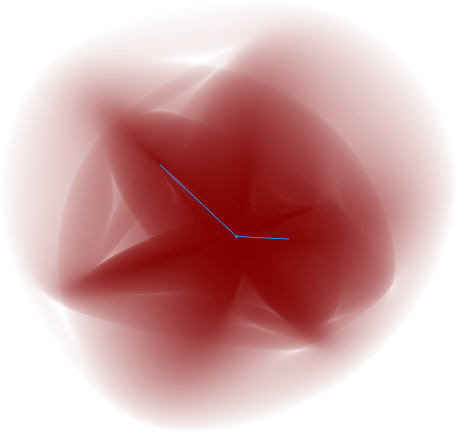
Do tracial joint spectral measures exist for tracial von Neumann algebras $(\mathcal{M}, \tau)$? That is, if $A, B \in (\mathcal{M}, \tau)$ are self-adjoint, does there exist a measure $\mu_{A,B}$ on $\mathbb{R}^2$, such that for $f : \mathbb{R} \to \mathbb{R}$ and $x, y \in \mathbb{R}$, one has

$$\tau(H(f)(xA + yB)) = \int_{\mathbb{R}^2} f(ax + by)\, \mathrm{d}\mu_{A,B}(a, b)?$$

Question

Is every 2-dimensional subspace of a non-commutative L_p-space isometric to a subspace of $L_p(0, 1)$?

Thank you!



Interactive demo (that generated the above 10×10 example):

shikhin.in/tjsm/tjsm.html

On the proof

1. Define $g(x) = \int_0^1 e^{tx}(1-t)/t \, \mathrm{d}t$ and consider the function

$$G : (x, y) \mapsto \operatorname{tr} g(xA + yB).$$

2. Prove that the tracial joint spectral measure coincides with the (distributional) Fourier transform of G outside 0.
3. Prove that $\hat{G}$ satisfies the formula by taking a test function φ and calculating

$$(\hat{G}, \varphi) = (G, \hat{\varphi}) = \int G(x, y)\hat{\varphi}(x, y) \, \mathrm{d}x \, \mathrm{d}y = \dots$$