Tracial joint spectral measures

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Spectral measures

$A \in M_n(\mathbb{C})$ is Hermitian

Spectral theorem

- A is diagonalizable
- A has real eigenvalues
- A has orthogonal eigenspaces

Spectral measures

$A \in M_n(\mathbb{C})$ is Hermitian

Spectral theorem

There exists a projection valued measure π_A on $\mathbb{R}$ such that

- $\pi_A(U \cap V) = \pi_A(U)\pi_A(V)$ for open $U, V \subset \mathbb{R}$,
- $A^k = \int_{\mathbb{R}} a^k \, d\pi_A(a)$ for $k \in \mathbb{N}$.

Spectral measures

$A \in M_n(\mathbb{C})$ is Hermitian

Spectral theorem

There exists a projection valued measure π_A on $\mathbb{R}$ such that

$$f(A) = \int_{\mathbb{R}} f(a)\,\mathrm{d}\pi_A(a)$$

for any f.

As

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \Rightarrow f(A) = \begin{bmatrix} f(a_1) & 0 \\ 0 & f(a_2) \end{bmatrix},$$

$\pi_A(\{a_i\})$ equals the projection to the i:th coordinate (with distincts a_i's)

Joint spectral measures

$A, B \in M_n(\mathbb{C})$ are **commuting** Hermitian.

Joint spectral theorem

There exists a projection valued measure $\pi_{A,B}$ on $\mathbb{R}^2$ such that

$$f(A, B) = \int_{\mathbb{R}} f(a, b)\, \mathrm{d}\pi_{A,B}(a, b)$$

for any f.

As

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \; B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \Rightarrow f(A, B) = \begin{bmatrix} f(a_1, b_1) & 0 \\ 0 & f(a_2, b_2) \end{bmatrix},$$

$\pi_{A,B}(\{(a_i, b_i)\})$ equals the projection to the i:th coordinate (with distinct (a_i, b_i)'s).

Noncommutative joint spectral measure

If A and B don't commute, what should $f(A, B)$ mean?

Weyl calculus

We should have

$$A^k = \int_{\mathbb{R}^2} a^k \, \mathrm{d}\pi_{A,B}(a,b)$$

for $k \in \mathbb{N}$.

Weyl calculus

We should have

$$A^k = \int_{\mathbb{R}^2} a^k \, \mathrm{d}\pi_{A,B}(a, b)$$
$$B^k = \int_{\mathbb{R}^2} b^k \, \mathrm{d}\pi_{A,B}(a, b)$$

for $k \in \mathbb{N}$.

Weyl calculus

We should have

$$A^k = \int_{\mathbb{R}^2} a^k \, \mathrm{d}\pi_{A,B}(a,b)$$

$$B^k = \int_{\mathbb{R}^2} b^k \, \mathrm{d}\pi_{A,B}(a,b)$$

$$(A+B)^k = \int_{\mathbb{R}^2} (a+b)^k \, \mathrm{d}\pi_{A,B}(a,b)$$

for $k \in \mathbb{N}$.

Weyl calculus

We should have

$$(xA + yB)^k = \int_{\mathbb{R}^2} (xa + yb)^k \, d\pi_{A,B}(a, b)$$

for $k \in \mathbb{N}$ for $x, y \in \mathbb{R}$.

Weyl calculus

Theorem

There exists a projection valued measure $\pi_{A,B}$ on $\mathbb{R}^2$ such that for any $x, y \in \mathbb{R}$ and $k \in \mathbb{N}$ one has,

$$(xA + yB)^k = \int_{\mathbb{R}^2} (xa + yb)^k \, \mathrm{d}\pi_{A,B}(a, b).$$

Weyl calculus

Theorem (Weyl, 1931)

There exists a ~~projection~~ matrix valued ~~measure~~ distribution ~~$\pi_{A,B}$~~ $\mathcal{W}_{A,B}$ on $\mathbb{R}^2$ such that for any $x, y \in \mathbb{R}$ and $k \in \mathbb{N}$ one has,

$$(xA + yB)^k = \int_{\mathbb{R}^2} (xa + yb)^k \, \mathrm{d}\mathcal{W}_{A,B}(a, b).$$

Weyl calculus

Theorem (Weyl, 1931)

There exists a ~~projection~~ matrix valued ~~measure~~ distribution ~~$\pi_{A,B}$~~ $\mathcal{W}_{A,B}$ on $\mathbb{R}^2$ such that for any $x, y \in \mathbb{R}$ and $k \in \mathbb{N}$ one has,

$$(xA + yB)^k = \int_{\mathbb{R}^2} (xa + yb)^k \, \mathrm{d}\mathcal{W}_{A,B}(a, b).$$

Anderson (1970): $\mathcal{W}_{A,B}$ is measure only if A and B commute.

Tracial joint spectral measure

Theorem

There exists a ~~projection valued~~ measure ~~$\pi_{A,B}$~~ $\mu_{A,B}$ on $\mathbb{R}^2$ such that for any $x, y \in \mathbb{R}$ and $k \in \mathbb{N}$ one has,

$$\operatorname{tr}(xA + yB)^k = \int_{\mathbb{R}^2} (xa + yb)^k \, \mathrm{d}\mu_{A,B}(a, b).$$

Tracial joint spectral measure

Theorem (H, 2023)

There exists a measure $\mu_{A,B}$ on $\mathbb{R}^2 \setminus \{0\}$ such that for any $x, y \in \mathbb{R}^2$ and $k \in \mathbb{N}_+$,

$$\operatorname{tr}(xA + yB)^k = k(k+1) \int_{\mathbb{R}^2} (xa + yb)^k \, \mathrm{d}\mu_{A,B}(a, b).$$

This $\mu_{A,B}$ is the **tracial joint spectral measure** of A and B.

Tracial joint spectral measure

Theorem (H, 2023)

For Hermitian $A, B \in M_n(\mathbb{C})$, there exists a unique measure $\mu_{A,B}$ on $\mathbb{R}^2 \setminus \{0\}$ such that for any $x, y \in \mathbb{R}^2$ and any $f : \mathbb{R} \to \mathbb{R}$,

$$\operatorname{tr} H(f)(xA + yB) = \int_{\mathbb{R}^2} f(ax + by)\, \mathrm{d}\mu_{A,B}(a, b),$$

where

$$H(f)(x) = \int_0^1 f(xt) \frac{1-t}{t}\, \mathrm{d}t.$$

$H(t^k) = t^k/(k(k+1))$.

Formula for tracial joint spectral measure

Theorem (H, 2023)

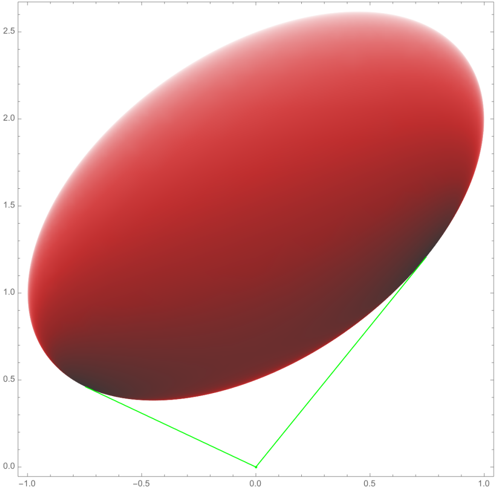
Decompose $\mu_{A,B} = \mu_c + \mu_s$ w.r.t. the Lebesgue measure ($\mu_c \ll m_2$, $\mu_s \perp m_2$). Then

$$\frac{\mathrm{d}\mu_c}{\mathrm{d}m_2}(a,b) = \frac{1}{2\pi}\sum_{i=1}^{n}\left|\operatorname{Im}\left(\lambda_i\left(\left(I - \frac{aA+bB}{a^2+b^2}\right)(bA-aB)^{-1}\right)\right)\right|,$$

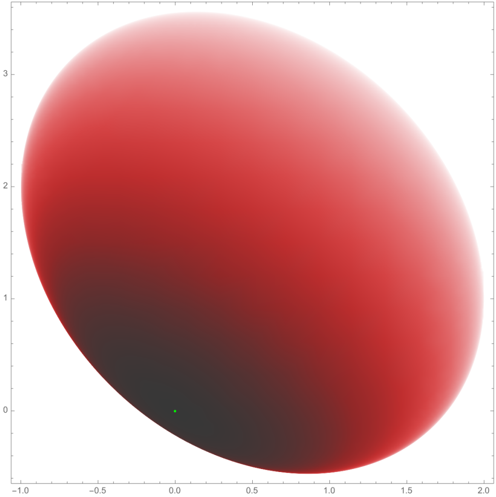
and for $\varphi \in C_c(\mathbb{R}^2 \setminus \{0\})$,

$$\int_{\mathbb{R}^2}\varphi(a,b)\,\mathrm{d}\mu_s(a,b) = \sum_{i=1}^{k}\int_0^1 \varphi\left(\frac{\langle Av_i, v_i\rangle}{\langle v_i, v_i\rangle}t, \frac{\langle Bv_i, v_i\rangle}{\langle v_i, v_i\rangle}t\right)\frac{1-t}{t}\,\mathrm{d}t.$$

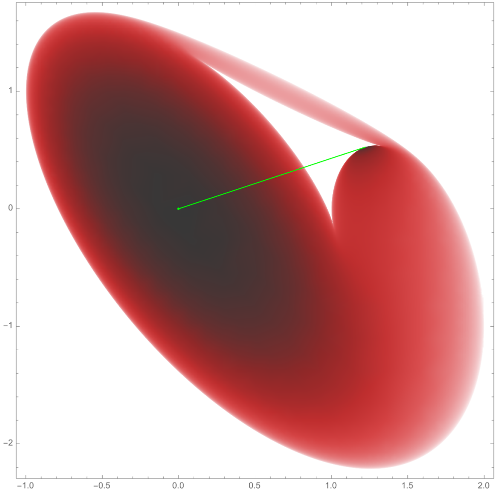
where $\{v_1, v_2, \ldots, v_k\}$ are eigenvectors of $A^{-1}B$ corresponding to the real eigenvalues.



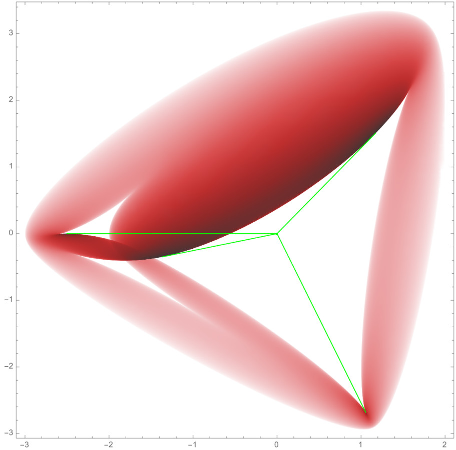
$$(A, B) = \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \right).$$



$$(A, B) = \left(\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ -2 & 2 \end{bmatrix} \right).$$



$$(A,B)=\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & -1\end{bmatrix},\begin{bmatrix}0 & 1 & 1\\ 1 & -1 & -1\\ 1 & -1 & 1\end{bmatrix}\right).$$



$$(A, B) = \left(\begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -2 & -2 \\ -1 & -1 & -2 & 2 \end{bmatrix} \right).$$

Definition (Schatten-p spaces)

For $p \geq 1$ and a compact operator A, define the S_p-norm with

$$\|A\|_{S_p} = \left(\sum_{i=1}^{\infty} \sigma_i(A)^p \right)^{1/p} = \left(\operatorname{tr}(A^* A)^{p/2} \right)^{1/p}.$$

If A is Hermitian, then

$$\|A\|_{S_p}^p = \sum_{i=1}^{n} |\lambda_i(A)|^p = \operatorname{tr} |A|^p.$$

Theorem (H, 2023)

For $p > 0$ and $A, B \in M_n(\mathbb{C})$, there exists functions $f, g \in L_p(0,1)$ such that for any $x, y \in \mathbb{R}$,

$$\|xA + yB\|_{S_p} = \|xf + yg\|_{L_p}.$$

Corollary (Hanner's inequality for Schatten-p)

For $p \geq 2$,

$$\|A+B\|_{S_p}^p + \|A-B\|_{S_p}^p \leq (\|A\|_{S_p} + \|B\|_{S_p})^p + |\|A\|_{S_p} - \|B\|_{S_p}|^p$$

Theorem (H, 2023)

For $p > 0$ and $A, B \in M_n(\mathbb{C})$, there exists functions $f, g \in L_p(0,1)$ such that for any $x, y \in \mathbb{R}$,

$$\|xA + yB\|_{S_p} = \|xf + yg\|_{L_p}.$$

Proof.

Tracial joint spectral measure of A and B applied to the function $t \mapsto |t|^p$ (for which $H(|t|^p) = |t|^p/(p(p+1))$) implies that for $x, y \in \mathbb{R}$,

$$\frac{\|xA + yB\|_{S_p}^p}{p(p+1)} = \frac{\operatorname{tr}|xA + yB|^p}{p(p+1)} = \int_{\mathbb{R}^2} |ax + by|^p \, \mathrm{d}\mu_{A,B}(a,b).$$

This means that we should choose $f, g \in L_p(\mu_{A,B})$ with $f = (a,b) \mapsto a$ and $g = (a,b) \mapsto b$. □

Theorem (H, 2023)

If $f : \mathbb{R} \to \mathbb{R}$ has non-negative k:th derivative, then for any Hermitian $A, B \in M_n(\mathbb{C})$ with $A \geq 0$, so does

$$t \mapsto \operatorname{tr} f(tA + B).$$

Proof.

Apply tracial joint spectral measure to $f(t) = t_+^{k-1}$. □

Theorem (H, 2023)

If $f : \mathbb{R} \to \mathbb{R}$ has non-negative k:th derivative, then for any Hermitian $A, B \in M_n(\mathbb{C})$ with $A \geq 0$, so does

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Proof.

Apply tracial joint spectral measure to $f(t) = t_+^{k-1}$. □

Applying this result to $f(t) = \exp(t)$ recovers a result of Stahl (formerly the BMV conjecture).

Theorem (Stahl, 2011)

Function $t \mapsto \operatorname{tr} \exp(B - tA)$ is a Laplace transform of a positive measure for Hermitian $A, B \in M_n(\mathbb{C})$ with $A \geq 0$.

Any non-negative bivariate polynomial p with $p(0,0) = 0$ gives rise to an (often non-trivial) inequality.

Example

If $p(a,b) = (a^2 + b^2 - a)^2$,

$$0 \leq 6 \int p(a,b) \, \mathrm{d}\mu_{A,B}(a,b) = \operatorname{tr}(A^2) - \operatorname{tr}(A^3) - \operatorname{tr}(AB^2) + \frac{3\operatorname{tr}(A^4) + 4\operatorname{tr}(A^2B^2) + 2\operatorname{tr}(ABAB) + 3\operatorname{tr}(B^4)}{10}.$$

Tracial joint spectral measures don't generalize to triplets of matrices.

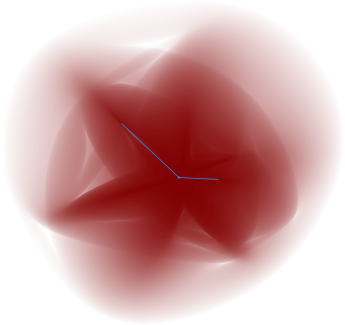
Theorem (H, 2022)

If $0 < p < \infty$, $p \neq 2$, the 3-dimensional space of 2×2 real symmetric matrices is ***not*** *isometric to a subspace of $L_p(0,1)$.*

Thank you!

Some questions:

- Von Neumann algebras? (cf. Connes)
- Proof without matrices? (cf. hyperbolic polynomials, Helton–Vinnikov)
- $[A, B]$ small $\Rightarrow$? (cf. Lin's theorem)
- Simpler proof? (cf. the proof of Stahl's theorem)
- Structure in higher dimensions?



Interactive demo (that generated the above 10×10 example):

shikhin.in/tjsm/tjsm.html

0. Define $g(x) = \int_0^1 (e^{tx} - 1)(1-t)/t \, \mathrm{d}t$ and consider the function

$$G : (x, y) \mapsto \operatorname{tr} g(xA + yB).$$

1. (easy) Prove that the tracial joint spectral measure coincides with the (distributional) Fourier transform of G outside 0, i.e.

$$\operatorname{tr} H(f)(xA + yB) = \int_{\mathbb{R}^2} f(ax + by) \hat{G}(a, b) \, \mathrm{d}a \, \mathrm{d}b.$$

2. (not so easy) Prove that $\hat{G}$ satisfies the formula by taking a test function φ and calculating

$$(\hat{G}, \varphi) = (G, \hat{\varphi}) = \int G(x, y) \hat{\varphi}(x, y) \, \mathrm{d}x \, \mathrm{d}y = \dots$$

Recall that $\mu_{A,B}$ has continuous part with density

$$\frac{d\mu_c}{dm_2}(a,b) = \frac{1}{2\pi}\sum_{i=1}^{n} \left|\mathrm{Im}\left(\lambda_i\left(C(a,b)\right)\right)\right|,$$

where $C(a,b)$ is some auxiliary matrix; and singular part satisfying

$$\int_{\mathbb{R}^2} \varphi(a,b)\, d\mu_s(a,b) = \sum_{i=1}^{k}\int_0^1 \varphi\left(\frac{\langle Av_i, v_i\rangle}{\langle v_i, v_i\rangle}t, \frac{\langle Bv_i, v_i\rangle}{\langle v_i, v_i\rangle}t\right)\frac{1-t}{t}\, dt.$$

Key identities:

1. For $\lambda \in \mathbb{C}$,

$$\lim_{M\to\infty}\int_{|t|<M} \log\left|1+\frac{\lambda}{t}\right| dt = \pi\left|\mathrm{Im}(\lambda)\right|.$$

2. For Hermitian $A, B \in M_n(\mathbb{C})$, if $\ker(B) = \mathrm{span}(v)$, then

$$\frac{\det(B+tA)}{\det(B+tI)} = \frac{\langle Av, v\rangle}{\langle v, v\rangle} + O(t) \text{ as } t \to 0.$$