Tracial joint spectral measures

Final public oral exam

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Princeton University

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Distances

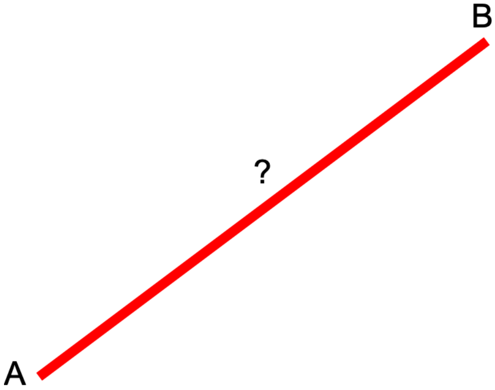


Figure: points A and B

Distances

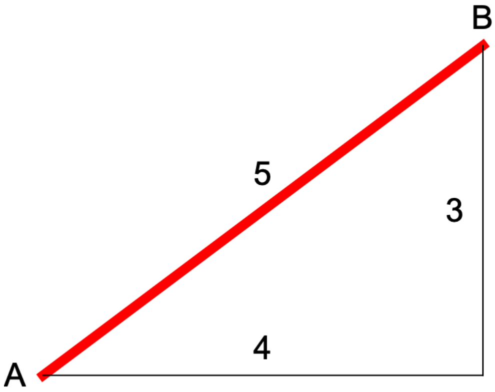


Figure: distance equals $\sqrt{4^2 + 3^2} = 5$

Distances

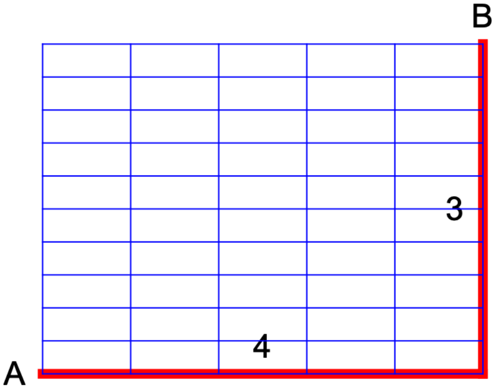
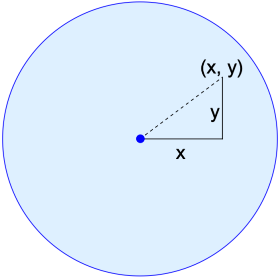


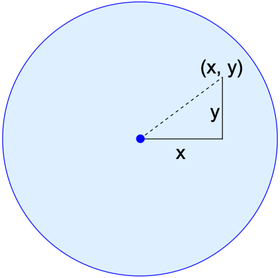
Figure: distance equals $4 + 3 = 7$

Balls



(a) $x^2 + y^2 \leq 1$

Balls



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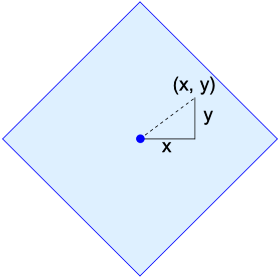
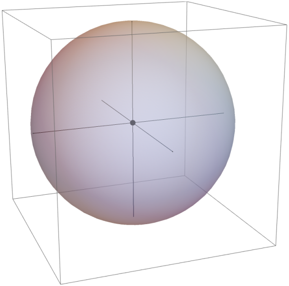


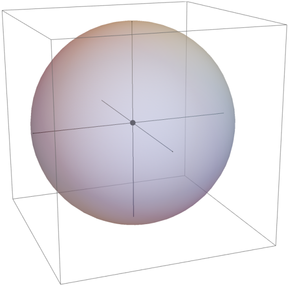
(b) $|x| + |y| \leq 1$

3-dimensional balls



(a) $x^2 + y^2 + z^2 \leq 1$

3-dimensional balls



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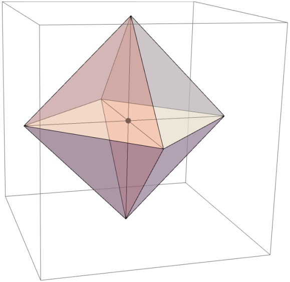


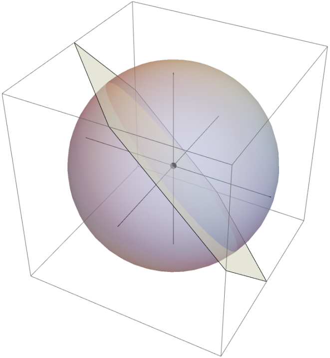
(b) $|x| + |y| + |z| \leq 1$

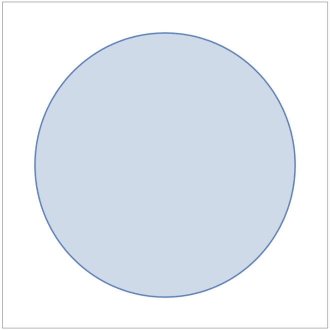
Higher dimension

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \mapsto \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2}$$

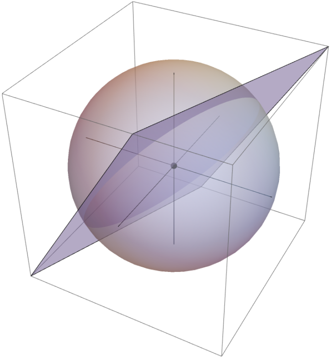
$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \mapsto |x_1| + |x_2| + |x_3| + |x_4| + |x_5| + |x_6| + |x_7| + |x_8| + |x_9|$$

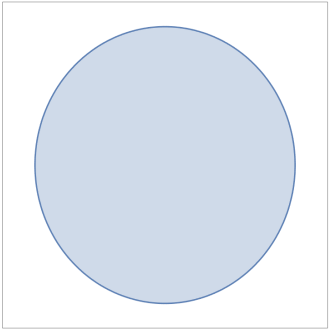
Slices



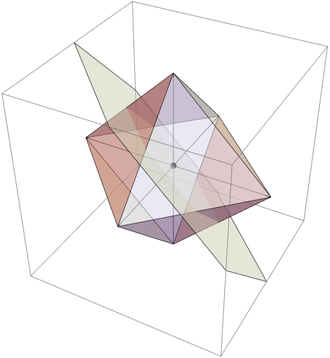


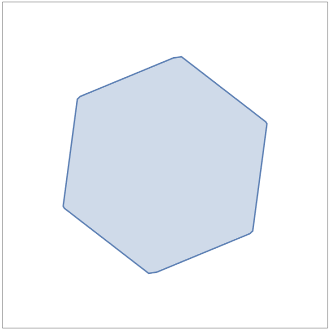
Slices



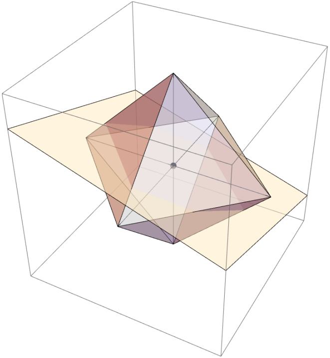


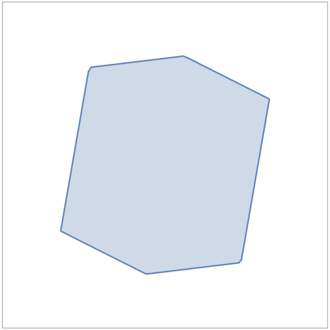
Slices



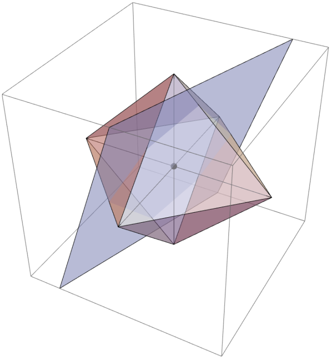


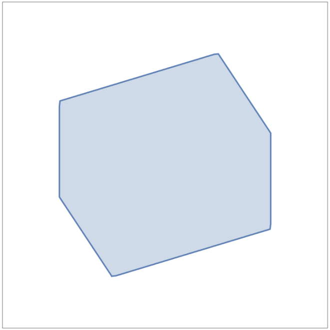
Slices



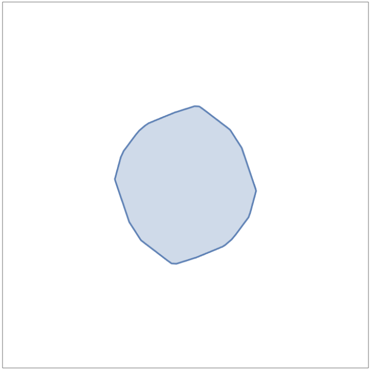


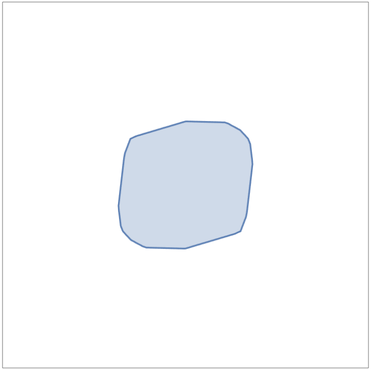
Slices





Slices of the 9-dimensional Manhattan ball





Are slices useful?

Yes: they can see features of the ball.

1. Corners (sharp edges)
2. Flat portions

Are slices useful?

Yes: they can see features of the ball.

1. Corners (sharp edges)
2. Flat portions
3. Smoothened out corners (sharp-ish edges)
4. Portions that are almost flat

ℓ_p-norm:

$$(x_1, x_2, x_3, x_4) \mapsto (|x_1|^p + |x_2|^p + |x_3|^p + |x_4|^p)^{1/p}$$

Schatten-p-norm:

$$\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix} \mapsto \text{some funky thing}$$

My thesis

Theorem

All slices of Schatten-p ball can also be found as slices of ℓ_p ball.

My thesis

Theorem

All slices of Schatten-p ball can also be found as slices of ℓ_p ball.

Proof.

Recipe: **Tracial joint spectral measures!** □

Definition (Schatten-p spaces)

For $p \geq 1$ and a compact operator A, define the S_p-norm with

$$\|A\|_{S_p} = \left(\sum_{i=1}^{\infty} \sigma_i(A)^p \right)^{1/p} = \left(\operatorname{tr}|A|^p\right)^{1/p} = \left(\operatorname{tr}(A^* A)^{p/2}\right)^{1/p}.$$

S_2 is the Hilbert–Schmidt norm.
S_∞ is the operator norm.
S_1 is the trace/nuclear norm.

Unitarily invariant norms

Norm $\|\cdot\|$ on $M_n(\mathbb{C})$ is **unitarily invariant**, if for any two unitaries U and V we have $\|UAV\| = \|A\|$ for any $A \in M_n(\mathbb{C})$.
Function $\mathbb{C}^n \to \mathbb{R}_+$ is a **symmetric gauge function** if

- Φ is a norm
- $\Phi((x_{\sigma(i)})_{i=1}^n) = \Phi((x_i)_{i=1}^n)$ for any permutation σ
- $\Phi((\omega_i x_i)_{i=1}^n) = \Phi((x_i)_{i=1}^n)$ for any $\omega_1, \omega_2, \ldots, \omega_n \in S^1 \subset \mathbb{C}$.

Theorem (von Neumann, 1937)

Norm $\|\cdot\|$ on $M_n(\mathbb{C})$ is unitarily invariant iff there exists a symmetric gauge function Φ such that

$$\|A\| = \Phi((\sigma_i(A))_{i=1}^n).$$

Theorem (von Neumann, 1937)

Norm $\|\cdot\|$ on $M_n(\mathbb{C})$ is unitarily invariant iff there exists a symmetric gauge function Φ such that

$$\|A\| = \Phi((\sigma_i(A))_{i=1}^n).$$

Lemma (von Neumann, 1937)

If Φ and Φ' are dual norms, i.e.

$$\sup_{u \in \mathbb{C}^n, \Phi'(u) \leq 1} \langle v, u \rangle = \Phi(v),$$

then

$$\sup_{B \in M_n(\mathbb{C}^n), \|B\|_{\Phi'} \leq 1} \operatorname{tr}(AB^*) = \|A\|_\Phi.$$

Similarities and differences

For $1 < p < \infty$,

1. S_p is dual to S_q if $1/p + 1/q = 1$
2. S_p is uniformly convex and uniformly smooth
3. S_p is the (complex) interpolation space of S_1 and S_∞
4. S_p has type $\min(2, p)$ and cotype $\max(2, p)$,

but for $p \neq 2$,

1. S_p is **not** isomorphic to a subspace of $L_p([0, 1])$ and in fact
2. S_p does **not** have an unconditional basis

Uniform convexity

Definition (Clarkson)

A Banach space $(X, \|\cdot\|)$ is uniformly convex, if for any $\varepsilon > 0$

$$\delta_X(\varepsilon) := \inf_{x,y \in S_X} \left\{ 1 - \left\| \frac{x+y}{2} \right\| \mid \|x - y\| \geq \varepsilon \right\} > 0.$$

Theorem (Clarkson, 1936)

ℓ_p and $L_p([0,1])$ are uniformly convex.

Clarkson's inequalities

$$\delta_X(\varepsilon) := \inf_{x,y \in S_X} \left\{ 1 - \left\| \frac{x+y}{2} \right\| \mid \|x-y\| \geq \varepsilon \right\} > 0.$$

Theorem (Clarkson, 1936)

If $1 < p \leq 2$ *and* $f, g \in L_p([0,1])$*, then*

$$\left\| \frac{f+g}{2} \right\|_{L_p}^q + \left\| \frac{f-g}{2} \right\|_{L_p}^q \leq \left(\frac{1}{2} \left(\|f\|_{L_p}^p + \|g\|_{L_p}^p \right) \right)^{q/p},$$

where $1/p + 1/q = 1$.

$$\delta_{L_p}(\varepsilon) \geq 1 - \left(1 - \left(\frac{\varepsilon}{2} \right)^q \right)^{1/q} \geq \frac{1}{q 2^q} \varepsilon^q$$

Theorem (Hanner, 1955)

If $1 \leq p \leq 2$ *and* $f, g \in L_p([0,1])$, *then*

$$\|f+g\|_{L_p}^p + \|f-g\|_{L_p}^p \geq (\|f\|_{L_p} + \|g\|_{L_p})^p + \big|\|f\|_{L_p} - \|g\|_{L_p}\big|^p.$$

Theorem (Hanner, 1955)

If $1 \leq p \leq 2$ *and* $f, g \in L_p([0,1])$, *then*

$$\|f+g\|_{L_p}^p + \|f-g\|_{L_p}^p \geq (\|f\|_{L_p} + \|g\|_{L_p})^p + |\|f\|_{L_p} - \|g\|_{L_p}|^p.$$

A question of Ball, Carlen and Lieb (1994)

Does Hanner's inequality generalize to S_p? Namely, for $1 \leq p \leq 2$, is the following true for $A, B \in M_n(\mathbb{C})$?

$$\|A+B\|_{S_p}^p + \|A-B\|_{S_p}^p \geq (\|A\|_{S_p} + \|B\|_{S_p})^p + |\|A\|_{S_p} - \|B\|_{S_p}|^p$$

Ball, Carlen and Lieb proved that this is true for $1 \leq p \leq 4/3$.

Embedding conjecture

Conjecture (H, 2022)

For $p \geq 1$ and $A, B \in M_n(\mathbb{C})$, there exists functions $f, g \in L_p([0,1])$ such that for any $x, y \in \mathbb{R}$,

$$\|xA + yB\|_{S_p} = \|xf + yg\|_{L_p}.$$

Embedding result!

Theorem (H, 2023)

For $p > 0$ and $A, B \in M_n(\mathbb{C})$, there exists functions $f, g \in L_p([0,1])$ such that for any $x, y \in \mathbb{R}$,

$$\|xA + yB\|_{S_p} = \|xf + yg\|_{L_p}.$$

Characterization of subspaces of L_p

$\mathbb{R}^k$ with norm $\|\cdot\|$ is isometric to a subspace of L_p iff there exists a (necessarily unique) measure μ_p on S^{k-1} such that for any $(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$,

$$\|(x_1, x_2, \ldots, x_k)\|^p = \int_{S^{k-1}} |x_1 t_1 + \ldots + x_k t_k|^p \, \mathrm{d}\mu_p(t_1, \ldots, t_k).$$

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Measure μ_p can be explicitly calculated for (Hermitian) 2×2 matrices ($\|(x_1, x_2)\| := \|x_1 A + x_2 B\|_{S_p}$), but for bigger matrices, this seems hopeless.

Better representing measure?

For $A, B \in M_n(\mathbb{C})$, does there exist a **natural** measure μ on $\mathbb{R}^2$ such that for any $(x_1, x_2) \in \mathbb{R}^2$,

$$\|x_1 A + x_2 B\|_{S_p}^p = \int_{\mathbb{R}^2} |x_1 t_1 + x_2 t_2|^p \, \mathrm{d}\mu(t_1, t_2)$$

for every $p > 0$?

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for every $p > 0$?

The measure μ is not unique, so how should it be chosen?

Simultaneous embedding to L_p?

For $A, B \in M_n(\mathbb{C})$, does there exist a measure μ on $\mathbb{R}^2$ such that for any $(x_1, x_2) \in \mathbb{R}^2$,

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Simultaneous embedding to L_p?

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for every $p > 0$?

No... μ is usually not a measure, but a distribution.

Better simultaneous embedding to L_p?

Does there exists a scaling function $c : \mathbb{R}_+ \to \mathbb{R}_+$ with the following property: for $A, B \in M_n(\mathbb{C})$, there exists a measure μ on $\mathbb{R}^2$ such that for any $(x_1, x_2) \in \mathbb{R}^2$,

$$\|x_1 A + x_2 B\|_{S_p}^p = c(p) \int_{\mathbb{R}^2} |x_1 t_1 + x_2 t_2|^p \,\mathrm{d}\mu(t_1, t_2)$$

for every $p > 0$?

What should $c(p)$ be?

Simultaneous embedding to L_p (H, 2023)

For $A, B \in M_n(\mathbb{C})$, there exists a measure μ on $\mathbb{R}^2$ such that for any $(x_1, x_2) \in \mathbb{R}^2$ and $p > 0$,

$$\|x_1 A + x_2 B\|_{S_p}^p = p(p+1) \int_{\mathbb{R}^2} |x_1 t_1 + x_2 t_2|^p \, \mathrm{d}\mu(t_1, t_2).$$

Tracial joint spectral measure

Theorem (H, 2023)

For **Hermitian** $A, B \in M_n(\mathbb{C})$, there exists a unique measure $\mu_{A,B}$ on $\mathbb{R}^2 \setminus \{0\}$ such that for any $x, y \in \mathbb{R}$ and $k \in \mathbb{N}_+$,

$$\operatorname{tr}(xA + yB)^k = k(k+1) \int_{\mathbb{R}^2} (ax + by)^k \, \mathrm{d}\mu_{A,B}(a, b).$$

This $\mu_{A,B}$ is the **tracial joint spectral measure** of A and B.

Tracial joint spectral measure

Theorem (H, 2023)

For Hermitian $A, B \in M_n(\mathbb{C})$, there exists a unique measure $\mu_{A,B}$ on $\mathbb{R}^2 \setminus \{0\}$ such that for any $x, y \in \mathbb{R}$ and any $f : \mathbb{R} \to \mathbb{R}$,

$$\operatorname{tr} H(f)(xA + yB) = \int_{\mathbb{R}^2} f(ax + by)\, \mathrm{d}\mu_{A,B}(a, b),$$

where

$$H(f)(x) = \int_0^1 f(xt) \frac{1-t}{t}\, \mathrm{d}t.$$

$H(t^k) = t^k/(k(k+1))$.
$H(|t|^p) = |t|^p/(p(p+1))$.

Formula for tracial joint spectral measures

Theorem (H, 2023)

Decompose $\mu_{A,B} = \mu_c + \mu_s$ w.r.t. the Lebesgue measure ($\mu_c \ll m_2$, $\mu_s \perp m_2$). Then

$$\frac{\mathrm{d}\mu_c}{\mathrm{d}m_2}(a,b) = \frac{1}{2\pi}\sum_{i=1}^{n}\left|\operatorname{Im}\left(\lambda_i\left(\left(I - \frac{aA+bB}{a^2+b^2}\right)(bA-aB)^{-1}\right)\right)\right|,$$

and for $\varphi \in C_c(\mathbb{R}^2 \setminus \{0\})$,

$$\int_{\mathbb{R}^2} \varphi(a,b)\,\mathrm{d}\mu_s(a,b) = \sum_{i=1}^{k}\int_0^1 \varphi\left(\frac{\langle Av_i, v_i\rangle}{\langle v_i, v_i\rangle}t, \frac{\langle Bv_i, v_i\rangle}{\langle v_i, v_i\rangle}t\right)\frac{1-t}{t}\,\mathrm{d}t.$$

where $\{v_1, v_2, \ldots, v_k\}$ are eigenvectors of $A^{-1}B$ corresponding to the real eigenvalues.

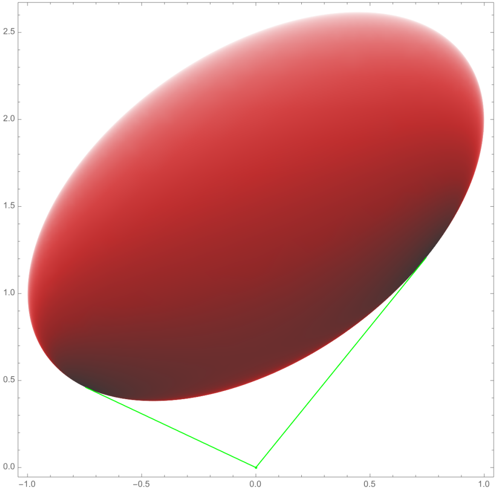


Figure: $(A, B) = \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \right).$

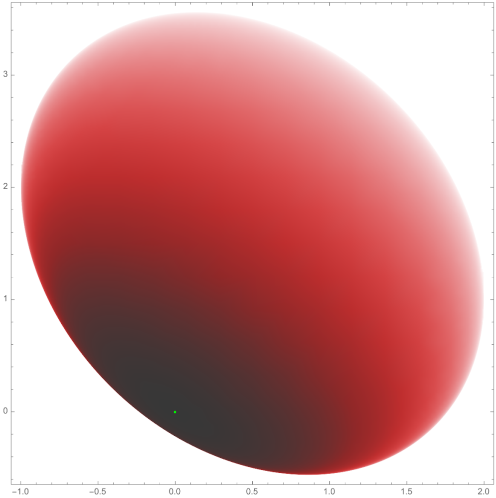


Figure: $(A, B) = \left(\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ -2 & 2 \end{bmatrix} \right).$

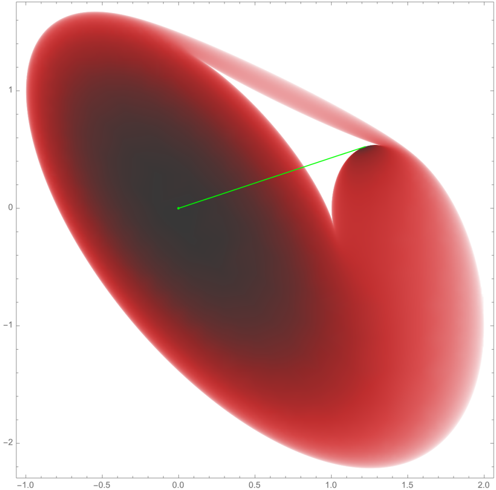


Figure: $(A, B) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \right).$

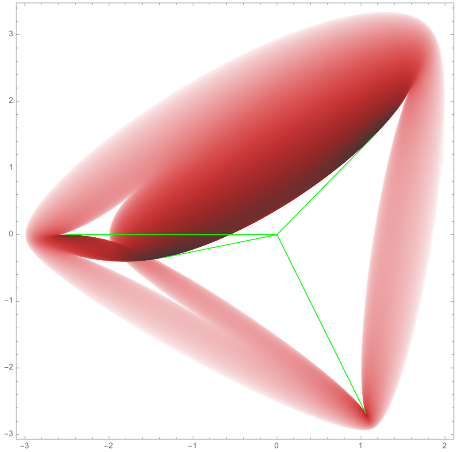


Figure: $(A, B) = \left(\begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -2 & -2 \\ -1 & -1 & -2 & 2 \end{bmatrix} \right).$

Basic properties of tracial joint spectral measures

1. The continuous part μ_c is supported on (a subset of) the joint numerical range

$$\mathcal{W}(A,B) = \{(\langle Av, v\rangle, \langle Bv, v\rangle) \mid v \in S^{n-1}\} \subset \mathbb{R}^2.$$

2. Singular part is supported on tangents from the origin to the boundary curve of the continuous part, *Kippenhahn curve*.

3.

$$\text{If} \qquad A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix},$$

$$\text{then} \qquad \mu_{A,B} = \mu_{A_1,B_1} + \mu_{A_2,B_2}.$$

4. The continuous part μ_c vanishes iff A and B commute.

Theorem (H, 2023)

For $p > 0$ and $A, B \in M_n(\mathbb{C})$, there exists functions $f, g \in L_p$ such that for any $x, y \in \mathbb{R}$,

$$\|xA + yB\|_{S_p} = \|xf + yg\|_{L_p}.$$

Proof.

Tracial joint spectral measure of A and B applied to the function $t \mapsto |t|^p$ implies that for $x, y \in \mathbb{R}$,

$$\frac{\|xA + yB\|_{S_p}^p}{p(p+1)} = \frac{\operatorname{tr}|xA + yB|^p}{p(p+1)} = \int_{\mathbb{R}^2} |ax + by|^p \, d\mu_{A,B}(a, b).$$

This means that we should choose $f, g \in L_p(\mu_{A,B})$ with $f = (a, b) \mapsto a$ and $g = (a, b) \mapsto b$. $\square$

Theorem (H, 2023)

If $f : \mathbb{R} \to \mathbb{R}$ has non-negative k:th derivative, then for any Hermitian $A, B \in M_n(\mathbb{C})$ with $A \geq 0$, so does

$$t \mapsto \operatorname{tr} f(tA + B).$$

Proof.

Apply tracial joint spectral measure to $f(t) = t_+^{k-1}$. □

Applying this result to $f(t) = \exp(t)$ recovers a result of Stahl (formerly the BMV conjecture).

Theorem (Stahl, 2011)

Function $t \mapsto \operatorname{tr} \exp(B - tA)$ is a Laplace transform of a positive measure for Hermitian $A, B \in M_n(\mathbb{C})$ with $A \geq 0$.

Any non-negative bivariate polynomial p with $p(0,0) = 0$ gives rise to an (often non-trivial) inequality.

Example

If $p(a,b) = (a^2 + b^2 - a)^2$,

$$0 \leq 6 \int p(a,b)\, \mathrm{d}\mu_{A,B}(a,b) = \operatorname{tr}(A^2) - \operatorname{tr}(A^3) - \operatorname{tr}(AB^2) + \frac{3\operatorname{tr}(A^4) + 4\operatorname{tr}(A^2B^2) + 2\operatorname{tr}(ABAB) + 3\operatorname{tr}(B^4)}{10}.$$

Tracial joint spectral measures don't generalize to triplets of matrices.

Theorem (H, 2022)

If $0 < p < \infty$, $p \neq 2$, the 3-dimensional space of 2×2 real symmetric matrices is ***not*** *isometric to a subspace of $L_p([0,1])$.*

Proposition (Complex case fails)

There are **no** functions $f, g \in L_8([0,1], \mathbb{C})$ such that for any $z, w \in \mathbb{C}$ one has

$$\left\| z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + w \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\|_{S_8} = \|zf + wg\|_{L_8}.$$

Generalizations?

The proof only works for matrices, and while a compactness argument can deal with compact operators on a Hilbert space:

Question

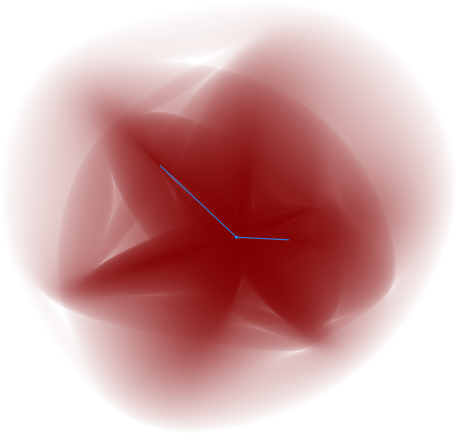
Do tracial joint spectral measures exist for tracial von Neumann algebras $(\mathcal{M}, \tau)$? That is, if $A, B \in (\mathcal{M}, \tau)$ are self-adjoint, does there exist a measure $\mu_{A,B}$ on $\mathbb{R}^2$, such that for $f : \mathbb{R} \to \mathbb{R}$ and $x, y \in \mathbb{R}$, one has

$$\tau(H(f)(xA + yB)) = \int_{\mathbb{R}^2} f(ax + by) \, \mathrm{d}\mu_{A,B}(a, b)?$$

Question

Is every 2-dimensional subspace of a non-commutative L_p-space isometric to a subspace of $L_p([0, 1])$?

Thank you!



Interactive demo (that generated the above 10×10 example):

shikhin.in/tjsm/tjsm.html

0. Define $g(x) = \int_0^1 (e^{tx} - 1)(1 - t)/t \, \mathrm{d}t$ and consider the function

$$G : (x, y) \mapsto \operatorname{tr} g(xA + yB).$$

1. (easy) Prove that the tracial joint spectral measure coincides with the (distributional) Fourier transform of G outside 0, i.e.

$$\operatorname{tr} H(f)(xA + yB) = \int_{\mathbb{R}^2} f(ax + by) \hat{G}(a, b) \, \mathrm{d}a \, \mathrm{d}b.$$

2. (not so easy) Prove that $\hat{G}$ satisfies the formula by taking a test function φ and calculating

$$(\hat{G}, \varphi) = (G, \hat{\varphi}) = \int G(x, y) \hat{\varphi}(x, y) \, \mathrm{d}x \, \mathrm{d}y = \dots$$

Recall that $\mu_{A,B}$ has continuous part with density

$$\frac{\mathrm{d}\mu_c}{\mathrm{d}m_2}(a,b) = \frac{1}{2\pi}\sum_{i=1}^{n} \left|\mathrm{Im}\left(\lambda_i\left(C(a,b)\right)\right)\right|,$$

where $C(a,b)$ is some auxiliary matrix; and singular part satisfying

$$\int_{\mathbb{R}^2} \varphi(a,b)\,\mathrm{d}\mu_s(a,b) = \sum_{i=1}^{k}\int_0^1 \varphi\left(\frac{\langle Av_i, v_i\rangle}{\langle v_i, v_i\rangle}t, \frac{\langle Bv_i, v_i\rangle}{\langle v_i, v_i\rangle}t\right)\frac{1-t}{t}\,\mathrm{d}t.$$

Key identities:

1. For $\lambda \in \mathbb{C}$,

$$\lim_{M\to\infty}\int_{|t|<M} \log\left|1+\frac{\lambda}{t}\right|\mathrm{d}t = \pi\left|\mathrm{Im}(\lambda)\right|.$$

2. For Hermitian $A, B \in M_n(\mathbb{C})$, if $\ker(B) = \mathrm{span}(v)$, then

$$\frac{\det(B+tA)}{\det(B+tI)} = \frac{\langle Av, v\rangle}{\langle v, v\rangle} + O(t) \text{ as } t \to 0.$$