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Identification of peer effects through social networks

Yann Bramoullé, Habiba Djebbari, Bernard Fortin *

CIRPÉE, Université Laval, Canada
Department of Economics, Université Laval, Canada

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A B S T R A C T

We provide new results regarding the identification of peer effects. We consider an extended version of the linear-in-means model where interactions are structured through a social network. We assume that correlated unobservables are either absent, or treated as network fixed effects. We provide easy-to-check necessary and sufficient conditions for identification. We show that endogenous and exogenous effects are generally identified under network interaction, although identification may fail for some particular structures. We use data from the Add Health survey to provide an empirical application of our results on the consumption of recreational services (*e.g.*, participation in artistic, sports and social activities) by secondary school students. Monte Carlo simulations calibrated on this application provide an analysis of the effects of some crucial characteristics of a network (*i.e.*, density, intransitivity) on the estimates of peer effects. Our approach generalizes a number of previous results due to Manski [Manski, C., 1993. Identification of endogenous social effects: The reflection problem. Review of Economic Studies 60 (3), 531–542], Moffitt [Moffitt, R., 2001. Policy interventions low-level equilibria, and social interactions. In: Durlauf, Steven, Young, Peyton (Eds.), Social Dynamics. MIT Press] and Lee [Lee, L.F., 2007. Identification and estimation of econometric models with group interactions, contextual factors and fixed effects. Journal of Econometrics 140 (2), 333–374].

1. Introduction

In recent years, studies on peer effects have literally exploded. They have been applied to topics as diverse as criminal activity (*e.g.*, Glaeser et al. (1996)), welfare participation (Bertrand et al., 2000), school achievement (*e.g.*, Sacerdote (2001)), participation in retirement plans (Saez and Duflo, 2003), and obesity (*e.g.*, Trogdon et al. (2008)).

One key challenge for the empirical literature on peer effects is to identify what drives the correlation between outcomes of individuals who interact together (see Blume and Durlauf (2005) and Soetevent (2006) for recent surveys). In a pioneer study, Manski (1993) distinguishes between *exogenous (or contextual) effects*, *i.e.*, the influence of exogenous peer characteristics, *endogenous effects*, *i.e.*, the influence of peer outcomes, and *correlated effects*, *i.e.*, individuals in the same reference group tend to behave similarly because they are alike or face a common environment.

Manski shows that two main identification problems arise in the context of a *linear-in-means* model.[1] First, it is difficult to distinguish real social effects (endogenous + exogenous) from correlated effects.[2] Second, even in the absence of correlated effects, simultaneity in behavior of interacting agents introduces a perfect collinearity between the expected mean outcome of the group and its mean characteristics. This *reflection* problem hinders

* Corresponding address: Department of Economics, Pavillon J.A. De Sève, 1025 Ave des Sciences - Humaines, Laval University, Quebec, Canada G1V 0A6. Tel.: +1 418 656 5678.
E-mail address: bernard.fortin@ecn.ulaval.ca (B. Fortin).

[1] In the linear-in-means model, the outcome of each individual depends linearly on his own characteristics, on the mean outcome of his reference group and on its mean characteristics. Most papers on social interactions have considered the linear-in-means model since it is naturally related to the standard simultaneous linear model (Moffitt, 2001). Notable exceptions are Brock and Durlauf (2001a, 2003) that exploit non-linearities emerging from discrete choice models to identify endogenous from exogenous effects under the assumption of no correlated effects, and Krauth (2006) and Brock and Durlauf (2007) that extent Brock and Durlauf (2001a) to account for correlated effects. Brock and Durlauf (2001b) provide a careful analysis of identification in both linear-in-means and discrete choice models.

[2] A number of studies have notably addressed this problem by exploiting data where individuals are randomly assigned to groups (removing any correlated effects), *e.g.*, Sacerdote (2001) and Zimmerman (2003), by imposing exclusion restrictions on the structural model, *e.g.*, Evans et al. (1992) and Graham and Hahn (2005), or by introducing variance restrictions on the error terms (Graham, 2008). In the latter case, the basic identifying assumption is that the variance matrix parameters are independent of the reference group size.

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the identification of the endogenous effect from the exogenous effects.[3]

One basic assumption that is usually made in the linear-in-means model, as well as in most peer effects models, is that individuals *interact in groups*. This means that the population is partitioned in groups, and that individuals are affected by all others in their group and by none outside of it. This type of interaction pattern is very particular and is not likely to represent most forms of relationship between individuals. Indeed, there is increasing recognition among economists in general of the role played by *social network* in structuring interactions among agents. A social network is a social structure made of nodes (which are generally individuals or organizations) that are tied by one or more specific types of interdependency, such as friendship, values, beliefs, conflict or trade. The resulting graph-based structures are often very complex. The analysis of social networks has been first developed by sociologists and has become a central field of research in sociology (*e.g.*, Wasserman and Faust (1994) and Freeman (2004)). In economics, a growing body of theoretical work explores how individual incentives give rise to networks and, in turn, can be shaped by them (*e.g.*, Jackson (2006)). At the empirical level, a few recent studies, including the present one, exploit datasets possessing rich information on relationships between agents in order to provide cleaner evidence on social effects.[4] It is thus natural to analyze the problem of identification under more general assumptions.

Our approach is inspired from the literature in spatial econometrics (*e.g.*, Case (1991) and Anselin et al. (2004)). We consider an extended version of the linear-in-means model where each individual has his own specific reference group, defined by the individuals whose mean outcome and characteristics influence his own outcome. Interactions are thus structured through a directed social network.[5] We show that relaxing the assumption of group interactions generally permits to separate endogenous and exogenous effects. Therefore, the second negative result of Manski (1993) is not robust to reference group heterogeneity.

Our main objective is to characterize the networks for which endogenous and exogenous effects are identifiable. We determine these structures both in the absence of correlated effects and when controlling for correlated effects in the form of network fixed effects. In both cases, we provide easy-to-check necessary and sufficient conditions for identification. When there are no correlated effects, we show that endogenous and exogenous effects are identified as soon as individuals do not interact in groups.[6] Thus, even the slightest departure from a groupwise structure is sufficient to obtain identification.

When correlated effects are present at the network level, it is natural to take them out through a *within* transformation similar to the one used in linear panel data models. However many transformations can be used for this purpose. We focus on two of them: the *local* transformation which expresses the model in deviation from the mean equation of the individual's neighbors and the *global* transformation which expresses it in deviation from the individual's network. We show that the global transformation is the one which imposes less restrictive conditions to obtain identification. Whatever the transformation used, degrees of freedom are lost, and identification now fails on some networks, such as the star. We still find that endogenous and exogenous effects can be distinguished on most networks.

Our analysis admits as special cases several models studied in the literature, among which Manski (1993), Moffitt (2001) and Lee (2007). These authors analyze different versions of the standard model with group interactions. For our purposes, Manski's model has the same properties as one where the individual is included when computing the mean of his group.[7] In this case, peer effects are not identified. In Moffitt's model, the individual is excluded from the mean and all groups have the same size. Peer effects are also not identified. In contrast, Lee's model considers interactions in groups with different sizes, and the individual is also excluded from the mean. He finds that variations in group sizes can yield identification. We show that these three results directly follow from our general conditions.

Our paper advances the methodology of the empirical estimation of peer effects. We provide a theoretical foundation behind a few recent attempts at identifying and estimating social effects (Laschever, 2005; De Georgi et al., 2007; Lin, 2007). Laschever (2005) applies a model of social interactions with multiple reference groups to the likelihood of post-war employment of World War I veterans.[8] De Georgi et al. (2007) also make use of multiple groups to identify peer effects in the educational choices of college students. As shown in this paper, a multiple reference group structure is only one of many structures of interaction for which social effects can be identified. Lin (2007) uses detailed data on friendship links to estimate peers' influence on students' outcomes. She can obtain separate estimates for endogenous and exogenous effects only because the friendship networks in her dataset satisfy our general identification conditions.

While our theoretical setting is perfectly general, we use a running empirical application to illustrate the basic concepts and motivate our results. Consider the consumption by a secondary school student of recreational activities such as participation in artistic, sports and social organizations and clubs. His recreational activities are assumed to depend not only on his own characteristics (*e.g.*, age, gender, family background), but also on his friends' mean characteristics (exogenous social effect) and their mean recreational activities (endogenous social effect). The latter effect may reflect conformity or simply the pleasure for the student to participate in recreational activities with friends who also participate in such activities. Moreover, the more (and the better) artistic, sports and social clubs are available in a school, the more likely students from this school will consume recreational activities (correlated effects).

While theoretically identified, a social interaction model can suffer from weak identification in practice. We investigate this

3 Empirical studies have addressed this issue in many ways: for instance by assuming that only one type of social effect exists (endogenous or exogenous), *e.g.*,Gaviria and Raphael (2001) and Trogdon et al. (2008), or by assuming that there exists an individual characteristic whose group-level analog does not play the role of a contextual variable, *e.g.*,Ioannides and Zabel (2003). These assumptions are often *ad hoc* and in any case cannot be tested when the model is exactly identified.

4 Dercon and De Weerdt (2006) study the network of risk-sharing relationships between households in a Tanzanian village. Conley and Udry (2005) look at how communication networks among farmers affect the adoption of a new technology. Goyal et al. (2006) analyze the network of coauthorships among economists. Calvó-Armengol et al. (2005) and Lin (2007) study the Add Health dataset and friendship networks among adolescents.

5 In a directed social network, the direction of influence from one node to another is taken into account. Any direction is disregarded in an undirected social network.

6 We also show that they may be identified under group interactions. See our discussion on Lee (2007) below.

7 More precisely, Manski develops a *linear-in-expectations* model, where the individual's outcome depends on the outcome expectation of his group and social equilibrium is assumed.

8 While Laschever's model is somewhat different from ours, the reason for identification is similar. Especially, an individual may belong to more than one reference group. In contrast, Cohen-Cole (2006) studies a context where groups are mutually exclusive. He allows for both within and between-group endogenous effects. He shows that identification usually holds in the presence of between-group exogenous effects. In future research, it could be interesting to generalize our approach to a setting where different social networks can simultaneously affect an individual's behavior.

issue using both real data and Monte Carlo simulations. Using Add Health dataset,[9] we provide estimation results on recreational activities by secondary school students when it is assumed that each individual's reference group is given by his best friends at school, as self-reported in the questionnaire. Our results show that the mean recreational activities by his friends have a positive and significant influence on a student's recreational activities. Moreover, some friends' characteristics such as their mean parents' participation in the labor market positively (and significantly) affect a student's recreational activities. Using the estimated parameters, we calibrate the model in order to perform Monte Carlo simulations. We analyze the effects of important characteristics of a network, such as its density and its level of intransitivity (defined below), on the quality of peer effects estimates.

The rest of the paper is organized as follows. Section 2 presents our basic linear-in-means model. It also provides a simple example inspired from panel data literature to illustrate that the presence of intransitivity in the network may help to identify the peer effects. Our general identification results are then presented and discussed. Section 3 addresses correlated effects in the form of network fixed effects. Section 4 presents our estimation results on recreational activities and Section 5 our Monte Carlo simulations. A brief discussion concludes.

2. Social effects and social networks

2.1. The basic model

Our presentation closely follows our empirical application on consumption of recreational services by secondary school students. Our model is an extension of the standard linear-in-means social interaction model in which we allow for student-specific friends' groups. Vectors are denoted with bold lower case letters and matrices with bold capital letters. Suppose we have a set of students i, $(i = 1, \dots, n)$. Let y_i be the level of recreational activities by student i. Let $\mathbf{x}_i$ be a $1 \times K$ vector of characteristics of i. For simplicity, we present the model with a unique characteristic $(K = 1)$, that is, family background (*e.g.*, parents' income). Results hold with any number of them (see the appendices for more details). Our main new assumption is as follows. Each student i may have a specific peers' group P_i of size n_i. This reference group (known by the modeler) contains all students whose recreational activities or family background may affect i's recreational activities. Except where otherwise specified, we assume that student i is excluded from his reference group, that is, $i \notin P_i$. This corresponds to the usual empirical formulation (*e.g.*, Sacerdote (2001), and Soetevent and Kooreman (2007)). A student is *isolated* if his friends' group is empty.[10] We assume that not all students are isolated. The collection of student-specific friends' groups defines a directed network between students.

Our results are consistent with two types of observations. First, they hold if we observe an i.i.d. sample of size L from a population of networks with a fixed and known structure.[11] Alternatively, they hold if we observe an i.i.d. sample from a population of networks with a stochastic but strictly exogenous structure. For notational simplicity, our results are presented for a fixed network; they can be easily adapted to the latter case (see discussion below). Also, to focus on the population model, we omit for the moment the network observation index l, $(l = 1, \dots, L)$.

We do not change any other assumption of the standard model (see Moffitt (2001)). Especially, we assume that a student's recreational activities may be affected by the mean recreational activities of his friends' group, by his family background, and by the mean family background of his friends' group. Formally, the structural model is given by:

$$y_i = \alpha + \beta \frac{\sum_{j \in P_i} y_j}{n_i} + \gamma x_i + \delta \frac{\sum_{j \in P_i} x_j}{n_i} + \epsilon_i, \qquad \mathbb{E}[\epsilon_i \mid \mathbf{x}] = 0,$$

where β captures the endogenous effect and δ the exogenous effect. It is standard to require that $|\beta| < 1$. Except for this restriction, our model does not impose any other constraints on the parameters. The error term ϵ_i reflects unobservable (to the modeler) characteristics associated with i. In this section, we assume the strict exogeneity of the regressors, that is, $\mathbb{E}[\epsilon_i \mid \mathbf{x}] = 0$, where $\mathbf{x}$ is an $n \times 1$ vector of family background. Thus we assume no correlated effects. This assumption is relaxed in Section 3. We make no further assumption on the error terms within a network. Especially, they are not necessarily i.i.d. or normally distributed.[12] Therefore our model is semiparametric, or "distribution-free".

We finally write the structural model in matrix notation. Our model is of the form

$$\mathbf{y} = \alpha \boldsymbol{\iota} + \beta \mathbf{G}\mathbf{y} + \gamma \mathbf{x} + \delta \mathbf{G}\mathbf{x} + \boldsymbol{\epsilon}, \qquad \mathbb{E}[\epsilon \mid \mathbf{x}] = 0, \tag{1}$$

where $\mathbf{y}$ is an $n \times 1$ vector of recreational activities for the l network, $\mathbf{G}$ is an $n \times n$ *interaction matrix* with $G_{ij} = 1/n_i$ if j is a friend of i, and 0 otherwise,[13] and $\boldsymbol{\iota}$ is an $n \times 1$ vector of ones. The variance matrix of the error terms is assumed to be unrestricted and therefore contains no identifying information. In particular, we do not impose homoskedasticity. We assume throughout that the expected outer product of $(\boldsymbol{\iota}, \mathbf{x})$ has full rank. Note also that the systematic part of (1) is similar to that of a spatial autoregressive (SAR) model (*e.g.*, Cliff and Ord (1981)) extended to allow for exogenous effects.[14] It is also an extension of Lee's (2007) model since, in its general version, it does not impose that students interact in groups.

The main result of the paper is to show that $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \delta)$ is identified given the moment restriction $\mathbb{E}[\boldsymbol{\epsilon} \mid \mathbf{x}] = 0$ *and* restrictions on $\mathbf{G}$. We relate in particular Manski's (1993) and Moffitt's (2001) non-identification and Lee's (2007) identification results to their maintained assumptions on the form of $\mathbf{G}$.

[9] The National Longitudinal Study of Adolescent Health (Add Health) is a school-based panel study of a nationally-representative sample of adolescents in grades 7–12 in the United States in 1994–95. A full description of the sample design, data, and documentation is available at: http://www.cpc.unc.edu/projects/addhealth.

[10] While an isolated individual is not affected by others, he may still affect others.

[11] This may include the case where one observes the same network of students at different points in time as long as the sample of observations on networks can be considered as i.i.d.

[12] This structural model can be derived from a choice-theoretic approach where each individual i chooses his outcome in order to maximize a quadratic utility function depending on his outcome and on his reference group's mean expected outcome and mean characteristics. This approach also assumes that social interactions have reached a noncooperative (Nash) equilibrium at which expected outcomes are realized.

[13] More generally, G_{ij} could capture the strength of the interaction between i and j, and decrease with social, or geographic, distance. Some of our results hold for arbitrary $\mathbf{G}$, others for matrices that are row-normalized.

[14] A standard spatial autoregressive (SAR) model consists of a spatially lagged version of the dependent variable $\mathbf{y}$ with regressors $\mathbf{X}$: $\mathbf{y} = \beta \mathbf{G}\mathbf{y} + \boldsymbol{X}\boldsymbol{\gamma} + \boldsymbol{\epsilon}$, where $\mathbf{G}$ is a predetermined spatial weighting matrix and where β and $\boldsymbol{\gamma}$ are parameters to be estimated. The error $\boldsymbol{\epsilon}$ may also have a SAR structure. For instance, such a model could be used to estimate the unemployment rate in State i as a weighted linear function of the unemployment rates in States geographically close to i and additional explanatory variables.

2.2. A simple example

The main intuition for our results can be conveyed through a simple example.[15] In addition, this example illustrates the formal similarity with panel data econometrics. Consider a network of students in infinite number arrayed on a line with each student being influenced only by his left-hand friend in his choice of recreational activities. The matrix **G** takes the following specific form:

$$G_{ij} = \begin{cases} 1, & j = i-1 \\ 0, & j \neq i-1 \end{cases}$$

which leads to the following structural model:

$$\begin{aligned} &y_i = \alpha + \beta y_{i-1} + \gamma x_i + \delta x_{i-1} + \epsilon_i, \\ &\mathbb{E}[\epsilon_i \mid x_{-\infty}, \ldots, x_{\infty}] = 0. \end{aligned} \tag{2}$$

Therefore, using the panel data model terminology, lags of x_i may be used as instruments for y_{i-1} even if ϵ_i is serially correlated of unspecified form in (2). This captures the intuition that the characteristics of the friends' friends of a student who are not his friends may serve as instruments for the actions of his own friends. Note that it is formally similar to the observation first made by Chamberlain (1984) that strictly exogenous regressors can be used to distinguish true state dependence (*i.e.*, the value of β) from correlated heterogeneity in the linear panel data model.

This example illustrates the case of a network in which we can find *intransitive triads*. These are sets of three students i, j, k such that i is affected by j and j is affected by k (that is, a triad), but i is not affected by k. Here, i, j, k forms an intransitive triad for any i when $j = i - 1$ and $k = i - 2$, since i is not directly affected by $i - 2$. We will show below that the presence of intransitive triads is a sufficient (but not necessary) condition for the identification of social effects in the absence of correlated effects.

Now consider the case where there may be network fixed effects potentially correlated with the family background of students. Indexing networks by l, assuming that the x_{li}'s are strictly exogenous *conditional on* α_l, and maintaining our other assumptions, we have

$$\begin{aligned} &y_{li} = \alpha_l + \beta y_{li-1} + \gamma x_{li} + \delta x_{li-1} + \epsilon_{li}, \\ &\mathbb{E}[\epsilon_{li} \mid x_{-l\infty}, \ldots, x_{l\infty}, \alpha_l] = 0. \end{aligned} \tag{3}$$

Define $\Delta z_{li} = z_{li} - z_{li-1}$, for $z = y, x, \epsilon$. Then differencing (3) gives

$$\Delta y_{li} = \beta \Delta y_{li-1} + \gamma \Delta x_{li} + \delta \Delta x_{li-1} + \Delta\epsilon_{li}.$$

Hence, one has

$$\mathbb{E}[\Delta\epsilon_{li} \mid \Delta x_{l-\infty}, \ldots, \Delta x_{l\infty}] = 0. \tag{4}$$

Therefore, lags $(\Delta x_{li-2}, \ldots, \Delta x_{l-\infty})$ can be used as valid identifying instruments. From (4), and for all j and i, moment restrictions of the form

$$\mathbb{E}[\Delta x_{lj}(\Delta y_{li} - \beta \Delta y_{li-1} - \gamma \Delta x_{li} - \delta \Delta x_{li-1})] = 0$$

are valid. The model thus generates internal conditions that ensure identification of social effects in spite of serial correlation of unspecified form and the endogeneity of lagged y. This is essentially Chamberlain's (1984) result for dynamic panel data models with regressors strictly exogenous conditional on the fixed effect. In the remaining part of the paper, we extend this example to arbitrary networks.

[15] We thank a referee for having suggested this example to us.

2.3. Reduced form and identification

We now write the restricted reduced form of model (1). Since $|\beta| < 1$, $\mathbf{I} - \beta\mathbf{G}$ is invertible.[16] We can write:

$$\mathbf{y} = \alpha(\mathbf{I} - \beta\mathbf{G})^{-1}\iota + (\mathbf{I} - \beta\mathbf{G})^{-1}(\gamma\mathbf{I} + \delta\mathbf{G})\mathbf{x} + (\mathbf{I} - \beta\mathbf{G})^{-1}\boldsymbol{\epsilon}, \tag{5}$$

where the intercept is simply $\alpha/(1 - \beta)$ if the student is not isolated, and α otherwise.

We say that *social effects are identified* if and only if the vector $\boldsymbol{\theta}$ of structural parameters can be uniquely recovered from the unrestricted reduced-form parameters in (5) (injective relationship). Our identification results are asymptotic in nature (see Manski (1995)). They characterize when social effects can, or cannot, be disentangled if we are not limited in the number of observations we can obtain.

It will be useful in the following to use a series expansion of (5). Since $(\mathbf{I} - \beta\mathbf{G})^{-1} = \sum_{k=0}^{\infty} \beta^k \mathbf{G}^k$ and assuming no isolated students, one has:

$$\mathbf{y} = \alpha/(1-\beta)\boldsymbol{\iota} + \gamma\mathbf{x} + (\gamma\beta + \delta)\sum_{k=0}^{\infty} \beta^k \mathbf{G}^{k+1}\mathbf{x} + \sum_{k=0}^{\infty} \beta^k \mathbf{G}^k \boldsymbol{\epsilon}. \tag{6}$$

Moreover, from (6), the expected mean friends' groups' recreational activities conditional on $\mathbf{x}$ can be written as:

$$\mathbb{E}(\mathbf{Gy} \mid \mathbf{x}) = \alpha/(1-\beta)\boldsymbol{\iota} + \gamma\mathbf{Gx} + (\gamma\beta + \delta)\sum_{k=0}^{\infty} \beta^k \mathbf{G}^{k+2}\mathbf{x}. \tag{7}$$

The remainder of the paper clarifies the conditions under which identification holds under network interaction.

2.4. Results

Our first result shows that identification is related to a simple property of the matrix **G**.

Proposition 1. *Suppose that $\gamma\beta + \delta \neq 0$. If the matrices* **I**, **G**, *and* $\mathbf{G}^2$ *are linearly independent social effects are identified. If the matrices* **I**, **G**, *and* $\mathbf{G}^2$ *are linearly dependent and no individual is isolated, social effects are not identified.*

It is worth noting that the first part of this result holds even if Eq. (1) is written with an arbitrary matrix **G**, while the second part holds as soon as **G** is row-normalized.[17]

The condition $\gamma\beta + \delta \neq 0$ is natural in this setting. As shown in (6), it means that family background of friends has *some* (direct and/or indirect) effect on a student's expected recreational activities. When it is violated, endogenous and exogenous effects are zero or exactly cancel out, and social effects are absent from the reduced form. The condition is satisfied as soon as γ and δ have the same sign, $\beta > 0$ and $\gamma \neq 0$. With several characteristics, it must be satisfied for at least one of them.

Proposition 1 can be given a natural interpretation in terms of instrumental variables. We show (see Appendix B) that when no student is isolated, the matrices **I**, **G**, and $\mathbf{G}^2$ are linearly dependent if and only if $\mathbb{E}(\mathbf{Gy}|\mathbf{x})$ is perfectly collinear with $(\boldsymbol{\iota}, \mathbf{x}, \mathbf{Gx})$. This perfect collinearity means that we cannot find a valid identifying instrument for **Gy** in the structural equation (1). In contrast, when $\mathbb{E}(\mathbf{Gy}|\mathbf{x})$ is not perfectly collinear with the regressors, the restrictions imposed by the network structure allow the model to

[16] See Case (1991), footnote 5.

[17] One easy way to check whether these three matrices are linearly independent is the following. First, vectorize each matrix, that is, stack its columns on top of each other. Second, verify whether the matrix formed by concatening these stacked vectors has rank three.

be identified. From (7), it is clear that the variables ($\mathbf{G}^2\mathbf{x}, \mathbf{G}^3\mathbf{x}, \ldots$) can be used as identifying instruments and therefore can be used to consistently estimate the parameters.[18] In our application, these instruments have a socioeconomic interpretation. For instance, $\mathbf{G}^2\mathbf{x}$ represents an $n \times 1$ vector of weighted averages of family background of the friends' friends of each student in the network.

Here an important remark is in order. Up to now, we have assumed that we observe an i.i.d. sample of ($\mathbf{y}_l, \mathbf{x}_l$) of size L ($l = 1, \ldots, L$) from a population of networks with a fixed and known structure (matrix $\mathbf{G}$ non-stochastic). Alternatively suppose that we observe an i.i.d. sample of ($\mathbf{y}_l, \mathbf{x}_l, \mathbf{G}_l$) of size L, where the matrices $\mathbf{G}_l$ are now stochastic but strictly exogenous, which means that $\mathbb{E}[\boldsymbol{\epsilon}_l \mid \mathbf{x}_l, \mathbf{G}_l] = \mathbf{0}$. Form a large network $\mathbf{G}$ by combining the different matrices: add any network $\mathbf{G}_l$ in support of the network's distribution as a diagonal block of the large network. Then, we can directly apply Proposition 1 to the network $\mathbf{G}$. This is the case, since for identification purposes, the size of the sample can be as large as needed. Propositions 4 and 6 below can be generalized in a similar way.

2.4.1. Group interactions

In this section, we apply Proposition 1 to analyze identification when students interact in groups. We focus particularly on models developed by Manski (1993), Moffitt (2001) and Lee (2007).

2.4.1.1. Non-identification in Manski's (1993) model. Let us first show how Proposition 1 covers Manski's first negative result discussed in the introduction. Suppose that students interact in groups (*e.g.,* their classroom), and also that the student is included when computing the mean. That is, there is a partition of the population in subsets $G_1, \ldots, G_m$ such that for any $i \in G_l$, $P_i = G_l$. This means that students are affected by all others in their classroom and by none outside of it. In this case, $\mathbf{G}$ is block diagonal with $\mathbf{G}^2 = \mathbf{G}$. The second part of Proposition 1 applies. From (7), the expected mean recreational activities of the friends' groups, $\mathbb{E}(\mathbf{Gy}|\mathbf{x})$, is given by $\alpha/(1-\beta)\boldsymbol{\iota} + (\gamma+\delta)/(1-\beta)\mathbf{Gx}$ and is therefore perfectly collinear with the mean family background of the group. Therefore, the list of valid instruments is limited to ($\boldsymbol{\iota}, \mathbf{x}, \mathbf{Gx}$). In this model, no matter the group sizes, social effects are not identified. The interpretation of this result is simple. As long as students interact in groups (including themselves), the mean family background of their friends is also equal to the mean family background of their friends' friends ($\mathbf{G}^2\mathbf{x} = \mathbf{Gx}$). In that case there is no identifying instrument for the mean recreational activities of students' friends.

2.4.1.2. Non-identification in Moffitt's (2001) model. Alternatively, suppose, as in Moffitt (2001), that students interact in groups, that groups have the same size s, and that the student is excluded when computing the mean. Denote $\mathbf{\Gamma}_s$ as the interaction matrix within a group. One has $\Gamma_{s,ij} = 1/(s-1)$ if $i \neq j$ and 0 otherwise. Again, the matrix $\mathbf{G}$ is block diagonal but this time with diagonal blocks given by $\mathbf{\Gamma}_s$. It is easy to see that $\mathbf{G}^2 = \frac{1}{s-1}\mathbf{I} + \frac{s-2}{s-1}\mathbf{G}$ if $s \geq 2$, and the second part of Proposition 1 again applies. Social effects are not identified.

2.4.1.3. Identification in Lee's (2007) model. Assume now that students interact in groups of different sizes. Assume also, with no loss of generality, that there are two groups and that these groups have sizes s_1 and s_2 with $s_1, s_2 \geq 2$. The interaction matrix $\mathbf{G}$ can be written as follows

$$\mathbf{G} = \begin{pmatrix} \mathbf{\Gamma}_{s_1} & 0 \\ 0 & \mathbf{\Gamma}_{s_2} \end{pmatrix}$$

Suppose that $\mathbf{G}^2 = \lambda_0\mathbf{I} + \lambda_1\mathbf{G}$.[19] The diagonal elements give $\lambda_0 = 1/(s_1-1) = 1/(s_2-1)$, hence $s_1 = s_2$. Therefore, if $s_1 \neq s_2$, the matrices $\mathbf{I}$, $\mathbf{G}$, and $\mathbf{G}^2$ are linearly independent, the first part of Proposition 1 applies. Moreover, using Eq. (7), one can easily show that $\mathbb{E}(\mathbf{Gy}|\mathbf{x}) = \alpha/(1-\beta)\boldsymbol{\iota} + b_0\mathbf{x} + b_1\mathbf{Gx} + b_2\mathbf{G}^2\mathbf{x}$, with $b_2 \neq 0$ when $\beta \neq 0$ and $\gamma\beta + \delta \neq 0$.[20] Therefore, the variable $\mathbf{G}^2\mathbf{x}$ can be used as a valid instrument for $\mathbf{Gy}$. In this model, social effects are identified. This result is related to Lee's (2007) model in the absence of group fixed effects.

Proposition 2. *Suppose that individuals interact in groups. If all groups have the same size, social effects are not identified. If (at least) two groups have different sizes, and if $\gamma\beta + \delta \neq 0$, social effects are identified.*

Identification arises thanks to the effects of the classroom size on reduced-form coefficients within each classroom. After some manipulations, Eq. (5) can be rewritten as follows:

$$y_i = \frac{\alpha}{1-\beta} + \left[\gamma + \frac{\beta(\gamma\beta+\delta)}{(1-\beta)(s_g - 1 + \beta)}\right] x_i + \frac{\gamma\beta+\delta}{(1-\beta)\left(1+\frac{\beta}{s_g-1}\right)} \bar{x}_i + \nu_i, \tag{8}$$

where s_g is the size of i's classroom, and $\bar{x}_i$ is the mean family background over all other students in the classroom and ν_i is the error term. Variation of reduced-form coefficients across classrooms of different size allows us to identify the structural model. The impact of s_g on these coefficients has an intuitive interpretation. The size of classrooms (s_g, for $g = 1, 2, \ldots, M$) varies within the network. Consider first the reduced-form coefficient on x_i in (8). It is the sum of a direct and an indirect effect. The direct effect is simply equal to γ, and captures the effect of i's family background on i's recreational activities. This effect is already present in the structural model. The indirect effect arises through feedback effects: when j and k belong to i's group, x_i affects j's recreational activities ($=y_j$), which in turn, affects i's recreational activities ($=y_i$).[21] The indirect effect decreases with s_g, and become negligible as s_g tends to infinity. This reflects the diminishing role that i plays, by himself, in determining other students' recreational activities when the size of the classroom grows. For a similar reason, the reduced-form coefficient on the mean family background of i's friends, $\bar{x}_i$, is *increasing* in the size of the classroom. As the role played by one student decreases, the mean family background of all the others become more important.

Thus, variations in group sizes create exogenous variations in the reduced-form coefficients across groups that lead to identification. Interestingly, Davezies et al. (2006) have shown that Lee's model is generically identified, even when all members in the classroom are not observed. Observe finally that, in this model, social effects may be identified when the individual is excluded in the computation of the group mean, but not when he is included.

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[18] One can show that the potential number of instruments cannot exceed a critical level smaller than or equal to the number of individuals in the network. Overidentification tests such as the one suggested by Lee (2003) for a SAR model could be implemented.

[19] Observe that, since at least one individual is not isolated, the matrices $\mathbf{I}$ and $\mathbf{G}$ are linearly independent.

[20] After few manipulations, one obtains: $b_2 = [(\gamma\beta+\delta)\beta]/(1-\lambda_2\beta-\lambda_1\beta^2-\lambda_0\beta^3)$, where $\lambda_0 = \frac{1}{(s_1-1)(s_2-1)}$, $\lambda_1 = \frac{s_1+s_2-3}{(s_1-1)(s_2-1)}$, and $\lambda_2 = \frac{s_1s_2-2(s_1+s_2)+3}{(s_1-1)(s_2-1)}$.

[21] This indirect effect itself has different channels: x_i affects y_j both directly through exogenous effects (δ), and indirectly through endogenous effects (β) via its effects on y_i (γ) and y_k (δ).

2.4.2. Network interactions

Suppose now that students interact through a network. In addition, suppose that we can find *an intransitive triad* in the network. Recall that this is a set of three individuals i, j, k such that i is affected by j and j is affected by k, but i is not affected by k. In this case, $G_{ik} = 0$ while $G^2_{ik} \geq G_{ij}G_{jk} > 0$. In contrast, $\mathbf{G}^2 = \lambda_0\mathbf{I} + \lambda_1\mathbf{G}$ implies that $G^2_{ik} = 0$. Therefore, the presence of an intransitive triad guarantees that $\mathbf{I}$, $\mathbf{G}$, and $\mathbf{G}^2$ are linearly independent. This means that $\mathbf{G}^2\mathbf{x}$ is an identifying instrument for $\mathbf{Gy}$, since x_k affects y_i but only indirectly, through its effect on y_j. This result applies as long as there are students whose friends' friends are not all their friends. This is a generalization of the first part of the example discussed in Section 2.2.

Most networks have intransitive triads. Some networks do not, however. They are called transitive, and are characterized by specific properties (*e.g.*, Bang-Jensen and Cutin (2000)). In transitive networks, the student's friends' friends are always his friends. We show in Appendix A how to apply Proposition 1 to analyze these networks. Interestingly, we find that identification generally holds on transitive networks. It relies then on the directed nature of the links.[22] In these networks, we can always find a group of students who are only friends to each other. Students in the group are not friends to students outside the group. Friendship need not be symmetric, however. We can also find students outside the group who are friends to students inside the group. This difference in social positions impacts the reduced-form equations. Especially, mean family background of students in the group affects students in and outside the group differently. This difference in magnitude can be exploited to identify social effects.

Formally, we show that when the network is transitive, $\mathbf{I}$, $\mathbf{G}$, and $\mathbf{G}^2$ are linearly independent if and only if $\mathbf{G}^2 \neq 0$.[23] In this case, the variable $\mathbf{G}^2\mathbf{x}$ can be used as an identifying instrument for $\mathbf{Gy}$. In the end, we obtain the following result.

Proposition 3. *Suppose that individuals do not interact in groups. Suppose that $\gamma\beta + \delta \neq 0$. If $\mathbf{G}^2 \neq 0$, social effects are identified. If $\mathbf{G}^2 = 0$, social effects are identified when $\alpha \neq 0$, but not when $\alpha = 0$.*

Overall, the results presented in this section show that endogenous and exogenous effects can usually be identified as soon as there is some variation in the reference groups. The results are clear-cut, reflecting the theoretical nature of identification conditions. In practice though, we expect identification to be weak if the network is close to being complete. We explore this issue in Section 5.

3. Correlated effects

In this section, we partially address the problem of correlated effects. We introduce unobserved variables common to students who belong to the same network.[24] These variables may be correlated with the family background of students, which introduces an additional identification problem. As in linear panel data models with fixed effects, we solve this problem by using appropriate differencing to eliminate unobserved variables.[25] We then ask whether endogenous and exogenous social effects can be disentangled. We characterize the necessary and sufficient conditions for identification. Not surprisingly, these conditions are more demanding than in the absence of correlated effects. Identification still holds in most networks, but it fails for some specific ones. We also find that the way common unobservables are eliminated matters. We provide the best possible condition for identification in this setting.

3.1. The model

We introduce network-specific unobservables in the previous model. For any network l and for any student i belonging to l,

$$y_{li} = \alpha_l + \beta\frac{\sum_{j\in P_i} y_{lj}}{n_i} + \gamma x_{li} + \delta\frac{\sum_{j\in P_i} x_{lj}}{n_i} + \epsilon_{li}, \qquad (9)$$
$$\mathbb{E}(\epsilon_{li} \mid \mathbf{x}_l, \alpha_l) = 0.$$

Eq. (9) generalizes Eq. (3) in our example discussed in Section 2.2. The network fixed effect α_l captures unobserved (by the modeler) variables that have common effects on the outcome of all students within the network (*e.g.*, same professors, similar preferences for recreational activities). Importantly, $\mathbb{E}(\alpha_l|\mathbf{x}_l)$ is allowed to be any function of $\mathbf{x}_l$ but we assume that $\mathbb{E}(\epsilon_{li} \mid \mathbf{x}_l, \alpha_l) = 0$. Thus, correlated unobservables may be present but we maintain the strict exogeneity of $\mathbf{x}_l$ conditional on α_l. This is a natural extension of the model of Lee (2007) to a network setting. A similar model has been estimated by Lin (2007).

Here an important remark is in order. In the version of our model with stochastic networks, we assume that the matrix $\mathbf{G}_l$ is exogenous conditional on α_l and $\mathbf{x}_l$, *i.e.*, $\mathbb{E}(\boldsymbol{\epsilon}_l \mid \alpha_l, \mathbf{x}_l, \mathbf{G}_l) = 0$. This condition fails to hold, for instance, if some unobserved characteristics affect both the likelihood to form links and the outcome and differs among individuals in the same network. Thus, if popular students are likely to interact with other popular students and to participate in many recreational activities, and if popularity is not observed and varies within a classroom, the network will not be exogenous conditional on α_l and $\mathbf{x}_l$. In this case, our approach will yield inconsistent estimates of social interactions.

Before studying how the reflection problem can be solved in this context, the standard approach is first to eliminate the network-specific unobservables. In analogy with the *within* transformation in panel data models, this can be done by taking appropriate differences between structural equations. However there are many transformations that can eliminate the unobservables. We next study two natural ways to do that. Throughout this section, we assume that no student is isolated. More generally, our results are valid for any row-normalized matrix $\mathbf{G}$.

3.2. Local differences

We first take local differences. We average Eq. (9) over all student i's friends, and subtract it from i's equation. This approach is local since it does not fully exploit the fact that the fixed effect

22 If links are undirected, a network without intransitive triads necessarily has a group structure. This case is covered by Proposition 2.

23 When $\mathbf{G}^2 = 0$, the first part of Proposition 1 does not apply. In these networks, a student either affects others or is affected by them. When $\alpha \neq 0$, the presence of isolated students still yields identification, see Appendix A.

24 In an earlier version of the paper (Bramoullé et al., 2006), we looked at a finer subdivision of the population that may be relevant in some applications. We assumed that unobserved variables are common to individuals in the same component of the network. A component is a maximal set of indirectly related individuals.

25 Additional assumptions on the error terms can help identification. Graham (2008) uses restrictions on the variance matrix to identify social interactions from correlated effects. In his example applied to the Project Star, a key assumption is that the variances of unobserved teacher quality and student characteristics, as well as the (exogenous + endogenous) social effect, are not affected by the classroom size.

is not only the same for all i's friends but also for all students of his network. Written in matrix notations, the structural model, on which a *within local transformation* is applied, becomes:

$$(\mathbf{I}-\mathbf{G})\mathbf{y}_l = \beta(\mathbf{I}-\mathbf{G})\mathbf{G}\mathbf{y}_l + \gamma(\mathbf{I}-\mathbf{G})\mathbf{x}_l + \delta(\mathbf{I}-\mathbf{G})\mathbf{G}\mathbf{x}_l + (\mathbf{I}-\mathbf{G})\boldsymbol{\epsilon}_l. \tag{10}$$

The corresponding reduced form is obtained as in Section 2 (see Eq. (5)).

$$(\mathbf{I}-\mathbf{G})\mathbf{y}_l = (\mathbf{I}-\beta\mathbf{G})^{-1}(\gamma\mathbf{I}+\delta\mathbf{G})(\mathbf{I}-\mathbf{G})\mathbf{x}_l + (\mathbf{I}-\beta\mathbf{G})^{-1}(\mathbf{I}-\mathbf{G})\boldsymbol{\epsilon}_l. \tag{11}$$

Our next result characterizes identification in this setting.

Proposition 4. *Consider model (10). Suppose that $\gamma\beta+\delta \neq 0$. Social effects are identified if and only if the matrices $\mathbf{I}$, $\mathbf{G}$, $\mathbf{G}^2$, and $\mathbf{G}^3$ are linearly independent.*

This condition is more demanding than the condition of Proposition 1. Some information has been lost to take into account the presence of correlated effects. This loss makes identification more difficult.

As in the previous section, Proposition 4 has a natural interpretation in terms of instrumental variables. We show in Appendix B (Result 2) that the matrices $\mathbf{I}$, $\mathbf{G}$, $\mathbf{G}^2$, and $\mathbf{G}^3$ are linearly dependent if and only if the expected value of the endogenous variable on the right-hand side of Eq. (10), $E[(\mathbf{I}-\mathbf{G})\mathbf{G}\mathbf{y}_l|\mathbf{x}_l]$, is perfectly collinear with the regressors $((\mathbf{I}-\mathbf{G})\mathbf{x}_l, (\mathbf{I}-\mathbf{G})\mathbf{G}\mathbf{x}_l)$. When this perfect collinearity holds, the structural model is clearly not identified. Again, from a series expansion of $E[(\mathbf{I}-\mathbf{G})\mathbf{G}\mathbf{y}_l|\mathbf{x}_l]$ similar to the one in Eq. (7), it is clear that when $E[(\mathbf{I}-\mathbf{G})\mathbf{G}\mathbf{y}_l|\mathbf{x}_l]$ is not perfectly collinear with the regressors, the variables $((\mathbf{I}-\mathbf{G})\mathbf{G}^2\mathbf{x}_l, (\mathbf{I}-\mathbf{G})\mathbf{G}^3\mathbf{x}_l, \ldots)$ can be used as valid instruments for $(\mathbf{I}-\mathbf{G})\mathbf{G}\mathbf{y}_l$. This result generalizes the one presented in the second part of Section 2.2 (see Eq. (4)).

We study the implications of Proposition 4 on the pattern of interactions among students. Consider group interactions and assume that each classroom forms a group. Three different classroom sizes are now necessary to obtain identification. With two classrooms of sizes s_1 and s_2, we can see that $\mathbf{I}$, $\mathbf{G}$, $\mathbf{G}^2$, and $\mathbf{G}^3$ are linearly dependent. More precisely,

$$\mathbf{G}^3 = \frac{1}{(s_1-1)(s_2-1)}\mathbf{I} + \frac{s_1+s_2-3}{(s_1-1)(s_2-1)}\mathbf{G} + \frac{s_1s_2-2(s_1+s_2)+3}{(s_1-1)(s_2-1)}\mathbf{G}^2.$$

In contrast, with three classroom sizes (or more), $\mathbf{I}$, $\mathbf{G}$, $\mathbf{G}^2$, and $\mathbf{G}^3$ are linearly independent, and social effects are identified (see Davezies et al. (2006)). This is, of course, confirmed by looking directly at the reduced form. Let s_g be the size of i's group. Eq. (11) becomes

$$y_{li} - \bar{y}_l = \frac{(s_g-1)\gamma-\delta}{(s_g-1)+\beta}(x_{li}-\bar{x}_l) + \frac{s_g-1}{s_g-1+\beta}(\epsilon_{li}-\bar{\epsilon}_l). \tag{12}$$

where means are computed over all students in i's classroom (see Eq. (2.5) in Lee (2007)). Only one composite parameter can now be recovered from the reduced form for each group size. Three sizes are thus needed to identify the three structural parameters.

Next, consider network interaction. Intransitive triads have a natural counterpart. Define the distance between two students i and j in the network as the number of friendship links connecting i and j in the shortest chain of students $i_1, \ldots, i_l$ such that i_1 is a friend of i, i_2 is a friend of $i_1, \ldots,$ and j is a friend of i_l. For instance, this distance is 1 between two students who are friends and 2 between two students who are not friends but who have a common friend (intransitive triad). Define the *diameter* of the network as the maximal friendship distance between any two students in the network (see Wasserman and Faust (1994)). Suppose that the diameter is greater than or equal to 3. Then we can find two students i and j separated by a friendship distance 3 in the network. In this case, $G^3_{ij} > 0$ while $G^2_{ij} = G_{ij} = 0$. Hence, no linear relation of the form $\mathbf{G}^3 = \lambda_0\mathbf{I} + \lambda_1\mathbf{G} + \lambda_2\mathbf{G}^2$ can exist. In our example of Section 2.2, the diameter of the network is infinite. This implies that we can always find two students separated by a friendship distance 3 (and more). Therefore, peer effects are identified even when there are correlated effects. One has the following result:

Corollary 1. *Consider model (10) and suppose that $\gamma\beta+\delta \neq 0$. If the diameter of the network is greater than or equal to 3, social effects are identified.*

This condition is satisfied in most networks. As in Section 2, it can be understood in terms of instrumental variables. From the series expansion of the model, the variable $(\mathbf{I}-\mathbf{G})\mathbf{G}^2\mathbf{x}_l$ is a valid identifying instrument for the right-hand side endogenous vector $(\mathbf{I}-\mathbf{G})\mathbf{G}\mathbf{y}_l$.

Identification fails, however, for a number of nontrivial networks of diameter lower than or equal to 2. This is notably the case for *complete bipartite networks*. In these graphs, the population of students is divided into two groups such that all students in one group are friends with all students in the other group, and there is no friendship links within groups. These include star networks, where one student, at the center, is friend with all other students, who are all friends only with him. It is easy to check that $\mathbf{G}^3 = \mathbf{G}$ for complete bipartite networks. By Proposition 4, social effects are not identified for these networks. To illustrate, consider star networks. Let $i = 1$ denote the center. Reduced-form equation (11) for star networks can be expressed as follows:

$$y_{l1} - \frac{1}{n-1}\sum_{j=2}^{n} y_{lj} = \frac{\gamma-\delta}{1+\beta}\left(x_{l1} - \frac{1}{n-1}\sum_{j=2}^{n} x_{lj}\right) + \nu_{l1}, \quad \text{or}$$

$$y_{li} - y_{l1} = \gamma(x_{lj} - x_{l1}) + \left(\gamma - \frac{\gamma-\delta}{1+\beta}\right) \times \left(x_{l1} - \frac{1}{n-1}\sum_{j=2}^{n} x_{lj}\right) + \nu_{li}, \quad \forall i \geq 2.$$

We can only recover the two composite parameters γ and $\frac{\gamma-\delta}{1+\beta}$ from the estimation of the reduced form. This makes the identification of the three structural parameters impossible.

We could not fully characterize the condition of Proposition 4 in terms of the geometry of the network. To gain some insight on this issue, we determined all the connected undirected networks for which identification fails when $n = 4, 5,$ and 6. They are depicted in Fig. 1. We observe two features. First, the number of networks for which identification fails is relatively low, even within the set of networks with diameter lower than or equal to 2. Second, all these networks exhibit a high degree of symmetry. We suspect that both features hold more generally. Linear dependence between $\mathbf{I}$, $\mathbf{G}$, $\mathbf{G}^2$, and $\mathbf{G}^3$ likely imposes strong restrictions on the network's geometry.

3.3. Global differences

We next take global differences. We average Eq. (9) over all students in i's network, and subtract from i's equation. In contrast with the previous section, the equation being subtracted is now identical for all students in the same network. Introduce the matrix $\mathbf{H}$ as follows: $\mathbf{H} = \frac{1}{n}(\iota\iota')$. Therefore, $\mathbf{I}-\mathbf{H}$ is the matrix which obtains the deviation from network means. We can write the

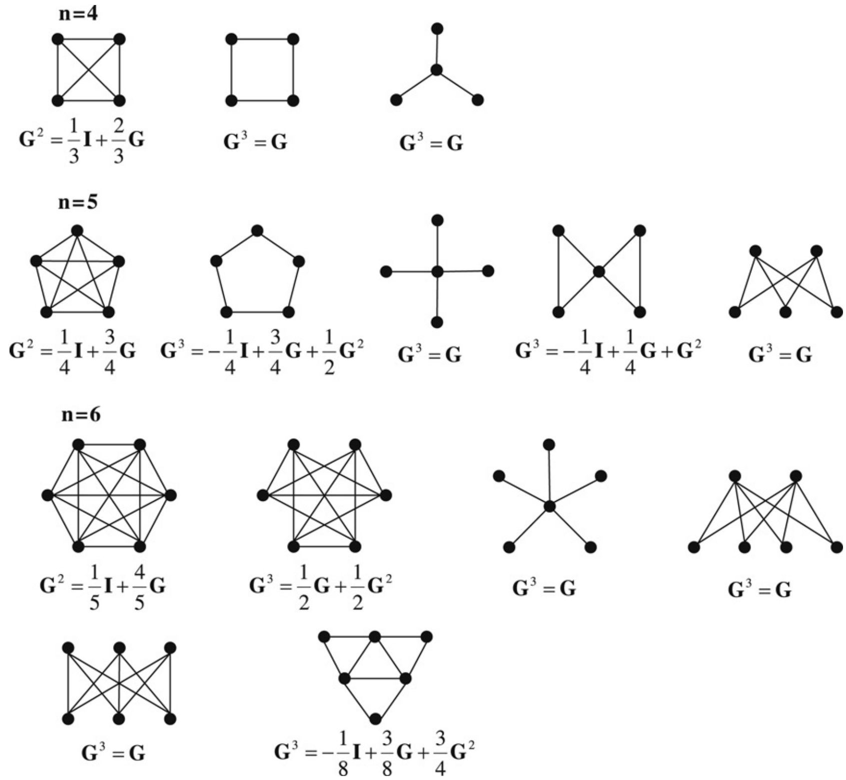

Fig. 1. Undirected graphs for which the matrices $\mathbf{I}$, $\mathbf{G}$, $\mathbf{G}^2$, and $\mathbf{G}^3$ are linearly dependent.

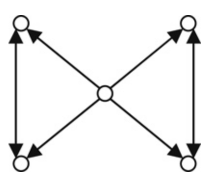

Fig. 2. A graph for which identification holds under global, but not local, differences.

structural model on which a *within global transformation* is applied as follows:

$$(\mathbf{I}-\mathbf{H})\mathbf{y}_l = \beta(\mathbf{I}-\mathbf{H})\mathbf{G}\mathbf{y}_l + \gamma(\mathbf{I}-\mathbf{H})\mathbf{x}_l + \delta(\mathbf{I}-\mathbf{H})\mathbf{G}\mathbf{x}_l + (\mathbf{I}-\mathbf{H})\boldsymbol{\epsilon}_l. \quad (13)$$

In this context, the restricted reduced form becomes:

$$(\mathbf{I}-\mathbf{H})\mathbf{y}_l = (\mathbf{I}-\mathbf{H})(\mathbf{I}-\beta\mathbf{G})^{-1}(\gamma\mathbf{I}+\delta\mathbf{G})\mathbf{x}_l + (\mathbf{I}-\mathbf{H})(\mathbf{I}-\beta\mathbf{G})^{-1}\boldsymbol{\epsilon}_l. \quad (14)$$

We characterize the condition under which this model is identified. Our result involves the rank of the matrix $\mathbf{I}-\mathbf{G}$. Since $(\mathbf{I}-\mathbf{G})\boldsymbol{\iota} = \mathbf{0}$, this rank is always lower than or equal to $n-1$.

Proposition 5. *Consider model (13). Suppose that $\gamma\beta+\delta \neq 0$. If the matrices $\mathbf{I}$, $\mathbf{G}$, $\mathbf{G}^2$, and $\mathbf{G}^3$ are linearly independent, social effects are identified. Next, suppose that $\mathbf{G}^3 = \lambda_0\mathbf{I}+\lambda_1\mathbf{G}+\lambda_2\mathbf{G}^2$. If* rank$(\mathbf{I}-\mathbf{G}) < n-1$ *and $2\lambda_0+\lambda_1+1 \neq 0$, social effects are identified. In contrast, if* rank$(\mathbf{I}-\mathbf{G}) = n-1$, *social effects are not identified.*

From a demonstration similar to the one used in Result 2 of Appendix B, one shows that the model is not identified if and only if $E[(\mathbf{I}-\mathbf{H})\mathbf{G}\mathbf{y}_l|\mathbf{x}_l]$ is perfectly collinear with the regressors $((\mathbf{I}-\mathbf{H})\mathbf{x}_l, (\mathbf{I}-\mathbf{H})\mathbf{G}\mathbf{x}_l)$. Moreover, the variables $((\mathbf{I}-\mathbf{H})\mathbf{G}^2\mathbf{x}_l, (\mathbf{I}-\mathbf{H})\mathbf{G}^3\mathbf{x}_l, \ldots)$ can be used as valid instruments to estimate the model consistently, when the model is identified.

Proposition 5 implies that if social effects are identified when taking local differences, they are also identified when taking global differences. For many networks, the conditions in Propositions 4 and 5 are in fact equivalent. Especially, we can check that rank$(\mathbf{I}-\mathbf{G}) = n-1$ for the networks depicted in Fig. 1, so that identification also fails for these structures when taking global differences. The two conditions are not *always* equivalent, however. Consider, for instance, the network presented in Fig. 2. For this graph, the matrix $\mathbf{G}$ satisfies $\mathbf{G}^3 = \mathbf{G}$, rank$(\mathbf{I}-\mathbf{G}) = 3 < n-1 = 4$ and $2\lambda_0+\lambda_1+1 = 2 \neq 0$. Applying our previous results, we see that identification holds under global differences but not under local ones. More generally, we have shown (see Bramoullé et al. (2006)) that Proposition 5 provides the best possible identification condition in this setting. If social effects are not identified when subtracting the network's average, they are never identified.

4. Empirical results

To illustrate our approach, we analyze econometric results on recreational activities by high school students in the US. We use the In-school Add Health data collected between September 1994 and April 1995. In the project, a sample of 80 high schools and 52 middle schools was selected. The study design makes sure that the sample is representative of US school with many respects (region, school size, school type, urbanicity, and ethnicity). All students in each school sampled have been asked to fill up the self-administrated questionnaire. The data provide information on variables such as social and demographic characteristics of the respondents, the education level and occupation of their parents, and their friendship links (*i.e.*, their best friends, up to 5 females and up to five males).

The left-hand side variable of our econometric model is an index of participation in recreational activities such as educational, artistic and sports organizations and clubs by a student.[26] This recreational activities index takes a value from 0 to 4, where 0 to 3 corresponds to the number of organizations and clubs of which the student is a member when the number is smaller than 4, and 4 when the number is 4 and over.[27] The right-hand side variables includes own characteristics (age and dummies for grade, gender, race, born in the US, mother present, father present, mother's level of education, father's level of education, and parents' participation in the labor market[28]), their best friends' mean characteristics (exogenous social variables), and their mean recreational activities index (endogenous social variable). Note that to construct our peer variables, we had to restrict the list of friends to those registered at the same secondary school.

In our empirical application, and following the second interpretation of our model (see Section 2.4), each school l is assumed to form a network with a stochastic but strictly exogenous interaction matrix $\mathbf{G}_l$, on the network fixed effect and the observed students' characteristics. We construct the block diagonal matrix $\mathbf{G}$ of the large network of all schools. We apply a local transformation to the model defined on the latter network so that it can be written as

$$(\mathbf{I}-\mathbf{G})\mathbf{y} = \beta(\mathbf{I}-\mathbf{G})\mathbf{G}\mathbf{y} + (\mathbf{I}-\mathbf{G})\boldsymbol{X}\boldsymbol{\gamma} + (\mathbf{I}-\mathbf{G})\boldsymbol{G}\boldsymbol{X}\boldsymbol{\delta} + \boldsymbol{\nu} \tag{15}$$

where $\mathbf{X}$ is the matrix of observations on students' own characteristics. Students are assumed to interact with their friends, not in groups as in most empirical studies on peer effects. It is easy to verify that the matrices $\mathbf{I}$, $\mathbf{G}$, $\mathbf{G}^2$, and $\mathbf{G}^3$ are linearly independent. Therefore, from Proposition 4, the model is identified and the matrices ($(\mathbf{I}-\mathbf{G})\mathbf{G}^2\mathbf{X}$, $(\mathbf{I}-\mathbf{G})\mathbf{G}^3\mathbf{X}, \ldots$) can be used as valid instruments. We suppose that errors in (15) are independent across observations but heteroskedastic.[29]

Social effects are estimated following a Generalized 2SLS strategy proposed in Kelejian and Prucha (1998) and refined in Lee (2003). This procedure yields an asymptotically optimal IV estimator when the errors are i.i.d. and reduces to a two-step estimation method in our case.[30] The first step consists in estimating a 2SLS using as instruments $\mathbf{S} = \left[(\mathbf{I}-\mathbf{G})\mathbf{X} \quad (\mathbf{I}-\mathbf{G})\mathbf{G}\mathbf{X} \quad (\mathbf{I}-\mathbf{G})\mathbf{G}^2\mathbf{X}\right]$. In our case, the model is overidentified, and we obtain $\widehat{\boldsymbol{\theta}}^{2SLS} = (\widetilde{\mathbf{X}}'\mathbf{P}\widetilde{\mathbf{X}})^{-1}\widetilde{\mathbf{X}}'\mathbf{P}\mathbf{y}$ where $\widetilde{\mathbf{X}} = \left[(\mathbf{I}-\mathbf{G})\mathbf{G}\mathbf{y} \quad (\mathbf{I}-\mathbf{G})\mathbf{X} \quad (\mathbf{I}-\mathbf{G})\mathbf{G}\mathbf{X}\right]$ is the matrix of explanatory variables and $\mathbf{P} = \mathbf{S}(\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}$ is the weighting matrix. The second step consists in estimating a 2SLS using as instruments $\widehat{\mathbf{Z}} = \mathbf{Z}(\widehat{\boldsymbol{\theta}}^{2SLS})$, with $\mathbf{Z}(\boldsymbol{\theta}) = \left[\mathbb{E}[(\mathbf{I}-\mathbf{G})\mathbf{G}\mathbf{y}(\boldsymbol{\theta}) \mid \mathbf{X},\mathbf{G}] \quad (\mathbf{I}-\mathbf{G})\mathbf{X} \quad (\mathbf{I}-\mathbf{G})\mathbf{G}\mathbf{X}\right]$. From the reduced-form equation, it follows that

$$\mathbb{E}[(\mathbf{I}-\mathbf{G})\mathbf{G}\mathbf{y}(\boldsymbol{\theta}) \mid \mathbf{X},\mathbf{G}] = \mathbf{G}(\mathbf{I}-\beta\mathbf{G})^{-1}[(\mathbf{I}-\mathbf{G})(\boldsymbol{X}\boldsymbol{\gamma} + \boldsymbol{G}\boldsymbol{X}\boldsymbol{\delta})].$$

[26] The questionnaire classifies these activities into three categories : (1) clubs and organizations, which include language club, book club, debate club, drama club, future farmers of America, history/math/science/computer club, music clubs (band, orchestra, chorus), dance/cheerleading and other clubs and organization; (2) sports clubs, (3) other, including newspaper, honor society, student council, yearbook.

[27] We used this definition to reduce the problem of measurement errors when a student reports a large number of activities.

[28] This dummy variable = 1 when the student reports living with at least one parent who participates in the labor market, 0 otherwise. This variable may be interpreted as a proxy for the student not being on welfare.

[29] Our approach thus imposes within-network independence of the error terms, which is a reasonable assumption given the presence of a network effect. An alternative approach in this setting would be to "cluster" at the network level (thus allowing to introduce random coefficients that could induce both heteroskedasticity and within-network dependence).

[30] Since we do not assume homoskedasticity, our estimates are consistent but not asymptotically optimal. However, our estimates of the standard errors of the estimated parameters do take the presence of heteroskedasticity into account.

Table 1
Descriptive statistics.

Variable	Mean	Standard deviation
Consumption of recreational services	2.122	1.267
Age	14.963	1.682
Female	0.535	0.499
Race is white only	0.619	0.486
White	0.674	0.469
Black	0.168	0.373
Asian	0.062	0.242
Native	0.052	0.221
Other	0.082	0.274
Born in the US	0.928	0.259
Mother present	0.929	0.257
Father present	0.779	0.415
Grade 6 to 8	0.263	0.440
Grade 9 or 10	0.406	0.491
Grade 11 or 12	0.331	0.471
Parents' participation in the labor market	0.965	0.184
Mother's level of education		
No high school (HS)	0.097	0.296
HS graduate	0.284	0.451
More than HS but no college/University degree	0.276	0.447
College/University graduate	0.206	0.404
Went to school but do not know the level	0.066	0.248
Father's level of education		
No high school (HS)	0.081	0.273
HS graduate	0.211	0.408
More than HS but no college/University degree	0.240	0.427
College/University graduate	0.178	0.383
Went to school but do not know the level	0.069	0.253
Number of observations		55 208

Since the model is just identified, we obtain

$$\widehat{\boldsymbol{\theta}}^{LEE} = (\widehat{\mathbf{Z}}'\widetilde{\mathbf{X}})^{-1}\widehat{\mathbf{Z}}'\mathbf{y}.$$

The variance matrix of the estimated parameters is consistently estimated by:

$$\widehat{V}(\widehat{\boldsymbol{\theta}}^{LEE}) = (\widehat{\mathbf{Z}}'\widetilde{\mathbf{X}})^{-1}\widehat{\mathbf{Z}}'\mathbf{D}\widehat{\mathbf{Z}}(\widetilde{\mathbf{X}}'\widehat{\mathbf{Z}}')^{-1},$$

where $\mathbf{D}$ is an $n \times n$ diagonal matrix with entries given by the squared residuals from this second step.[31]

Descriptive statistics of our sample are provided in Table 1. The total number of observations is 55 208. The average age of the students is 14.9 and 53.5% are female. Also, in 96.5% of households, at least one parent participates in the labor market. Table 2 presents the estimation results. Even though the set of exogenous social effects corresponds exactly to the one's own characteristics, the model can be estimated and many coefficients are significant. As regards own characteristics, our results show in particular that the recreational activities index decreases with age and with being white, but rises with being female and with the parents' participation in the labor market. The exogenous social effects are significant for a number of variables. In particular, our results indicate that a student's recreational activities index decreases with the mean age of his friends but rises with their mean parents' participation in the labor market. One interpretation for the latter result is that working friends' parents may help a student to have a better access to recreational activities (*e.g.*, through contacts with people responsible for clubs or organizations, or

[31] While our estimating approach ignores problems raised by a count dependent variable, it has the advantage of convenience, given the presence of an endogenous right-hand variable. In this, we follow the approach used in several published models on peer effects with nonperfectly continuous response variables, such as those on juvenile cigarette or drug use (*e.g.*, Gaviria and Raphael (2001)) or on Grade Point Average (*e.g.*, Sacerdote (2001)). Note also that the estimated standard errors of the coefficients handle the inherent heteroskedasticity of the count model.

Table 2
Consumption of recreational activities.

	Variable	Coeff	Std err	Asymp. t-stat
Own characteristics $= (\mathbf{I}-\mathbf{G})\mathbf{x}$	Age	−0.0225	0.0113	−1.9840
	Female	0.2130	0.0148	14.4136
	Race is white only	−0.1057	0.0204	−5.1874
	Born in the US	−0.0516	0.0333	−1.5527
	Mother present	−0.0125	0.0358	−0.3488
	Father present	−0.0185	0.0289	−0.6381
	Grade 9 or 10	0.0108	0.0963	0.1124
	Grade 11 or 12	0.0213	0.1021	0.2091
	Mother is HS grad	−0.0046	0.0274	−0.1687
	Father is HS grad	0.0470	0.0287	1.6394
	Mother: more than HS but no college degree	0.1464	0.0293	4.9963
	Father: more than HS but no college degree	0.1669	0.0304	5.4894
	Mother is college/University grad	0.1374	0.0331	4.1483
	Father is college/University grad	0.1274	0.0311	4.0985
	Mother went to school but do not know level	−0.0096	0.0381	−0.2517
	Father went to school but do not know level	−0.0665	0.0380	−1.7480
	Parents' participation in the labor market	0.0834	0.0400	2.0866
Exogenous social effects $= (\mathbf{I}-\mathbf{G})\mathbf{Gx}$	Age	−0.0611	0.0202	−3.0247
	Female	0.0079	0.0479	0.1654
	Race is white only	−0.0190	0.0447	−0.4256
	Born in the US	0.0420	0.0655	0.6407
	Mother present	0.1071	0.0638	1.6798
	Father present	−0.1087	0.0532	−2.0447
	Grade 9 or 10	−0.0344	0.1859	−0.1848
	Grade 11 or 12	0.0997	0.1941	0.5136
	Mother is HS grad	−0.0474	0.0504	−0.9407
	Father is HS grad	0.1719	0.0546	3.1469
	Mother: more than HS but no college degree	−0.0380	0.0676	−0.5622
	Father: more than HS but no college degree	0.0911	0.0667	1.3669
	Mother is college/University grad	−0.0310	0.0808	−0.3834
	Father is college/University grad	0.1238	0.0606	2.0435
	Mother went to school but do not know level	−0.0940	0.0702	−1.3379
	Father went to school but do not know level	0.1500	0.0749	2.0033
	Parents' participation in the labor market	0.1508	0.0728	2.0722
Endogenous social effects $= (\mathbf{I}-\mathbf{G})\mathbf{Gy}$	Endogenous effect	0.4667	0.2556	1.8259
Number of observations	55 208			

Note: Excluded categories: Grade 6 to 8, Mother: no HS, Father: no HS. The model is estimated using the Generalized 2SLS proposed in Kelejian and Prucha (1998) and refined in Lee (2003).

simply through financial help). The endogenous social effect is positive and significant at the 10% level (asymptotic $t = 1.82$). This indicates that a one point increase in the mean recreational activities index of student's friends induces him to increase his recreational activities index by 0.466.

5. Monte Carlo simulations

We provide Monte Carlo simulations calibrated on our econometric results to study how the strength of the identification is affected by the density of the graph and its level of intransitivity. Following Wasserman and Faust (1994), we define density as the ratio of the number of links over the total number of possible links. Thus, it describes the average probability that any two students are connected. We expect identification to be weak if the graph is close to being complete (*i.e.*, when its density is close to one). The level of intransitivity is the ratio of the number of intransitive triads over the number of triads. The intransitivity level lies between 0 and 1; it equals 0 only when the network is transitive.

We find that structural parameters are better estimated when the density of the graph is small. The impact of intransitivity on the precision of estimators is however more complex and often nonmonotonic with intermediate to high density.

5.1. Calibration

Running our Monte Carlo simulations using the whole sample of schools is intractable, given the large size of the dataset. Rather, the procedure we adopt is the following. First, we pick the median school in our sample. Its size is 240 students. Also, its density is 0.0201, its level of intransitivity is 0.2103 and the average number of links per student is 4.81. (The corresponding numbers are 7.43×10^{-5}, 0.1978 and 4.10 for the complete sample). Second, we assume that all students have the same characteristics except for their parents' participation in the labor market. Third, to increase the variability of the latter variable in our subsample, we convert it into a continuous measure of parents' labor income (named $\mathbf{x}$ from there on). More precisely, to take into account the mass of probability at $\mathbf{x} = \mathbf{0}$ we draw a 240×1 random vector in a Bernoulli distribution with $p = 227/240$ (13 students report to live with parents who are not working). Then, for the students who are predicted to live with at least one parent who works, we draw a vector of $\mathbf{x}$ from a log-normal distribution with $\mathbb{E}(\mathbf{x})$ normalized to 1 and $\sigma^2_{\mathbf{x}} = 3$. Fourth, the values of the parameters β, γ and δ are those obtained by estimation on the complete sample. Finally, the parameter α is calculated from the equation: $\alpha = \bar{\mathbf{y}} + \beta\overline{\mathbf{Gy}} + \gamma\bar{\mathbf{x}} + \delta\overline{\mathbf{Gx}}$, where the bar indicates that the vector is computed at the median school mean. One has:

$$\alpha = 0.7683, \qquad \beta = 0.4666, \qquad \gamma = 0.0834, \qquad \delta = 0.1507.$$

In this model, $\boldsymbol{\epsilon}$ is a 240×1 vector of error terms distributed as $n(\mathbf{0}, \boldsymbol{\Sigma})$. We set $\boldsymbol{\Sigma} = \sigma^2\mathbf{I}$ and $\sigma^2 = 0.1$. Finally, we generate the 240×1 vector of endogenous variables $\mathbf{y}$ using the reduced-form equation (5).

In the simulations, the model is still estimated using the Generalized 2SLS approach. However, we assume homoskedastic errors in the estimation of the variance matrix.

Table 3
Network simulations for Erdös–Renyi graphs (1000 draws).

	$\hat{\gamma}$ (S.E)	$\hat{\delta}$ (S.E)	$\hat{\beta}$ (S.E)
0.01	0.0833	0.1511	0.4663
	0.0032	0.0049	0.0113
0.02	0.0834	0.1512	0.4669
	0.0031	0.0056	0.0238
0.03	0.0834	0.1506	0.4671
	0.0031	0.0103	0.0892
0.04	0.0832	0.1508	0.465
	0.0031	0.0163	0.1679
0.05	0.0836	0.1516	0.4566
	0.0031	0.0222	0.2273
0.06	0.0833	0.151	0.4654
	0.0031	0.024	0.2531
0.07	0.0834	0.1514	0.4635
	0.0031	0.0275	0.2605
0.08	0.0834	0.1512	0.4649
	0.0031	0.0285	0.3195
0.09	0.0835	0.1516	0.4543
	0.0031	0.0351	0.4162
0.10	0.0833	0.1505	0.4677
	0.0031	0.0344	0.4112
0.20	0.0836	0.1509	0.4556
	0.0032	0.0938	1.1006
0.30	0.0834	0.1554	0.4289
	0.0032	0.1516	1.8064
0.40	0.0834	0.1447	0.5488
	0.0033	0.2692	3.228
0.50	0.0832	0.1261	0.8232
	0.0071	1.8709	23.8413
0.60	0.0845	−0.3838	7.5082
	0.2619	119.9752	1 574.4134
0.70	0.0835	0.1084	0.6689
	0.1259	105.2318	1 409.2152
0.80	0.0831	0.9454	−9.2928
	0.213	152.0559	1 914.4321
0.90	0.0919	−2.3448	32.6466
	16.8469	5955.1294	75 380.768
1.00	0.1026	4.7828	−1.3294
	∞	∞	∞

Table 4
Small-world graphs, $k = 3, 5$ (1000 draws).

k	p	Intransitivity	$\hat{\gamma}$ (S.E.)	$\hat{\delta}$ (S.E.)	$\hat{\beta}$ (S.E.)
3	0.01	0.0309	0.1022	0.1891	0.3366
			0.5979	1.2195	4.0886
3	0.05	0.2140	0.0835	0.1514	0.4654
			0.0052	0.0099	0.0299
3	0.1	0.4239	0.0833	0.1510	0.4666
			0.0040	0.0067	0.0205
3	0.2	0.5902	0.0833	0.1509	0.4662
			0.0037	0.0061	0.0198
3	0.4	0.8713	0.0836	0.1506	0.4664
			0.0032	0.0049	0.0122
3	0.6	0.9370	0.0834	0.1508	0.4662
			0.0031	0.0043	0.0102
3	0.8	0.9912	0.0835	0.1509	0.4673
			0.0031	0.0055	0.0110
3	1.0	0.9904	0.0835	0.1507	0.4672
			0.0031	0.0042	0.0108
5	0.01	0.0455	0.0838	0.1524	0.4611
			0.0124	0.0538	0.1548
5	0.05	0.1290	0.0838	0.1522	0.4629
			0.0068	0.0268	0.0783
5	0.1	0.2994	0.0832	0.1509	0.4667
			0.0047	0.0157	0.0553
5	0.2	0.5020	0.0835	0.1510	0.4658
			0.0037	0.0108	0.0373
5	0.4	0.7913	0.0835	0.1509	0.4666
			0.0033	0.0075	0.0386
5	0.6	0.9237	0.0833	0.1507	0.4652
			0.0031	0.0069	0.0392
5	0.8	0.9772	0.0835	0.1507	0.4679
			0.0031	0.0072	0.0250
5	1.0	0.9840	0.0835	0.1505	0.4669
			0.0031	0.0066	0.0204

5.2. Networks

To analyze the impact of density and intransitivity, we consider two types of networks. We first look at the standard Erdös and Rényi (1959) model of random graphs. Link are i.i.d. and each pair of students is connected with the same probability d. In this case, the expected density is equal to d while the expected level of intransitivity is $1 - d$. When $d = 1$, the network is complete and students interact in groups. Therefore social effects are not identified. We examine how the strength of identification changes when d varies from 0.01 to 1 by increments of .01 from 0.01 to 0.1 and by increments of .1 from 0.1 to 1.

Erdös–Rényi random graphs provide a natural starting point, but their structure is very specific. Especially, a lower d corresponds to a lower density and a higher intransitivity. In order to disentangle both effects, we introduce a second type of graphs. We adapt the small-world procedure of Watts and Strogatz (1998) to our context. We start from disjoint complete subgraphs of size k. (Hence k divides the population size n). For instance, if $k = 12$ in our school of 240 students, this means that there are 12 complete groups of 20 students. Then, with probability p each link is rewired at random. Since the total number of links is constant, density is fixed and equal to $\frac{k-1}{n-1}$ ($= 0.046$ in our example with $k = 12$). We let the probability p vary. When p equals 0, the level of intransitivity is 0. This is the case since students interact in groups. When p equals 1, we obtain an Erdös–Rényi graph. As p increases, the expected level of intransitivity increases. We let k and p take the following values: $k \in \{3, 5, 10, 12, 15, 20\}$ and $p \in \{0.01, 0.05, 0.1, 0.2, 0.4, 0.6, 0.8, 1\}$. In summary, the second type of graphs allows us to look at how intransitivity affects the strength of identification, holding density constant. Also, they can be used to analyze the impact of density, holding intransitivity constant.

5.3. Results

Table 3 reports the estimation results for Erdös–Rényi graphs. The probability of link formation, or expected density, d, is given in column 1. For each level of density, we pick one graph. We look at 1000 draws for the vectors $\boldsymbol{\epsilon}$ and $\mathbf{x}$. For each draw, we estimate the structural parameters. Columns 2–4 of Table 3 report the average estimates and the average standard errors over the 1000 draws. Estimates of the exogenous and the endogenous effects are respectively shown in columns 3 and 4.

We find that precision is a decreasing function of density. When the density is smaller than or equal to 0.07, the bias on the estimates of both peer effects is relatively small and the precision is relatively good especially on the exogenous effect. As the density of the graph increases, precision worsens. None of the two estimates are significant on average at the 10% level when the density reaches 0.20. As expected, when the density is equal to one (complete graph), the estimation procedure diverges.

The estimation results for small-world graphs are given in Tables 4–6. Each of these tables are composed of two panels. To each panel corresponds a specific value of k. Within each panel, density is fixed and as one goes down the columns, p, hence intransitivity, increases. For each value of k and p, we pick one graph. Again, for each graph we report the estimate averages over 1000 draws.

Results are more complex than for Erdös–Rényi graphs. When density is low ($k \in \{3, 5\}$), precision is an increasing function of intransitivity almost everywhere. However, for intermediate to high levels of density $k \in \{10, 12, 15, 20\}$, the relationship is nonmonotonic. In any case, starting from a situation where peer effects are not identified ($p = 0$), a slight increase in the

Table 5
Small-world graphs, $k = 10, 12$ (1000 draws).

k	p	Intransitivity	$\hat{\gamma}$ (S.E.)	$\hat{\delta}$ (S.E.)	$\hat{\beta}$ (S.E.)
10	0.01	0.0407	0.0831	0.1478	0.4743
			0.0067	0.0649	0.1678
10	0.05	0.1394	0.0836	0.1531	0.4601
			0.0050	0.0420	0.1165
10	0.1	0.2983	0.0833	0.1495	0.4670
			0.0037	0.0251	0.0799
10	0.2	0.4959	0.0832	0.1502	0.4674
			0.0036	0.0198	0.0794
10	0.4	0.7888	0.0834	0.1512	0.4611
			0.0032	0.0159	0.1051
10	0.6	0.9104	0.0834	0.1510	0.4640
			0.0031	0.0133	0.1125
10	0.8	0.9545	0.0835	0.1508	0.4666
			0.0031	0.0151	0.1327
10	1.0	0.9615	0.0834	0.1517	0.4574
			0.0031	0.0162	0.1519
12	0.01	0.0253	0.0825	0.1407	0.4889
			0.0199	0.2636	0.6622
12	0.05	0.1722	0.0836	0.1522	0.4607
			0.0043	0.0392	0.1111
12	0.1	0.2926	0.0834	0.1507	0.4642
			0.0037	0.0292	0.0966
12	0.2	0.5302	0.0832	0.1509	0.4629
			0.0034	0.0202	0.0910
12	0.4	0.7695	0.0834	0.1516	0.4585
			0.0032	0.0177	0.1245
12	0.6	0.9035	0.0834	0.1511	0.4636
			0.0031	0.0181	0.1696
12	0.8	0.9464	0.0834	0.1502	0.4696
			0.0031	0.0170	0.1653
12	1.0	0.9551	0.0833	0.1509	0.4618
			0.0031	0.0167	0.1734

Table 6
Small-world graphs, $k = 15, 20$ (1000 draws).

k	p	Intransitivity	$\hat{\gamma}$ (S.E.)	$\hat{\delta}$ (S.E.)	$\hat{\beta}$ (S.E.)
15	0.01	0.0253	0.0833	0.1509	0.4594
			0.0093	0.1471	0.3656
15	0.05	0.1535	0.0834	0.1521	0.4593
			0.0039	0.0413	0.1183
15	0.1	0.2969	0.0835	0.1520	0.4578
			0.0035	0.0304	0.1046
15	0.2	0.4767	0.0833	0.1508	0.4626
			0.0034	0.0286	0.1120
15	0.4	0.7496	0.0833	0.1510	0.4596
			0.0032	0.0228	0.1506
15	0.6	0.8905	0.0833	0.1518	0.4586
			0.0031	0.0219	0.2034
15	0.8	0.9363	0.0835	0.1509	0.4725
			0.0031	0.0234	0.2505
15	1.0	0.9421	0.0834	0.1511	0.4674
			0.0031	0.0236	0.2395
20	0.01	0.0237	0.0842	0.1698	0.4211
			0.0146	0.3085	0.7344
20	0.05	0.1482	0.0833	0.1497	0.4676
			0.0045	0.0703	0.1870
20	0.1	0.2875	0.0834	0.1492	0.4686
			0.0035	0.0395	0.1207
20	0.2	0.4789	0.0834	0.1515	0.4605
			0.0033	0.0312	0.1194
20	0.4	0.7522	0.0834	0.1525	0.4494
			0.0032	0.0257	0.1710
20	0.6	0.8742	0.0834	0.1532	0.4455
			0.0031	0.0295	0.2663
20	0.8	0.9133	0.0834	0.1518	0.4491
			0.0031	0.0298	0.3042
20	1.0	0.9197	0.0835	0.1521	0.4494
			0.0031	0.0298	0.3140

level of intransitivity holding density constant greatly improves identification. As regards the density, a glance across the tables indicates that for a given level of intransitivity, the precision of the estimates is everywhere a decreasing function of density. Overall, these results confirm the role played by the network's structure on the identification of peer effects.

6. Conclusion

In this paper, we characterize the conditions under which endogenous and exogenous social effects are identified in a linear-in-means model with general interaction structure. Our analysis shows that both effects can usually be distinguished with network data, although identification may fail for specific networks. At the empirical level, we demonstrate the *feasibility* of our identification strategy by estimating a model of consumption of recreational activities by secondary school students using Add Health data. We show in particular that endogenous and exogenous social effects (using the network of friends at school) can be disentangled. This empirical application is in line with a few recent papers that have already exploited the presence of a social network to estimate peer effects.[32] Also, when the model is identified, we show, from Monte Carlo simulations, that characteristics of a network, such as its density and its level of intransitivity, may strongly affect the quality of the estimates of the peer effects.

Our results suggest that there are high benefits to analyzing network data. On the other hand, collecting comprehensive data on relationships between agents can be very costly. The development of electronic databases has, in some instances, dramatically lowered this cost. Thus, studies on co-authorship networks among scientists are linked to the availability of literature databases (see Newman (2001) and Goyal et al. (2006)). *Sampling the network* provides a different way to reduce these costs. This can be done in many ways, *e.g.*, see Rothenberg (1995). It would be interesting to study how our identification results apply in such settings. Can the econometrician still recover endogenous and exogenous effects if he only knows a limited part of the network?

Finally, the problems of correlated effects and the endogeneity of link formation remain central. Experiments, or natural randomizations, provide one answer. In an experimental context, our results could guide empirical research. For instance, appropriate structures of interactions could be imposed on participants in the lab. With non-experimental data, taking differences between endogenous variables can eliminate certain types of unobserved variables, as done in Section 3. Alternatively, two-stage estimations could be attempted on network data. The likelihood of link formation could be estimated in a first step, and social effects conditional on links formed in a second step. This would require to have some understanding on how relationships emerge, hence could be fruitfully combined with theoretical models of network formation.[33]

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32 See Laschever (2005), Lin (2007), De Georgi et al. (2007), and our discussion in the introduction.

33 See Ioannides and Soetevent (2007) and Weinberg (2007) for first steps in this direction.

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Appendix A

Proof of Proposition 1. Consider two sets of structural parameters $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ leading to the same reduced form. It means that $\alpha(\mathbf{I}-\beta\mathbf{G})^{-1}\boldsymbol{\iota} = \alpha'(\mathbf{I}-\beta'\mathbf{G})^{-1}\boldsymbol{\iota}$ and $(\gamma\mathbf{I}+\delta\mathbf{G})(\mathbf{I}-\beta\mathbf{G})^{-1} = (\gamma'\mathbf{I}+\delta'\mathbf{G})(\mathbf{I}-\beta'\mathbf{G})^{-1}$. Multiply the second equality by $(\mathbf{I}-\beta\mathbf{G})(\mathbf{I}-\beta'\mathbf{G})$. Since $\forall a$, $(\mathbf{I}-a\mathbf{G})^{-1}\mathbf{G} = \mathbf{G}(\mathbf{I}-a\mathbf{G})^{-1}$, this is equivalent to

$$(\gamma-\gamma')\mathbf{I} + (\delta-\delta'+\gamma'\beta-\gamma\beta')\mathbf{G} + (\delta'\beta-\delta\beta')\mathbf{G}^2 = 0. \quad \text{(A.1)}$$

Suppose first that $\mathbf{I}$, $\mathbf{G}$, and $\mathbf{G}^2$ are linearly independent. Then, $\gamma = \gamma'$, $\delta+\gamma'\beta = \delta'+\gamma\beta'$, and $\delta'\beta = \delta\beta'$. Suppose first that $\delta'\beta \neq 0$. There exists $\lambda \neq 0$ such that $\beta' = \lambda\beta$, $\delta' = \lambda\delta$. Substituting yields $\delta'+\gamma\beta' = \lambda(\delta+\gamma\beta) = \delta+\gamma\beta$. Since $\delta+\gamma\beta \neq 0$, $\lambda = 1$, hence $\beta' = \beta$, $\delta' = \delta$. Suppose next that $\delta'\beta = 0$. Since $\delta+\gamma\beta \neq 0$, we cannot have $\beta = \delta = 0$ or $\beta' = \delta' = 0$. Thus, either $\beta = \beta' = 0$ and by the last equation $\delta = \delta' = 0$, or $\delta = \delta' = 0$ and by the last equation (and since $\gamma \neq 0$ because $\delta+\gamma\beta \neq 0$), $\beta = \beta' = 0$. To conclude, observe that $\alpha(\mathbf{I}-\beta\mathbf{G})^{-1}\boldsymbol{\iota} = \alpha'(\mathbf{I}-\beta\mathbf{G})^{-1}\boldsymbol{\iota}$ implies that $\alpha\boldsymbol{\iota} = \alpha'\boldsymbol{\iota}$, hence $\alpha = \alpha'$.

Next, suppose that $\mathbf{I}$, $\mathbf{G}$, and $\mathbf{G}^2$ are linearly dependent, and that no student is isolated. The latter property implies that $\mathbf{G}\boldsymbol{\iota} = \boldsymbol{\iota}$, and $\alpha(\mathbf{I}-\beta\mathbf{G})^{-1}\boldsymbol{\iota} = \alpha/(1-\beta)\boldsymbol{\iota}$. Three equations only need to be satisfied for $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ to yield the same reduced form. Therefore, the model is not identified.

Suppose that there are K characteristics and that parameters γ_k and δ_k are associated with characteristic k. Structural parameters $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ lead to the same reduced form iff $\alpha(\mathbf{I}-\beta\mathbf{G})^{-1}\boldsymbol{\iota} = \alpha'(\mathbf{I}-\beta'\mathbf{G})^{-1}\boldsymbol{\iota}$ and $\forall k$, $(\gamma_k\mathbf{I}+\delta_k\mathbf{G})(\mathbf{I}-\beta\mathbf{G})^{-1} = (\gamma_k'\mathbf{I}+\delta_k'\mathbf{G})(\mathbf{I}-\beta'\mathbf{G})^{-1}$. Suppose that for some k_0, $\delta_{k_0}+\gamma_{k_0}\beta \neq 0$. If $\mathbf{I}$, $\mathbf{G}$, and $\mathbf{G}^2$ are linearly independent, by the previous argument $\alpha = \alpha'$, $\beta = \beta'$, $\gamma_{k_0} = \gamma_{k_0}'$ and $\delta_{k_0} = \delta_{k_0}'$. Then, for any other k, multiplying by $\mathbf{I}-\beta\mathbf{G}$ leads to $\gamma_k\mathbf{I}+\delta_k\mathbf{G} = \gamma_k'\mathbf{I}+\delta_k'\mathbf{G}$ hence $\boldsymbol{\theta} = \boldsymbol{\theta}'$. If $\mathbf{I}$, $\mathbf{G}$, and $\mathbf{G}^2$ are linearly dependent, we can find $\boldsymbol{\theta} \neq \boldsymbol{\theta}'$ leading to the same reduced form. □

Proof of Proposition 3. Suppose that students do not interact in groups and that the network is transitive. We make use of standard properties of directed graphs, see *e.g.*, Bang-Jensen and Cutin (2000). Say that there is a path between i and j in the network if $\mathbf{G}_{ij} > 0$ or if there exist $i_1, \ldots, i_l$ such that $\mathbf{G}_{ii_1}\mathbf{G}_{i_1i_2}\ldots\mathbf{G}_{i_lj} > 0$. A cycle is a path between i and i. A strong component of the network is a maximal set S of students such that there is a path between any two students in S. The original network induces an acyclic network on the strong components. Thus, there always exist a strong component S such that $\forall i \in S$, $P_i \subset S$. Transitive directed graphs admit a simple characterization, see Proposition 4.3.1 in Bang-Jensen and Cutin (2000). Especially, their strong components are complete, and the relations between strong components are also complete. Formally, let S and S' be two strong components. Then, $\forall i \neq j \in S$, $\mathbf{G}_{ij} > 0$. And, if there is a path between $i \in S$ and $i' \in S'$, then for all $j \in S$ and $j' \in S'$, $\mathbf{G}_{jj'} > 0$.

We know that there exists two strong components S_1 and S_2 of sizes s_1, s_2 such that: (1) $\forall i \in S_1$, $P_i = S_1 - \{i\}$, and (2) $\forall i \in S_2$, $P_i = S \cup S_2 - \{i\}$ where S is a group size $s \geq s_1$ such that $S_1 \subset S$ and $S_2 \cap S = \emptyset$. S_1 is a strong component, which is not affected by any other strong component. S_2 is a strong component affected by S_1. We can find $S_1 \neq S_2$ since students do not interact in groups. Suppose first that $s_1, s_2 \geq 2$. For any $i \neq j \in S_1$, $\mathbf{G}_{ij} = \frac{1}{s_1-1}$ while for any $i \neq j \in S_2$, $\mathbf{G}_{ij} = \frac{1}{s_2+s-1}$. If $\mathbf{G}^2 = \lambda_0\mathbf{I}+\lambda_1\mathbf{G}$, then $\forall i$, $(\mathbf{G}^2)_{ii} = \lambda_0$. In addition, $(\mathbf{G}^2)_{ii} = \sum_{j\in N}\mathbf{G}_{ij}\mathbf{G}_{ji}$. Since S_1 is a strong component, if $i \in S_1$ and $\mathbf{G}_{ij}\mathbf{G}_{ji} > 0$, then $j \in S_1$. The same is true for S_2. Therefore, if $i \in S_1$, $(\mathbf{G}^2)_{ii} = \sum_{j\in S_1}\mathbf{G}_{ij}\mathbf{G}_{ji} = \frac{1}{s_1-1}$. In contrast, if $i \in S_2$, $(\mathbf{G}^2)_{ii} = \sum_{j\in S_2}\mathbf{G}_{ij}\mathbf{G}_{ji} = \frac{s_2-1}{(s_2+s-1)^2}$. Since $s_2+s-1 > s_1-1$, we have $\frac{s_2-1}{(s_2+s-1)^2} < \frac{1}{s_1-1}$, which is a contradiction.

Suppose next that s_1 or s_2 is equal to 1. It means that for some i, $(\mathbf{G}^2)_{ii} = 0$. Hence $\lambda_0 = 0$ and for all i, $(\mathbf{G}^2)_{ii} = 0$. Therefore, all the strong components of the network have size 1, and the network is acyclic. In this case, there must exist a pair of students i and j such that: (1) i is affected by j, (2) j is isolated, and (3) all the students affecting i are isolated. Then, $\mathbf{G}_{ij} = \frac{1}{n_i}$ and $(\mathbf{G}^2)_{ij} = \sum_{k\in N}\mathbf{G}_{ik}\mathbf{G}_{kj} = 0$. Therefore, $\lambda_1 = 0$, hence $\mathbf{G}^2 = 0$. Therefore, if $\mathbf{G}^2 \neq 0$, $\mathbf{I}$, $\mathbf{G}$, and $\mathbf{G}^2$ are linearly independent.

Finally, suppose that $\mathbf{G}^2 = 0$. These networks are characterized by the fact that: (1) no relation is reciprocal, *i.e.*, $j \in P_i \Rightarrow i \notin P_j$, and (2) they do not have any triad, *i.e.*, sets of three different students i, j, k such that $i \in P_j$ and $j \in N_k$. Two sets of structural coefficients lead to the same reduced form if $\gamma = \gamma'$, $\delta+\gamma'\beta = \delta'+\gamma\beta'$, and $\alpha(\mathbf{I}-\beta\mathbf{G})^{-1}\boldsymbol{\iota} = \alpha'(\mathbf{I}-\beta'\mathbf{G})^{-1}\boldsymbol{\iota}$. The last condition becomes $\alpha\boldsymbol{\iota}-\alpha\beta'\boldsymbol{G}\boldsymbol{\iota} = \alpha'\boldsymbol{\iota}-\alpha'\beta\boldsymbol{G}\boldsymbol{\iota}$. There must exist one isolated and one non-isolated student in the network. When i is isolated, $(\boldsymbol{G}\boldsymbol{\iota})_i = 0$ and $\alpha = \alpha'$. When i is not isolated, $(\boldsymbol{G}\boldsymbol{\iota})_i = 1$, and $\alpha(1-\beta') = \alpha'(1-\beta)$. Under the assumption that $\alpha \neq 0$, we have $\beta = \beta'$, hence $\delta = \delta'$ and social effects are identified. □

Proof of Proposition 4. Two sets of structural parameters (β, γ, δ) and $(\beta', \gamma', \delta')$ lead to the same reduced form of $(\mathbf{I}-\mathbf{G})\mathbf{y}$ if and only if $(\mathbf{I}-\beta\mathbf{G})^{-1}(\gamma\mathbf{I}+\delta\mathbf{G})(\mathbf{I}-\mathbf{G}) = (\mathbf{I}-\beta'\mathbf{G})^{-1}(\gamma'\mathbf{I}+\delta'\mathbf{G})(\mathbf{I}-\mathbf{G})$. This is equivalent to

$$(\gamma-\gamma')\mathbf{I} + [\delta-\delta'-(\gamma-\gamma')+\gamma'\beta-\gamma\beta']\mathbf{G} - [\delta-\delta' + \beta'(\delta-\gamma)-\beta(\delta'-\gamma')]\mathbf{G}^2 + (\beta'\delta-\beta\delta')\mathbf{G}^3 = 0.$$

Suppose first that $\mathbf{I}$, $\mathbf{G}$, $\mathbf{G}^2$, and $\mathbf{G}^3$ are linearly independent. Then, $\gamma = \gamma'$, $\delta+\gamma\beta = \delta'+\gamma\beta'$, and $\beta'\delta = \beta\delta'$. By the same argument as in the proof of Proposition 1, $\beta = \beta'$ and $\delta = \delta'$. Suppose next that $\mathbf{I}$, $\mathbf{G}$, $\mathbf{G}^2$, and $\mathbf{G}^3$ are linearly dependent. If $\mathbf{G}^2 = \lambda_0\mathbf{I}+\lambda_1\mathbf{G}$, only two equations must be satisfied for (β, γ, δ) and $(\beta', \gamma', \delta')$ to lead to the same reduced form, hence social effects are not identified. If $\mathbf{G}^3 = \lambda_0\mathbf{I}+\lambda_1\mathbf{G}+\lambda_2\mathbf{G}^2$, (β, γ, δ) and $(\beta', \gamma', \delta')$ lead to the same reduced form of $(\mathbf{I}-\mathbf{G})\mathbf{y}$ if and only if the following three equations are satisfied

$$\gamma-\gamma'+\lambda_0(\beta'\delta-\beta\delta') = 0$$
$$\delta-\delta'-(\gamma-\gamma')+\gamma'\beta-\gamma\beta'+\lambda_1(\beta'\delta-\beta\delta') = 0$$
$$-(\delta-\delta')-\beta'(\delta-\gamma)+\beta(\delta'-\gamma')+\lambda_2(\beta'\delta-\beta\delta') = 0.$$

Since no student is isolated, $\mathbf{G}\boldsymbol{\iota} = \boldsymbol{\iota}$, and $\lambda_0+\lambda_1+\lambda_2 = 1$. This means that the third equation can be simply obtained by summing the first two. Hence two equations only need to be satisfied, and social effects are not identified.

With K characteristics, $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ lead to the same reduced form iff $\forall k$, $(\mathbf{I}-\beta\mathbf{G})^{-1}(\gamma_k\mathbf{I}+\delta_k\mathbf{G})(\mathbf{I}-\mathbf{G}) = (\mathbf{I}-\beta'\mathbf{G})^{-1}(\gamma_k'\mathbf{I}+\delta_k'\mathbf{G})(\mathbf{I}-\mathbf{G})$. Suppose that $\delta_{k_0}+\gamma_{k_0}\beta \neq 0$. If $\mathbf{I}$, $\mathbf{G}$, $\mathbf{G}^2$, and $\mathbf{G}^3$ are linearly independent, then $\beta = \beta'$, $\gamma_{k_0} = \gamma_{k_0}'$ and $\delta_{k_0} = \delta_{k_0}'$. For any other k, multiplying by $\mathbf{I}-\beta\mathbf{G}$ leads to $(\gamma_k-\gamma_k')\mathbf{I}+(\delta_k-\delta_k'+\gamma_k'-\gamma_k)\mathbf{G}+(\delta_k'-\delta_k)\mathbf{G}^2 = 0$, hence $\gamma_k = \gamma_k'$ and $\delta_k = \delta_k'$. □

Proof of Proposition 5. We first show that social effects are identified if and only if the following condition is true. If $\mu_0\mathbf{I}+\mu_1\mathbf{G}+\mu_2\mathbf{G}^2$ has identical rows, then $\mu_0 = \mu_1 = \mu_2 = 0$.

Suppose that the condition holds. In the proof, we omit the index l for clarity. Eq. (14) becomes:

$$\mathbf{y} - \frac{1}{n}\left(\sum_{i=1}^{n} y_i\right)\boldsymbol{\iota} = (\mathbf{I}-\beta\mathbf{G})^{-1}(\gamma\mathbf{I}+\delta\mathbf{G})\mathbf{x} - \varphi(\mathbf{x}, \boldsymbol{\theta})\boldsymbol{\iota} + \boldsymbol{\nu}'$$

where $\boldsymbol{\theta} = (\beta, \gamma, \delta)$, and $\varphi(\mathbf{x}, \boldsymbol{\theta}) = \frac{1}{n}\iota'(\mathbf{I} - \beta\mathbf{G})^{-1}(\gamma\mathbf{I} + \delta\mathbf{G})\mathbf{x}$ is linear in $\mathbf{x}$. Next suppose that (β, γ, δ) and $(\beta', \gamma', \delta')$ lead to the same reduced form. It means that $\forall\mathbf{x}$, $(\mathbf{I} - \beta\mathbf{G})^{-1}(\gamma\mathbf{I} + \delta\mathbf{G})\mathbf{x} - \varphi(\mathbf{x}, \boldsymbol{\theta})\boldsymbol{\iota} = (\mathbf{I} - \beta'\mathbf{G})^{-1}(\gamma'\mathbf{I} + \delta'\mathbf{G})\mathbf{x} - \varphi(\mathbf{x}, \boldsymbol{\theta}')\boldsymbol{\iota}$. Multiplying by $\mathbf{I} - \beta\mathbf{G}$ and $\mathbf{I} - \beta'\mathbf{G}$ gives $\forall\mathbf{x}$, $[(\gamma - \gamma')\mathbf{I} + (\delta - \delta' + \gamma'\beta - \gamma\beta')\mathbf{G} + (\beta'\delta - \beta\delta')\mathbf{G}^2]\mathbf{x} = (1 - \beta)(1 - \beta')[\varphi(\mathbf{x}, \boldsymbol{\theta}) - \varphi(\mathbf{x}, \boldsymbol{\theta}')]\boldsymbol{\iota}$. This means that the matrix $(\gamma - \gamma')\mathbf{I} + (\delta - \delta' + \gamma'\beta - \gamma\beta')\mathbf{G} + (\beta'\delta - \beta\delta')\mathbf{G}^2$ has identical rows. Thus, $\gamma = \gamma'$, $\delta + \gamma'\beta = \delta' + \gamma\beta'$, and $\beta'\delta = \beta\delta'$, and, using the argument in the proof of Proposition 1, social effects are identified.

Conversely, suppose that the condition does not hold. There exist μ_0, μ_1, μ_2 not all equal to zero such that $\mu_0\mathbf{I} + \mu_1\mathbf{G} + \mu_2\mathbf{G}^2$ has identical rows. We follow the previous reasoning in reverse. We can find $\boldsymbol{\theta} \neq \boldsymbol{\theta}'$ such that $\mu_0 = \gamma - \gamma'$, $\mu_1 = \delta - \delta' + \gamma'\beta - \gamma\beta'$, and $\mu_2 = \beta'\delta - \beta\delta'$. Then, $(\gamma - \gamma')\mathbf{I} + (\delta - \delta' + \gamma'\beta - \gamma\beta')\mathbf{G} + (\beta'\delta - \beta\delta')\mathbf{G}^2$ has identical rows. There exists (r_j) such that $\forall\mathbf{x}$, $[(\gamma - \gamma')\mathbf{I} + (\delta - \delta' + \gamma'\beta - \gamma\beta')\mathbf{G} + (\beta'\delta - \beta\delta')\mathbf{G}^2]\mathbf{x} = (1 - \beta)(1 - \beta')(\sum_{j=1}^n r_j x_j)\boldsymbol{\iota}$. Dividing by $\mathbf{I} - \beta\mathbf{G}$ and $\mathbf{I} - \beta'\mathbf{G}$ means that $\forall\mathbf{x}$, $(\mathbf{I} - \beta\mathbf{G})^{-1}(\gamma\mathbf{I} + \delta\mathbf{G})\mathbf{x} - (\mathbf{I} - \beta'\mathbf{G})^{-1}(\gamma'\mathbf{I} + \delta'\mathbf{G})\mathbf{x} = (\sum_{j=1}^n r_j x_j)\boldsymbol{\iota}$. Averaging over the network yields $\varphi(\mathbf{x}, \boldsymbol{\theta}) - \varphi(\mathbf{x}, \boldsymbol{\theta}') = \sum_{j=1}^n r_j x_j$. Therefore, $\forall\mathbf{x}$, $(\mathbf{I} - \beta\mathbf{G})^{-1}(\gamma\mathbf{I} + \delta\mathbf{G})\mathbf{x} - \varphi(\mathbf{x}, \boldsymbol{\theta})\boldsymbol{\iota} = (\mathbf{I} - \beta'\mathbf{G})^{-1}(\gamma'\mathbf{I} + \delta'\mathbf{G})\mathbf{x} - \varphi(\mathbf{x}, \boldsymbol{\theta}')\boldsymbol{\iota}$. This means that $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ have the same reduced form, hence social effects are not identified.

Next, suppose that $\mu_0\mathbf{I} + \mu_1\mathbf{G} + \mu_2\mathbf{G}^2$ has identical rows. Since $\mathbf{G}\boldsymbol{\iota} = \boldsymbol{\iota}$, multiplying by $\mathbf{G}$ leaves the matrix unchanged. Thus, $\mu_0\mathbf{I} + \mu_1\mathbf{G} + \mu_2\mathbf{G}^2 = \mu_0\mathbf{G} + \mu_1\mathbf{G}^2 + \mu_2\mathbf{G}^3$. If $\mathbf{I}$, $\mathbf{G}$, $\mathbf{G}^2$, and $\mathbf{G}^3$ are linearly independent, then $\mu_0 = \mu_1 = \mu_2 = 0$ and social effects are identified. If $\mathbf{G}^3 = \lambda_0\mathbf{I} + \lambda_1\mathbf{G} + \lambda_2\mathbf{G}^2$, and $\mu_2 = 0$, then $\mu_0 = \mu_1 = 0$. If $\mu_2 \neq 0$, we can set $\mu_2 = 1$. This yields $\mu_0 = \lambda_0$ and $\mu_1 = \lambda_0 + \lambda_1$. In other words, either $\lambda_0\mathbf{I} + (\lambda_0 + \lambda_1)\mathbf{G} + \mathbf{G}^2$ has identical rows and social effects are not identified, or it does not have identical rows and social effects are identified. In the first case, introduce $\mathbf{M} = \lambda_0\mathbf{I} + (\lambda_0 + \lambda_1)\mathbf{G} + \mathbf{G}^2$. Notice that $\mathbf{GM} = \lambda_0\mathbf{G} + (\lambda_0 + \lambda_1)\mathbf{G}^2 + \mathbf{G}^3 = \mathbf{M}$. It means that any column $\mathbf{M}_j$ of $\mathbf{M}$ satisfies $\mathbf{GM}_j = \mathbf{M}_j$. Suppose that $\text{rank}(\mathbf{I} - \mathbf{G}) = n - 1$. Then, $\dim Ker(\mathbf{I} - \mathbf{G}) = 1$. Therefore, $\mathbf{M}_j = \xi_j\boldsymbol{\iota}$ and all rows of $\mathbf{M}$ are equal to $(\xi_1, \ldots, \xi_{n_C})$. Thus, social effects are not identified. Conversely, suppose that $\mathbf{M}$ has identical rows. Take a vector $\mathbf{u}$ such that $\mathbf{Gu} = \mathbf{u}$. Then, $\mathbf{Mu} = (2\lambda_0 + \lambda_1 + 1)\mathbf{u}$. Since $\mathbf{M}$ has identical rows, $\mathbf{Mu}$ has identical elements. As soon as $2\lambda_0 + \lambda_1 + 1 \neq 0$, $u_i = u_j$ and $\dim Ker(\mathbf{I} - \mathbf{G}) = 1$, hence $\text{rank}(\mathbf{I} - \mathbf{G}) = n - 1$.

With K characteristics, $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ lead to the same reduced form iff $\forall k$, $(\mathbf{I} - \mathbf{H})(\gamma_k\mathbf{I} + \delta_k\mathbf{G})(\mathbf{I} - \beta\mathbf{G})^{-1} = (\mathbf{I} - \mathbf{H})(\gamma_k'\mathbf{I} + \delta_k'\mathbf{G})(\mathbf{I} - \beta'\mathbf{G})^{-1}$. Suppose that $\delta_{k_0} + \gamma_{k_0}\beta \neq 0$. If $\mathbf{G}$ satisfies the condition of Proposition 5, then $\beta = \beta'$, $\gamma_{k_0} = \gamma_{k_0}'$ and $\delta_{k_0} = \delta_{k_0}'$. For any other k, right multiplying by $\mathbf{I} - \beta\mathbf{G}$ leads to $(\mathbf{I} - \mathbf{H})(\gamma_k\mathbf{I} + \delta_k\mathbf{G}) = (\mathbf{I} - \mathbf{H})(\gamma_k'\mathbf{I} + \delta_k'\mathbf{G})$ hence $\gamma_k = \gamma_k'$ and $\delta_k = \delta_k'$.

Finally, to extend Proposition 5 to a setting with stochastic networks, we just have to replace $n - 1$ by $n - d$ where d is the number of distinct networks in support of the network's distribution and n is the size of the large network $\mathbf{G}$. $\square$

Appendix B. Perfect collinearity and identification

Result 1. *Suppose that $\gamma\beta + \delta \neq 0$ and that no student is isolated. Then,*

(1) $\forall\mathbf{x}$, $\mathbb{E}(\mathbf{Gy}|\mathbf{x}) = \lambda_0\boldsymbol{\iota} + \lambda_1\mathbf{x} + \lambda_2\mathbf{Gx} \Rightarrow \mathbf{G}^2 = \mu_0\mathbf{I} + \mu_1\mathbf{G}$.

(2) $\mathbf{G}^2 = \mu_0\mathbf{I} + \mu_1\mathbf{G}$ *and* $\mu_0\beta \neq -1 \Rightarrow \forall\mathbf{x}$, $\mathbb{E}(\mathbf{Gy}|\mathbf{x}) = \lambda_0\boldsymbol{\iota} + \lambda_1\mathbf{x} + \lambda_2\mathbf{Gx}$.

Proof. Recall, from (1), that $\mathbf{y} = \frac{\alpha}{1-\beta}\boldsymbol{\iota} + (\mathbf{I} - \beta\mathbf{G})^{-1}(\gamma\mathbf{I} + \delta\mathbf{G})\mathbf{x} + (\mathbf{I} - \beta\mathbf{G})^{-1}\boldsymbol{\epsilon}$. Multiplying by $\mathbf{G}$ and taking the expectation yields:

$$\mathbb{E}(\mathbf{Gy}|\mathbf{x}) = \frac{\alpha}{1-\beta}\boldsymbol{\iota} + (\mathbf{I} - \beta\mathbf{G})^{-1}(\gamma\mathbf{G} + \delta\mathbf{G}^2)\mathbf{x}.$$

One can then see that $\forall\mathbf{x}$, $\mathbb{E}(\mathbf{Gy}|\mathbf{x}) = \lambda_0\boldsymbol{\iota} + \lambda_1\mathbf{x} + \lambda_2\mathbf{Gx}$ is equivalent to $\lambda_0 = \frac{\alpha}{1-\beta}$ and

$$\lambda_1\mathbf{I} + (\lambda_2 - \beta\lambda_1 - \gamma)\mathbf{G} - (\beta\lambda_2 + \delta)\mathbf{G}^2 = 0.$$

If $\mathbf{I}$, $\mathbf{G}$, $\mathbf{G}^2$ are linearly independent, then $\lambda_1 = 0$, $\lambda_2 = \gamma$, hence $\beta\gamma + \delta = 0$, which is not possible. This shows that $\forall\mathbf{x}$, $\mathbb{E}(\mathbf{Gy} \mid \mathbf{x}) = \lambda_0\boldsymbol{\iota} + \lambda_1\mathbf{x} + \lambda_2\mathbf{Gx} \Rightarrow \mathbf{G}^2 = \mu_0\mathbf{I} + \mu_1\mathbf{G}$.

Reciprocally, suppose that $\mathbf{G}^2 = \mu_0\mathbf{I} + \mu_1\mathbf{G}$. We want to find λ_1 and λ_2 such that $\beta\lambda_2 + \delta \neq 0$, $\lambda_1 = \mu_0(\beta\lambda_2 + \delta)$ and $\lambda_2 - \beta\lambda_1 - \gamma = \mu_1(\beta\lambda_2 + \delta)$. This is equivalent to $\lambda_1 - \mu_0\beta\lambda_2 = \mu_0\delta$ and $\beta\lambda_1 + (\mu_1\beta - 1)\lambda_2 = -\gamma - \mu_1\delta$. This system has a unique solution in λ_1 and λ_2 if and only if $\mu_1\beta - 1 + \mu_0\beta^2 \neq 0$. Since $\mathbf{G}^2\boldsymbol{\iota} = \boldsymbol{\iota}$, $\mu_0 + \mu_1 = 1$ and the last condition is equivalent to $\mu_0\beta \neq -1$. If $\mu_0\beta = -1$, then the system has no solution since $\gamma\beta + \delta \neq 0$.

With K characteristics, denote by $\mathbf{x}^k$ the vector associated with characteristic k. We can show through the same reasoning that $\forall\mathbf{x}^1, \ldots, \mathbf{x}^K$, $\mathbb{E}(\mathbf{Gy}|\mathbf{x}) = \lambda_0\boldsymbol{\iota} + \sum_{k=1}^K(\lambda_{1k}\mathbf{x}^k + \lambda_{2k}\mathbf{Gx}^k) \Rightarrow \mathbf{G}^2 = \mu_0\mathbf{I} + \mu_1\mathbf{G}$ and that $\mathbf{G}^2 = \mu_0\mathbf{I} + \mu_1\mathbf{G}$ and $\mu_0\beta \neq -1 \Rightarrow \forall\mathbf{x}^1, \ldots, \mathbf{x}^K$, $\mathbb{E}(\mathbf{Gy}|\mathbf{x}) = \lambda_0\boldsymbol{\iota} + \sum_{k=1}^K(\lambda_{1k}\mathbf{x}^k + \lambda_{2k}\mathbf{Gx}^k)$. Observe that here the variables $(\boldsymbol{\iota}, \mathbf{x}^1, \ldots, \mathbf{x}^K, \mathbf{Gx}^1, \ldots, \mathbf{Gx}^K, \mathbf{G}^2\mathbf{x}^1, \ldots, \mathbf{G}^2\mathbf{x}^K, \ldots)$ can be used to instrument for $\mathbf{Gy}$. $\square$

Result 2. *Suppose that $\gamma\beta + \delta \neq 0$ and that no student is isolated. Then,*

(1) $\forall x$, $\mathbb{E}[(\mathbf{I} - \mathbf{G})\mathbf{Gy}|\mathbf{x}] = \lambda_0(\mathbf{I} - \mathbf{G})\mathbf{x} + \lambda_1(\mathbf{I} - \mathbf{G})\mathbf{Gx} \Rightarrow \mathbf{G}^3 = \mu_0\mathbf{I} + \mu_1\mathbf{G} + \mu_2\mathbf{G}^2$.

(2) $\mathbf{G}^3 = \mu_0\mathbf{I} + \mu_1\mathbf{G} + \mu_2\mathbf{G}^2$ *and* $\mu_0\beta(1 + \beta) + \mu_1\beta \neq -1 \Rightarrow \forall x$, $\mathbb{E}[(\mathbf{I} - \mathbf{G})\mathbf{Gy}|\mathbf{x}] = \lambda_0(\mathbf{I} - \mathbf{G})\mathbf{x} + \lambda_1(\mathbf{I} - \mathbf{G})\mathbf{Gx}$.

Proof. Similarly, $\mathbb{E}[(\mathbf{I} - \mathbf{G})\mathbf{Gy}|\mathbf{x}] = (\mathbf{I} - \beta\mathbf{G})^{-1}(\gamma\mathbf{G} + \delta\mathbf{G}^2)(\mathbf{I} - \mathbf{G})\mathbf{x}$. Therefore, one can see that $\forall\mathbf{x}$, $\mathbb{E}[(\mathbf{I} - \mathbf{G})\mathbf{Gy}|\mathbf{x}] = \lambda_0(\mathbf{I} - \mathbf{G})\mathbf{x} + \lambda_1(\mathbf{I} - \mathbf{G})\mathbf{Gx}$ is equivalent to

$$\lambda_0\mathbf{I} + (\lambda_1 - (1 + \beta)\lambda_0 - \gamma)\mathbf{G} + (\gamma - \delta - \lambda_1 - \beta(\lambda_1 - \lambda_0))\mathbf{G}^2 + (\beta\lambda_1 + \delta)\mathbf{G}^3 = 0.$$

If $\mathbf{I}$, $\mathbf{G}$, $\mathbf{G}^2$, $\mathbf{G}^3$ are linearly independent, then $\lambda_0 = 0$, $\lambda_1 = \gamma$, hence $\beta\gamma + \delta = 0$, which is not possible.

Reciprocally, suppose that $\mathbf{G}^3 = \mu_0\mathbf{I} + \mu_1\mathbf{G} + \mu_2\mathbf{G}^2$. We want to find λ_1 and λ_2 such that $\beta\lambda_1 + \delta \neq 0$, $-\lambda_0 = \mu_0(\beta\lambda_1 + \delta)$, $-(\lambda_1 - (1 + \beta)\lambda_0 - \gamma) = \mu_1(\beta\lambda_1 + \delta)$, and $(-\gamma + \delta + \lambda_1 + \beta(\lambda_1 - \lambda_0)) = \mu_2(\beta\lambda_1 + \delta)$. Since $\mu_0 + \mu_1 + \mu_2 = 1$, one can see that the third equation is a simple linear combination of the first two. Thus, only the first two have to be satisfied. They are equivalent to $\lambda_0 + \mu_0\beta\lambda_1 = -\mu_0\delta$ and $(1 + \beta)\lambda_0 - (1 + \mu_1\beta)\lambda_1 = \mu_1\delta - \gamma$. This system as a unique solution if and only if $\mu_0\beta(1 + \beta) + \mu_1\beta \neq -1$. As in the previous result, the generalization to K characteristics is straightforward. Here, the variables $((\mathbf{I} - \mathbf{G})\mathbf{x}^1, \ldots, (\mathbf{I} - \mathbf{G})\mathbf{x}^K, (\mathbf{I} - \mathbf{G})\mathbf{Gx}^1, \ldots, (\mathbf{I} - \mathbf{G})\mathbf{Gx}^K, (\mathbf{I} - \mathbf{G})\mathbf{G}^2\mathbf{x}^1, \ldots, (\mathbf{I} - \mathbf{G})\mathbf{G}^2\mathbf{x}^K, \ldots)$ can be used to instrument for $(\mathbf{I} - \mathbf{G})\mathbf{Gy}$. $\square$

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