Estimate the parameters of a dynamic optimization problem: when to replace engine of a bus? This is a another practical example of an *optimal stopping* problem, which is easily applicable to dynamic economic decisions involving discrete choice variables (0=don't replace engine, 1=replace engine).

Engine replacement problem features a standard tradeoff: (i) there are fixed costs associated with replacing the engine, but new engine has lower associated future maintenance costs; (ii) by not replacing the engine, you avoid the fixed replacement costs, but suffer higher future maintenance costs.

General problem fits into "optimal stopping" framework: there is a "critical" cutoff mileage level x^* below which no replacement takes place, but above which replacement will take place. (Just like in Pakes (1986) patents problem, where there is a cutoff return $\bar{r}$ beyond which patent will be renewed.)

Compare with Pakes (1986):

1. Rust paper is infinite horizon: focus on *stationary* solutions. This means that value function, and optimal decision functions are just functions of the state variable, which is mileage, and not explicitly functions of time.

2. In Pakes problem, once you "stop" (allow your patent to expire) your problem ends. In Rust problem, once you "stop" (ie. replace your engine), your bus becomes good as new, as if your mileage were "reset" to zero. This problem is *regenerative*.

Note: A common distinction between papers with infinite-horizon and finite-horizon DO problems is that stationarity (or *time homogeneity*) is assumed for infinite-horizon problems, and they are solved using value function iteration. Finite-horizon problems are non-stationary, and solved by backward induction starting from the final period.

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BEHAVIORAL MODEL

Control variable:

$$i_t = \begin{cases} 1 & \text{if HZ replaces} \\ 0 & \text{otherwise.} \end{cases}$$

For simplicity. we describe the case where there is only one bus (in the paper, buses are treated as independent entities).

HZ chooses the (infinite) sequence $\{i_1, i_2, i_3, \ldots, i_t, i_{t+1}, \ldots\}$ to maximize discounted expected utility stream:

$$\max_{\{i_1,i_2,i_3,\ldots,i_t,i_{t+1},\ldots\}} E \sum_{t=1}^{\infty} \beta_{t-1} u\left(x_t, i_t; \theta\right) \tag{1}$$

where

- x_t is the state variable of the problem, which is the mileage of the bus. Assume that evolution of mileage is stochastic (from HZ's point of view) and follows

$$x_{t+1} \begin{cases} \sim G(x'|x_t) & \text{if } i_t = 0 \text{ (don't replace engine in period } t\text{)} \\ = 0 & \text{if } i_t = 1\text{: once replaced, bus is good as new} \end{cases} \tag{2}$$

  and $G(x'|x)$ is the conditional probability distribution of next period's mileage x' given that current mileage is x. HZ knows G; econometrician knows the form of G, up to a vector of parameters which are estimated. ($x' - x$ is assumed to follow multinomial distribution, with unknown probabilities, which are estimated.)

- Since mileage evolves randomly, this implies that even given a sequence of replacement choices $\{i_1, i_2, i_3, \ldots, i_t, i_{t+1}, \ldots\}$, the corresponding sequence of mileages $\{x_1, x_2, x_3, \ldots, x_t, x_{t+1}, \ldots\}$ is still random. The expection in Eq. (1) is over this stochastic sequence of mileages.

Define value function:

$$V(x_t) = \max_{i_\tau,\ \tau=t+1,t+2,\ldots} E_t \left[\sum_{\tau=t+1}^{\infty} \beta_{\tau-t} u\left(x_t, i_t; \theta\right) | x_t \right]$$

where maximum is over all possible sequences of $\{i_{t+1}, i_{t+2}, \ldots\}$. Note that we have imposed stationarity, so that the value function $V(\cdot)$ is a function of t only indirectly, through the value that the state variable x takes during period t.

What helps us (as usual) is the principle of optimality, which allows us to break down the DO problem into an (infinite) sequence of single-period decisions, characterized by Bellman's equation:

$$i_t = i^*(x_t; \theta) = \text{argmax}_i \left\{ u(x_t, i; \theta) + \beta E_{x'|x_t} V(x') \right\}$$

where the value function is

$$\begin{aligned} V(x) &= \max_{i=1,0} \left\{ u(x, i; \theta) + \beta E_{x'|x} V(x') \right\} \\ &= \max \left\{ u(x, 0; \theta) + \beta E_{x'|x} V(x'), u(x, 1; \theta) + \beta V(0). \right\} \end{aligned} \tag{3}$$

Parametric assumptions on utility flow:

$$u(x, i; \theta) = -c\left((1-i) * x; \theta\right) - i * RC + \epsilon_i$$

where

- $c(\cdots)$ is the maintenance cost function, which is presumably increasing in x (higher x means higher costs)
- RC denotes the "lumpy" fixed costs of adjustment. The presence of these costs implies that HZ won't want to replace the engine every period.
- ϵ_i, $i = 0, 1$ are structural errors, which represents factors which affect HZ's replacement choice i_t in period t, but are unobserved by the econometrician. As Rust remarks (bottom, pg. 1008), you need this in order to generate a positive likelihood for your observed data. Without these ϵ's, we observed as much as HZ does, and model will not be able to explain situations where (say) mileage was 20,000, but in one case HZ replaces, and in second case HZ doesn't replace.

As remarked earlier, these assumption imply a very simple type of optimal decision rule $i^*(x; \theta)$: in any period t, you replace when $x_t \geq x^*$, where x^* is some optimal cutoff mileage level, which is fixed across all periods t.

Important parameters are (i) parameters of maintenance cost function $c(\cdots)$; (ii) replacement cost RC; and (iii) parameters of mileage transition function $G(x'|x)$.

ECONOMETRIC MODEL

Data: observe $\{i_t, x_t\}$, $t = 1, \ldots, T$ for 62 buses. Treat buses as homogeneous and independent (ie. replacement decision on bus i is not affected by replacement decision on bus j).

Likelihood function:

$$\begin{aligned}
& l\left(x_1, \ldots, x_T, i_t, \ldots, i_T | x_0, i_0; \theta\right) \\
= & \prod_{t=1}^{T} Prob\left(i_t, x_t | x_0, i_0, \ldots, x_{t-1}, i_{t-1}; \theta\right) \\
= & \prod_{t=1}^{T} Prob\left(i_t, x_t | x_{t-1}, i_{t-1}; \theta\right) \\
= & \prod_{t=1}^{T} Prob\left(i_t | x_t; \theta\right) \times Prob\left(x_t | x_{t-1}, i_{t-1}; \theta_3\right).
\end{aligned} \tag{4}$$

The third line arises from the Markov feature of the mileage transition probability (2), and the state-contingent nature of the optimal policy function. The last equality arises due to the **conditional independence** assumption: conditional on x_t, the i_t are independent over time. (Note: conditional on x_t, the source of randomness in i_t is due to the structural shock ϵ_t.)

Given the factorization above, we can estimate in two steps:

1. Estimate θ_3, the parameters of the Markov transition probabilities for mileage, conditional on non-replacement of engine (i.e., $i_t = 0$). (Recall that $x_{t+1} = 0$ wp1 if $i_t = 1$.)

We assume a discrete distribution for $\Delta x_t \equiv x_{t+1} - x_t$, the incremental mileage between any two periods:

$$\Delta x_t = \begin{cases} [0, 5000) & \text{w/prob } p \\ [5000, 10000) & \text{w/prob } q \\ [10000, \infty) & \text{w/prob } 1 - p - q \end{cases}$$

so that $\theta_3 \equiv \{p, q\}$, with $0 < p, q < 1$ and $p + q < 1$.

This first step can be executed separately from the more substantial second step—

2. Estimate θ, parameters of maintenance cost function $c(\cdots)$ and engine replacement costs

Expand the expression for $Prob(i_t = 1|x_t; \theta)$ equals

$$\begin{aligned} & Prob\left\{-c(0;\theta) - RC + \epsilon_{1t} + \beta V(0) > -c(x_t;\theta) + \epsilon_{0t} + \beta E_{x'|x_t, i_t=0} V(x')\right\} \\ =& Prob\left\{\epsilon_{1t} - \epsilon_{0t} > c(0;\theta) - c(x_t;\theta) + \beta\left[EV(x) - V(0)\right] + RC\right\} \\ =& \frac{\exp\left(-c(0;\theta) - RC + \beta V(0)\right)}{\exp\left(-c(0;\theta) - RC + \beta V(0)\right) + \exp\left(-c(x_t;\theta) + \beta EV(x')\right)} \end{aligned}$$

where the last line follows if we assume that ϵ_{1t} and ϵ_{0t} are independent, and each is distributed iid TIEV, also independently over time. This is called a "dynamic logit" model, in the literature.

Define

$$\tilde{V}(x, i) = \begin{cases} V(0) & \text{if } i = 1 \\ V(x) & \text{if } i = 0. \end{cases}$$

Then

$$Prob\left(i_1, \ldots, i_T | x_1, \ldots, x_T; \theta\right) = \prod_t \frac{\exp\left(u(x_t, i_t; \theta) + \beta E\tilde{V}(x', i_t)\right)}{\sum_{i=0,1} \exp\left(u(x_t, i; \theta) + \beta E\tilde{V}(x', i)\right)}. \tag{5}$$

This simplification is mainly due to the logit assumption on the errors, as well as the independence of the errors over time.

ESTIMATION METHOD

The second-step of the estimation procedures is via a "nested fixed point algorithm".

Outer loop: search over different parameter values $\hat{\theta}$.

Inner loop: For $\hat{\theta}$, we need to compute the value function $V(x;\hat{\theta})$. After $V(x;\hat{\theta})$ is obtained, we can compute the LL fxn in Eq. (5).

COMPUTATIONAL DETAILS

Compute value function $V(x;\hat{\theta})$ by iterating over Bellman's equation (3).

Difference is that the HZ problem is stationary, so that the value function is the same each period.

A clever feature in Rust's paper is that he iterates over the *expected* value function $EV(x, i) \equiv E_{x',\epsilon'|x,i} V(x', \epsilon'; \theta)$. The reason for this is that you avoid having to calculate the value function at values of ϵ_0 and ϵ_1, which are additional state variables. He iterates over the following equation (which is Eq. 4.14 in his paper):

$$EV(x, i) = \int_y \log \left\{ \sum_{j \in C(y)} \exp \left[u(y, j; \theta) + \beta EV(y, j) \right] \right\} p(dy|x, i) \tag{6}$$

Somewhat awkward notation: here "EV" denotes a function (*not* the expectation of $V(x, i)$). Here x, i denotes the *previous* period's mileage and replacement choice, and y, j denote the *current* period's mileage and choice (as will be clear below).

This equation can be derived from Bellman's equation (3):

$$\begin{aligned}
V(y, \epsilon; \theta) =& \max_{j \in 0,1} \left[u(y, j; \theta) + \epsilon + \beta EV(y, j) \right] \\
\Rightarrow E_{y,\epsilon} \left[V(y, \epsilon; \theta) \mid x, i \right] \equiv EV(x, i; \theta) =& E_{y,\epsilon|x,i} \left\{ \max_{j \in 0,1} \left[u(y, j; \theta) + \epsilon + \beta EV(y, j) \right] \right\} \\
=& E_{y|x,i} E_{\epsilon|y,x,i} \left\{ \max_{j \in 0,1} \left[u(y, j; \theta) + \epsilon + \beta EV(y, j) \right] \right\} \\
=& E_{y|x,i} \log \left\{ \sum_{j=0,1} \left[u(y, j; \theta) + \beta EV(y, j) \right] \right\} \\
=& \int_y \log \left\{ \sum_{j=0,1} \left[u(y, j; \theta) + \beta EV(y, j) \right] \right\} p(dy|x, i).
\end{aligned}$$

The next-to-last equality follows due to extreme value assumption on the ϵ's, and additivity of utility in the error terms. In the above display, x and i denote the mileage and choice in the "previous" period, and y and j denote the mileage and choice in the "current" period.

The iteration procedure: Let τ index the iterations. Let $EV^\tau(x, i)$ denote the expected value function during the τ-th iteration. Let the values of the state variable x be discretized into a grid of points, which we denote $\vec{r}$.

- $\tau = 0$: Start from an initial guess of the expected value function $EV(x,i)$. Common way is to start with $EV(x,i) = 0$, for all $x \in \vec{r}$, and $i = 0, 1$.
- $\tau = 1$: Use Eq. (6) and $EV^0(x;\theta)$ to calculate, at each $x \in \vec{r}$, and $i \in \{0,1\}$.

$$
\begin{aligned}
V^1(x,i) =& \int_y \log \left\{ \sum_{j \in C(y)} \exp\left[u(y,j;\theta) + \beta EV^0(y,j)\right] \right\} p(dy|x,i) \\
=& p \cdot \int_x^{x+5000} \log \left\{ \sum_{j \in C(y)} \exp\left[u(y,j;\theta) + \beta EV^0(y,j)\right] \right\} dy + \\
& q \cdot \int_{x+5000}^{x+10000} \log\{\cdots\}\, dy + (1-p-q) \cdot \int_{x+10000}^{\infty} \log\{\cdots\}\, dy.
\end{aligned}
$$

Now check: is $EV^1(x,i)$ close to $EV^0(x,i)$? One way is to check whether

$$sup_{x,i}|EV^1(x,i) - EV^0(x,i)| < \eta$$

where η is some very small number (say 0.0001). If so, then you are done. If not, then go to next iteration $\tau = 2$.

References

Pakes, A. (1986): “Patents as Options: Some Estimates of the Value of Holding European Patent Stocks,” *Econometrica*, 54(4), 755–84.

Rust, J. (1987): “Optimal Replacement of GMC Bus Engines: An Empirical Model of Harold Zurcher,” *Econometrica*, 55, 999–1033.