

# A Dynamic Level- $k$ Model in Games

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*Backward induction is perhaps the most widely accepted principle for predicting behavior in extensive-form games. In experiments, however, players frequently violate this principle. An alternative is a 2-parameter “dynamic level- $k$ ” model, where players choose a rule from a rule hierarchy. The rule hierarchy is iteratively defined such that the level- $k$  rule is a best-response to the level- $(k - 1)$  rule and the level- $\infty$  rule corresponds to backward induction. Players choose rules based on their best guesses of others’ rules and use past plays to improve their guesses. The model captures two systematic violations of backward induction, namely limited induction and time unraveling and helps to resolve paradoxical behaviors in the centipede game, repeated prisoner’s dilemma, and chain store game, three canonical games used to demonstrate the failure of backward induction. The dynamic level- $k$  model can also be considered as a tracing procedure for backward induction because the former always converges to the latter in the limit.*

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# I. Introduction

Many economic situations, where players interact with one another over multiple stages, can be modeled as extensive-form games. Economists frequently invoke the principle of backward induction to predict behavior in these games. Under backward induction, players reason backward, replace each subgame by its optimal payoff, always choose rationally at each subgame, and use this iterative process to determine a sequence of optimal actions. Each player follows this procedure betting on others doing the same. This “divide and conquer” algorithm simplifies the game analysis and generates a sharp prediction of game play at every subgame.

However, there are several canonical games showing that people often violate backward induction. One of these games is the so-called “centipede” game (Rosenthal, 1981) (See top panel of Figure 1). In this game, there are 2 players (A and B) and 4 decision stages. Players are endowed with an initial pot of \$5. In Stage I, Player A has the property rights to the pot. He can choose either to end the game by taking 80% of the pot (and leaving the remaining 20% to Player B) or to allow it to double by passing the property rights to Player B. In Stage II, it is now Player B’s turn to make a similar decision. Player B must now decide whether to end the game by taking 80% of \$10 or to let the pot double again by passing the property rights back to Player A. This social exchange process leads to large financial gains as long as both players surrender their property rights at each stage. At Stage IV, Player B can either take 80% of \$40 (i.e., \$32) or pass and be left with 20% of a pot of \$80 (i.e., \$16).

The principle of backward induction generates a sharp prediction for this game by starting the analysis in the very last stage. Player B should take at Stage IV because 80% of \$40 is larger than 20% of \$80. Anticipating this, Player A should take at Stage III since 80% of \$20 is larger than 20% of \$40. Continuing with this line of logic, backward induction makes a surprising prediction: Player A should take immediately in Stage I and Outcome 4 occurs with probability 1 (i.e. Outcomes 0-3 should not occur). Moreover, the same prediction holds even if the game were to continue for more stages and with more dramatic potential gains. For example, the bottom panel of Figure 1 is the same game with 6 stages and the same sharp prediction holds (i.e., Outcome 6 occurs with probability 1). Introspection suggests that this prediction is unlikely to occur. This is so because as long as the game proceeds to

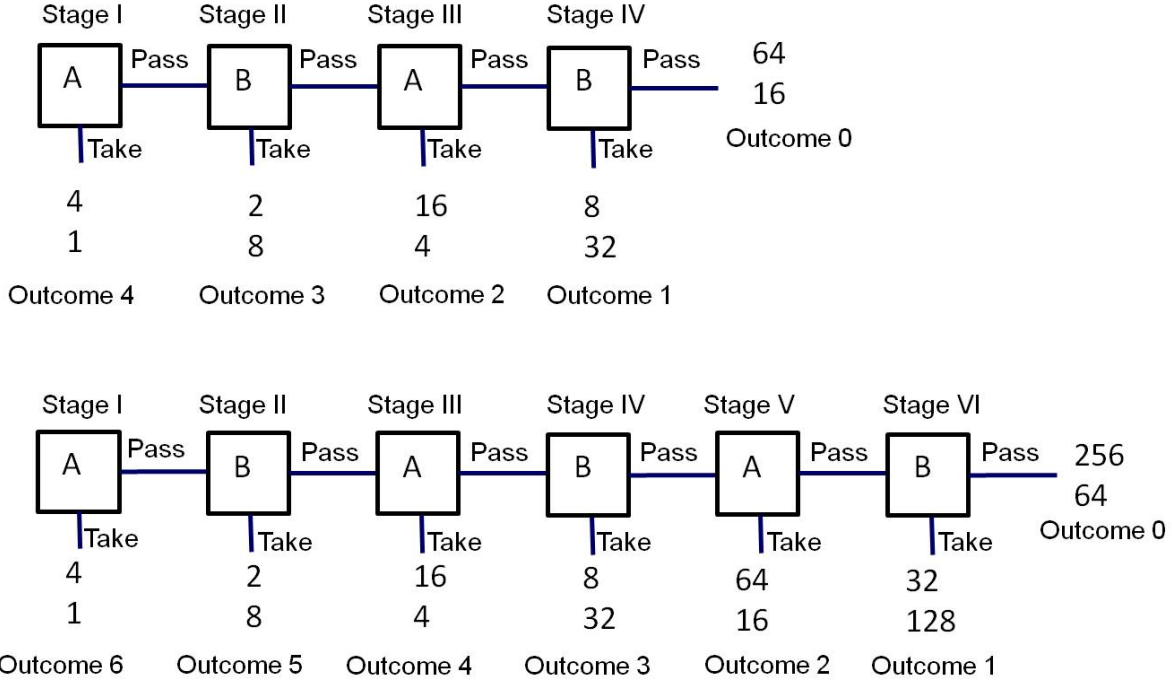


Figure 1: 4-stage (top) and 6-stage (bottom) centipede games.

Stage III, both players would have earned more money than the backward induction outcome.

Indeed, very few subjects (about 6%) choose to take immediately in experimental centipede games (McKelvey and Palfrey, 1992). Many subjects choose to take in the intermediate stages, approximately half way through the games (i.e., leading to Outcomes 2-3 in 4-stage and Outcomes 3-4 in 6-stage games). Clearly, this pattern of behavior runs counter to backward induction. Moreover, the observed behavior frequently leads to higher cash earnings for all subjects. Other well-known extensive-form games where backward induction performs poorly include the repeated prisoner's dilemma (Axelrod, 1984) and the chain store market entry game (Selten, 1978). In the latter two cases, the unique prediction prescribed by backward induction was rarely played especially when subjects were inexperienced.

There are two stylized facts concerning the violations of backward induction. First, players violate backward induction less in an extensive game with fewer subgames (or stages). That is, players' behaviors deviate less from backward induction in simpler extensive-form games. For instance, we observe fewer violations of backward induction in 4-stage than in 6-stage

game (see Figure 1). We call this behavioral tendency *limited induction*. Second, players unravel over time as they play the same extensive-form game repeatedly. That is, players' behaviors converge towards backward induction over time. For instance, we observe fewer violations of backward induction in Round 10 than in Round 1 of the experimental centipede games in Figure 1. This behavioral tendency is termed *time unraveling*. The inability of backward induction to account for the two empirical stylized facts poses modeling challenges for economists.

In this paper, we propose an alternative to backward induction, a “dynamic level- $k$ ” model, that generalizes backward induction and accounts for limited induction and time unraveling. In the dynamic level- $k$  model, players choose a level- $k$  rule,  $L_k$  ( $k = 0, 1, 2, 3 \dots$ ), from a set of iteratively defined rules and the chosen rule prescribes an action at each subgame (Stahl and Wilson, 1995; Stahl, 1996; Ho et al. 1998; Costa-Gomes et al. 2001; Costa-Gomes and Crawford, 2006; Crawford and Iriberri, 2007a, 2007b). Players choose a rule based on their beliefs of others' rules and update their beliefs based on game history (cf. Stahl, 2000). The rule hierarchy is defined such that the level- $k$  rule best-responds to the level- $(k - 1)$  and the level- $\infty$  corresponds to backward induction.

Players are heterogenous in that they have different initial guesses of others' rules and consequently choose different initial rules. The distribution of the initial guesses are assumed to follow a Poisson distribution. These initial guesses are updated based on game history and lead to players choosing different rules over time.

We prove that the dynamic level- $k$  model can account for limited induction and time unraveling properties in the centipede game, the finitely repeated prisoner's dilemma, and the chain store game. Consequently our model can explain passing in the centipede game, cooperation in the repeated prisoner's dilemma, and fighting by the incumbent in the chain-store game. All these behaviors are considered paradoxical under backward induction but are predicted by the dynamic level- $k$  model. In addition, the dynamic level- $k$  model is able to capture the empirical stylized fact that behaviors will eventually converge to backward induction over time, and hence the former can be considered as a tracing procedure for the latter.

We fit our model using experimental data on the centipede game from McKelvey and Palfrey

(1992) and find that our model fits the data significantly better than backward induction and static level- $k$  model. We rule out two alternative explanations including the reputation-based model of Kreps et al. (1982) and a model allowing for social preferences (Fehr and Schmidt, 1999). Overall, it appears that the dynamic level- $k$  model can be an empirical alternative to backward induction.

The rest of the paper is organized as follows. Section II discusses the backward induction principle and its violations. Section III formulates the dynamic level- $k$  model and applies it to explain paradoxical behaviors in centipede game, iterated prisoner's dilemma, and chain store game. Section IV fits the dynamic level- $k$  model to data from experimental centipede game and rules out two alternative explanations. Section V concludes.

## II. Violations of Backward Induction

Backward induction uses an iterative process to determine an optimal action at each subgame. The predictive success of this iterative reasoning process hinges on players' complete confidence in others applying the same logic in arriving at the backward induction outcomes. If players have doubts about others applying backward induction, it is in their best interest to deviate from backward induction's prescription. Indeed, subjects do and profitably so in many experiments.

If a player  $i$  chooses a behavioral rule  $L^i$  that is different from backward induction ( $L_\infty$ ), one would like to develop a formal measure to quantify this deviation. Consider an extensive-form game  $G$  with  $S$  subgames. We can define the deviation for a set of behavioral rules  $L^i (i = 1, \dots, I)$ , one for each player, in game  $G$  as:

$$\delta(L^1, \dots, L^I, G) = \frac{1}{S} \sum_{s=1}^S \left[ \frac{1}{N_s} \sum_{i=1}^{N_s} D_s(L^i, L_\infty) \right], \quad (\text{II.1})$$

where  $D_s(L^i, L_\infty)$  is 1 if player  $i$  chooses an action at subgame  $s$  that is different from the prescription of backward induction and 0 otherwise, and  $N_s$  is the number of players who are active at subgame  $s$ . Note that the measure varies from 0 to 1, where 0 indicates that players' actions perfectly match the predictions of backward induction and 1 indicates that

none of the players' actions agree with the predictions of backward induction.

Let's illustrate the deviation measure using a 4-stage centipede game. Let the behavioral rules adopted by player A and B be  $L^A = \{P, -, T, -\}$  and  $L^B = \{-, P, -, T\}$  respectively (that is, player A will pass in Stage I and take in Stage III, and player B will pass in Stage II and take in Stage IV). Then the game will end in Stage III (i.e., Outcome 2). The deviation will be  $\delta(L^A, L^B, G) = \frac{1}{4}[1 + 1 + 0 + 0] = \frac{1}{2}$ . Similarly, if  $L^A = \{P, -, T, -\}$  and  $L^B = \{-, T, -, T\}$ , then the game will end in Stage Stage II (i.e., Outcome 3). This gives  $\delta(L^A, L^B, G) = \frac{1}{4}[1 + 0 + 0 + 0] = \frac{1}{4}$ , which is smaller. Note that the latter behavioral rules are closer to backward induction than the former behavioral rules.

Using the above deviation measure, we can formally state the two systematic violations of backward induction as follows:

1. *Limited Induction:* Consider two extensive-form games  $G$  and  $G'$  where  $G'$  is a proper subgame of  $G$ . The deviation from backward induction is equal or larger in  $G$  than in  $G'$ . That is, the deviation from backward induction increases in  $S$ . Formally, for a group of players who adopt the same set of behavioral rules  $(L^i, i = 1, \dots, I)$  in games  $G$  and  $G'$ , we have  $\delta(L^1, \dots, L^I, G) \geq \delta(L^1, \dots, L^I, G')$ . Consequently, a good behavioral model must predict a larger deviation in  $G$  than in  $G'$  to be behaviorally plausible.
2. *Time Unraveling:* If an extensive-form game  $G$  is played repeatedly over time, the deviation from backward induction at time  $t$  converges to zero as  $t \rightarrow \infty$ . That is, time unraveling implies  $\delta(L^1(t), \dots, L^I(t), G) \rightarrow 0$  as  $t \rightarrow \infty$ . Therefore, game outcomes will be consistent with backward induction after sufficiently many repetitions.

Let's illustrate limited induction and time unraveling using data from McKelvey and Palfrey (1992). These authors conducted an experiment to study behavior in 4-stage and 6-stage centipede games. Each subject was assigned to one of these games and played the same game in the same role 10 times. For each observed outcome in a game play, we can compute

the deviation from backward induction using Equation (II.1).<sup>1</sup>

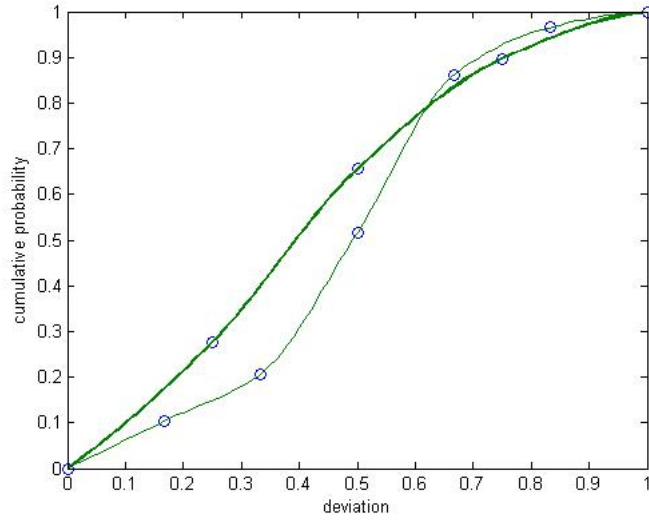


Figure 2: Deviations from backward induction in the first round of the 4-stage and 6-stage centipede games (Data from McKelvey and Palfrey, 1992).

Figure 2 plots the cumulative distributions of deviations from backward induction in the first round of the 4-stage and 6-stage games respectively. The thick line corresponds to the 4-stage game and the thin line corresponds to the 6-stage game. The curve for the 6-stage game generally lies to the right of the curve for the 4-stage game except for high deviation values.<sup>2</sup> The plot is suggestive that the limited induction property holds in this data set.

Figure 3 plots the cumulative distributions of deviations from backward induction in the first and the final round of the 4-stage game. The thick line corresponds to the first round and the thin line corresponds to the final round of game plays (similar results occur for the 6-stage game). As shown, the curve for the first round lies to the right of the curve for the

<sup>1</sup>Since subjects did not indicate what they would have chosen in every stage (i.e., data were not collected using the strategy method), we did not observe what the subjects would have done in subsequent stages if the game ended in an earlier stage. In computing the deviation, we assume that subjects always choose to take in stages beyond where the game ends. Therefore, the derived measure is a conservative estimate of the deviation from backward induction.

<sup>2</sup>The empirical proportions are (0,0.28,0.38,0.24,0.10) for Outcomes 4,3,2,1,0 in the 4-stage game and (0,0.10,0.10,0.31,0.35,0.10,0.03) for Outcomes 6,5,4,3,2,1,0 in the 6-stage game.

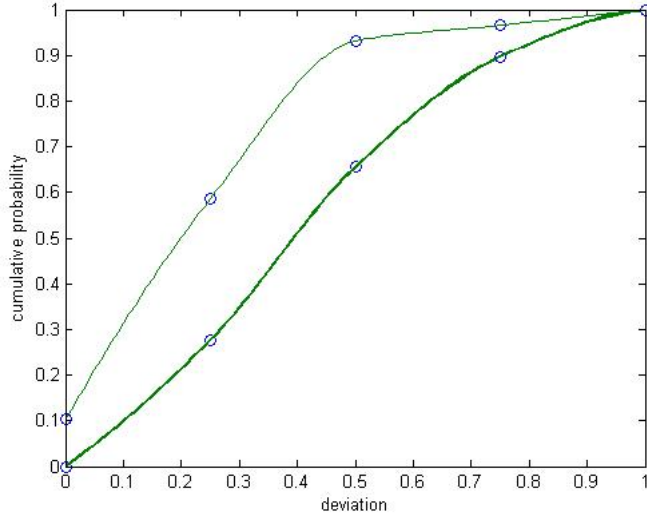


Figure 3: Deviations from backward induction in the first and the last round of the 4-stage centipede game (Data from McKelvey and Palfrey, 1992).

final round.<sup>3</sup> This suggests that the deviation from backward induction decreases over time.

We shall use the deviation measure to establish the main theoretical results below. Specifically we shall show that the deviation under dynamic level- $k$  model is smaller in simpler extensive-form games (i.e., limited induction property holds) and converges to zero over time (i.e., satisfies the time unraveling property).

### III. Dynamic Level- $k$ Model

We consider a 2-player extensive-form game that has  $S$  subgames. Players are indexed by  $i$  ( $i = 1, 2$ ) and subgames by  $s$  ( $s = 1, \dots, S$ ). Players are assumed to adopt a rule which prescribes an action at each subgame  $s$ . For example, in the centipede game studied by McKelvey and Palfrey (1992),  $S$  is either 4 or 6 and a rule Player A adopts in a 4-stage game can be  $L^A = \{P, -, T, -\}$ , which specifies that the player will pass in stage I and take in stage III.

Players choose a rule from a rule hierarchy. Rules are denoted by  $L_k$  ( $k = 0, 1, 2, 3, \dots$ )

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<sup>3</sup>The empirical proportions for Outcomes 4,3,2,1,0 are (0,0.28,0.38,0.24,0.10) and (0.10,0.48,0.35,0.03,0.03) in the first and the last rounds respectively.



and are generated from iterative best-responses. In general,  $L_k$  is a best-response to  $L_{k-1}$  at every subgame, and  $L_\infty$  corresponds to backward induction. Under this rule hierarchy, the deviation from backward induction (see equation (I.1)) is smaller if player  $i$  adopts a higher level rule (while others keep their rule at the same level). In fact, *ceteris paribus*, the deviation from backward induction is monotonically decreasing in  $k$ . In other words, the level of a player's rule captures its closeness to backward induction.

Note that  $L_k$  prescribes the same behavior as backward induction in any extensive-form game with  $k$  or fewer subgames. In this regard,  $L_k$  can be viewed as a limited backward induction rule that only works for simpler games. Putting it this way, a higher level rule is more likely to coincide with backward induction in a wider class of games so that the level of a rule measures its degree of resemblance to backward induction.

If every player is certain about others' rationality, all players will choose  $L_\infty$ . However, if players have doubts about others' rationality, it is not in their best interest to apply backward induction. Instead, they should form beliefs over rules adopted by others<sup>4</sup> and choose a best-response rule in order to maximize their expected payoffs. Therefore, under the model, players are subjective expected utility maximizers. In this regard, the dynamic level- $k$  model is similar to the notion of rationalizable strategic behavior (Bernheim, 1984; Pearce, 1984; Reny, 1992) except that players always choose a rule from the defined rule hierarchy.

In a typical laboratory experiment, players frequently play the same extensive-form game repeatedly. After each game round, players observe the rules used by their opponents and update their beliefs by tracking the frequencies of rules played by opponents in the past. Let player  $i$ 's rule counts at the end of round  $t$  be  $N^i(t) = (N_0^i(t), \dots, N_S^i(t))$  where  $N_k^i(t)$  is the cumulative count of rule  $L_k$  that has been used by opponents at the end of round  $t$  (Camerer and Ho, 1999; Ho et al., 2007). Note that for an extensive-form game with  $S$  subgames, all rules of level  $S$  or higher will prescribe the same action at each subgame and hence we pool them together and collectively call them  $L_S$ . Given these rule counts, player  $i$  forms a belief

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<sup>4</sup>Subjects' beliefs may depend on their knowledge of their opponents' rationality. Players who play against opponents who are known to be sophisticated will adopt a higher level rule. For example, Palacios-Huerta and Volijc (forthcoming) show that many chess players playing against other equally sophisticated chess players take immediately in the centipede game.

$B^i(t) = (B_0^i(t), \dots, B_S^i(t))$  where

$$B_k^i(t) = \frac{N_k^i(t)}{\sum_{k'=0}^S N_{k'}^i(t)}. \quad (\text{III.1})$$

$B_k^i(t)$  is player  $i$ 's belief of the probability that her opponent will play  $L_k$  in round  $t + 1$ . The updating equation of the cumulative count at the end of round  $t$  is given by:

$$N_k^i(t) = N_k^i(t - 1) + I(k, t) \cdot 1, \quad \forall k \quad (\text{III.2})$$

where  $I(k, t) = 1$  if player  $i$ 's opponent adopts rule  $L_k$  in round  $t$  and 0 otherwise. Therefore, players update their beliefs based on the history of game plays. This updating process is consistent with Bayesian updating involving a multinomial distribution with a Dirichlet prior (Fudenberg and Levine, 1998, Camerer and Ho, 1999). As a consequence of the updating process, players may adopt a different best-response rule in round  $t + 1$ .<sup>5</sup>

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<sup>5</sup>The above updating rule assumes that subjects observe rules chosen by opponents. This is possible if the strategy method is used to elicit subjects' contingent action at each subgame. When the opponents' chosen rules are not observed, the updating process is still a good approximation because subjects may have a good guess of their opponents' chosen rules in most simple games (e.g., centipede games). More generally, the updating of  $N_k^i(t)$  depends on whether player  $i$  adopts a higher or lower level rule than her opponent. If the opponent uses a higher level rule (e.g., the opponent takes before the player in the centipede game), then we have like the above:

$$N_k^i(t) = N_k^i(t - 1) + I(k, t) \cdot 1 \quad (\text{III.3})$$

where  $I(k, t) = 1$  if opponent adopts an action that is consistent with  $L_k$  in round  $t$  and 0 otherwise. If player  $i$  adopts a higher level rule  $k^*$  (e.g., takes before the opponent in the centipede game), the player can only infer that the opponent has chosen some rule that is below  $k^*$ . Then we have:

$$N_k^i(t) = N_k^i(t - 1) + I(k \leq k^*) \cdot \frac{N_{k^*}^i(t - 1)}{\sum_{k'=0}^{k^*} N_{k'}^i(t - 1)} \quad (\text{III.4})$$

where  $I(k \leq k^*) = 1$  if  $k \leq k^*$  and 0 otherwise. This updating process assigns a belief weight to all lower level rules that are consistent with the observed outcome. The weight assigned to each consistent rule is proportional to its prior belief weight. For this alternative updating process, the main results for the centipede game (i.e., Theorems 1 and 2) still go through. However, we are not able to prove the same results for the repeated prisoner's dilemma and the chain store game.

Player  $i$  chooses the optimal rule  $L_{k^*}$  in round  $t + 1$  from the rule hierarchy  $\{L_0, L_1, \dots, L_S\}$  based on belief  $B^i(t)$  in order to maximize expected payoffs. Let  $a_{ks}$  be the specified action of rule  $L_k$  at subgame  $s$ . Player  $i$  believes that action  $a_{k's}$  will be chosen with probability  $B_{k'}^i(t)$  by the opponent. Hence, the optimal rule chosen by player  $i$  is:

$$k^* = \operatorname{argmax}_{k=1,\dots,S} \sum_{s=1}^S \left\{ \sum_{k'=1}^S B_{k'}^i(t) \cdot \pi_i(a_{ks}, a_{k's}) \right\}, \quad (\text{III.5})$$

where  $\pi_i(a_{ks}, a_{k's})$  is player  $i$ 's payoff at subgame  $s$  if  $i$  chooses rule  $L_k$  and the opponent chooses  $L_{k'}$  rule (cf. Camerer et al., 2004).

Note that we model learning across game rounds but not across stages within a game round. This is clearly an approximation. For a general extensive-form game, a player could potentially update her belief about the opponent's rule within a round as the game unfolds. Specifically, a player may rule out a potential rule used by her opponent if an observed choice by the opponent at a particular stage is inconsistent with that rule. For example, in the 4-stage centipede game, if player B observes player A passing in stage I, player B will infer that player A's chosen rule must be level 3 or lower. Nevertheless, we believe that the assumption is a good starting point for 2 reasons. First, in the 3 games we study below (centipede game, repeated prisoner's dilemma, and chain-store game), the posterior belief at the end of a game round remains the same whether or not within-round learning is modeled explicitly. This is so because no new information is revealed once a player's opponent chooses an action that is contrary to the prior belief. For instance, in the centipede game, when a player is surprised by an opponent who takes earlier than expected, the game ends immediately and no additional information is revealed.<sup>6</sup> Second, within-round learning frequently generates prediction that is contrary to observed behavior. For example, in the centipede game, the second player who expects the first player to take immediately will be surprised if the latter passes. If we incorporate within-round learning, this will lead the second player to put more weights on the lower level rules. As a consequence, the second player will be more likely to pass, which generally runs counter to the observed data.

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<sup>6</sup>Similarly, in the repeated Prisoner's dilemma, players' subsequent actions after a defection by some player does not contain new information as long as players choose from the rule hierarchy specified below. In the chain store game, once the incumbent shares the market, subsequent actions by entrants will provide no new information as long as they choose from the defined rule hierarchy.

We need to determine player  $i$ 's initial belief  $B^i(0)$ . We define  $N^i(0)$  such that  $N_k^i(0) = \beta$  for some  $k$  and 0 otherwise where the parameter  $\beta$  captures the weight assigned to initial belief. In other words, player  $i$  places all the initial weight on a particular rule  $k$  and zero weight on all other rules. Different players have different guesses about others' rationality and hence place the initial weight on a different  $k$ . A player who places the initial weight on  $k$  will choose an initial rule of  $L_{k+1}$  in round 1. The heterogeneity of players' initial beliefs is captured by a Poisson distribution. The proportion of players who hold initial belief  $k$  is given by:

$$\phi(k) = \frac{e^{-\lambda} \cdot \lambda^k}{k!}; \quad k = 0, \dots, S. \quad (\text{III.6})$$

For example, a  $\phi(0)$  proportion of players believe that their opponents will play  $L_0$  and thus choose  $L_1$ . Similarly, a  $\phi(k)$  proportion of players believe that their opponents will play  $L_k$  and best-respond with  $L_{k+1}$ . All players best-respond given their beliefs in order to maximize their expected payoffs. Note that no players will choose  $L_0$  under our model. That is,  $L_0$  only occurs in the minds of the higher level players.<sup>7</sup>

The dynamic level- $k$  model, similar to the cognitive hierarchy (CH) model, uses a Poisson distribution to capture player heterogeneity. The single parameter Poisson approach allows these models to be tractably used as a building block for theoretical analysis. However, the parsimony of a parametric approach comes with an empirical limitation. Specifically, the Poisson distribution often implies a non-negligible proportion of  $L_0$  players in empirical applications of the CH model (see Camerer et al., 2004). In contrast, using a general discrete distribution, Costa-Gomes and Crawford (2006) and Crawford and Iriberri (2007a, 2007b) show that the estimated proportion of  $L_0$  players is frequently zero. In agreement with the above observation, our proposed dynamic level- $k$  model assumes that there are no  $L_0$  players. Unlike the CH model, which captures heterogeneity in players' rules, the dynamic level- $k$  model captures heterogeneity in players' beliefs of others' rules and allows players to best-respond to their beliefs. For example,  $\phi(0)$  represents the proportion of  $L_0$  players in

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<sup>7</sup>In the centipede game, this implies that the dynamic level- $k$  does not admit occurrence of outcome 0 in 4-stage and 6-stage games, where players pass all the way. No model with only self-interested players can account for such behavior. Our model can easily be extended by incorporating social preferences in order to capture this kind of phenomenon.

the CH model, but it represents, in our model, the proportion of players who believe that their opponents play  $L_0$  and thus choose  $L_1$ . Therefore, our proposed dynamic level- $k$  model excludes  $L_0$  players.<sup>8</sup> In this way, the dynamic level- $k$  model retains the parsimony of the Poisson distribution while improving its empirical validity.

The dynamic level- $k$  model is different from the static level- $k$  and cognitive hierarchy models in 3 fundamental aspects. First, players in our model are not born with a specific thinking type. That is, players in our model are cognitively capable of choosing any rule but always choose the level that maximizes their expected payoff. In other words, players in our model are not constrained by reasoning ability. Second, players in our model may be aware of others who adopt higher level rules than themselves. In other words, a player who chooses  $L_k$  may recognize that there are others who choose  $L_{k+1}$  or higher but still prefers choose  $L_k$  because there is a large majority of players who are  $L_{k-1}$  or below. On the other hand, the static level- $k$  and CH models assume players always believe they are the highest level thinkers (i.e., the opponents are always of a lower level rule). Third, unlike the static level- $k$  and CH models, players in the dynamic level- $k$  model may change their rules as they collect more information and update beliefs about others. Specifically, a player who interacts with opponents of higher level rules may advance to a higher rule. Similarly, a players who interacts with opponents of lower level rules may switch to a lower level rule in order to maximize their expected payoffs.

In summary, the dynamic level- $k$  model has 2 parameters,  $\beta$  and  $\lambda$ . The parameter  $\lambda$  captures the degree of heterogeneity in initial belief and the parameter  $\beta$  captures the strength of the initial belief, which in turn determines players' sensitivity to game history. The 2-parameter dynamic level- $k$  model nests several well-known special cases. When  $\lambda = \infty$ , it reduces to backward induction. If  $\beta = \infty$ , players have a stubborn prior and never respond to game history. This reduces our model to a variant of the static level- $k$  model.<sup>9</sup> Consequently, we can empirically test whether these special cases are good approximations of behavior using

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<sup>8</sup>It is possible to generalize this setup by introducing a segment of non-strategic players (level 0). Let  $\pi$  be the proportion of non-strategic players and  $1 - \pi$  be the proportion of strategic players (level 1 and higher). As described above, the Poisson model can be used to describe the distribution of strategic players. Consequently, the dynamic level- $k$  model is simply a special case with  $\pi = 0$ .

<sup>9</sup>Note that under the dynamic level- $k$  model, while players may believe others are level-0, they themselves never play level-0 rules.

the standard generalized likelihood principle.

Below, we apply the dynamic level- $k$  model to explain violations of backward induction in three canonical games. In each case, we prove that the dynamic level- $k$  model can account for the limited induction and time unraveling properties of the data.

## A. Centipede game

McKelvey and Palfrey (1992) assume there is a proportion of players who always pass in every subgame. To avoid over-fitting, we assume that  $L_0$  corresponds to the same strategy. That is, we choose  $L_0 = \{P_1, P_2, \dots, P_{S-1}, P_S\}$ . A player who believes her opponent uses  $L_0$  will maximize her payoff by adopting  $L_1 = \{P_1, P_2, \dots, P_{S-1}, T_S\}$ . In general,  $L_k$  best-responds to  $L_{k-1}$  so that  $L_k = \{P_1, \dots, P_{S-k}, T_{S-k+1}, \dots, T_S\}$ . Therefore in a centipede game with  $S$  subgames,  $L_S = L_{S+1} = \dots = L_\infty$ . Put differently, all rules  $L_S$  or higher prescribe the same action at each subgame as  $L_\infty$  (the backward induction rule). Consequently, we pool all these higher level rules together and collectively call them  $L_S$ .

Under the dynamic level- $k$  belief model, players  $i$  ( $i = A, B$ ) choose rules from the rule hierarchy  $L_k$  ( $k = 0, 1, 2, 3, \dots$ ). Denote player  $i$ 's rule in time  $t$  as  $L^i(t)$ . Then  $\delta(L^A(0), L^B(0), G)$  is the deviation from backward induction  $L_\infty$  in game  $G$  at time 0. For example, if Players A and B both play  $L_2$ , the deviation is  $\frac{1}{2}$  in a 4-stage and  $\frac{2}{3}$  in a 6-stage centipede game. Let  $G_4$  and  $G_6$  be the 4-stage and 6-stage games respectively. Then we have the following theorem:

**Theorem 1** *In the centipede game, the dynamic level- $k$  model implies that the limited induction property is satisfied. Formally, if the initial distribution of beliefs is the same in both games (i.e., having the same  $\lambda$ ), then the expected deviation from backward induction in  $G_4$ ,  $\delta(L^A(0), L^B(0), G_4)$ , is smaller than in  $G_6$ ,  $\delta(L^A(0), L^B(0), G_6)$ .*

*Proof: See appendix.*

Theorem 1 suggests that the dynamic level- $k$  model gives rise to a smaller deviation from backward induction in a game with a smaller number of subgames. This result is consistent with the data presented in Figure 2. The Appendix gives the detailed proof but the

basic idea of the proof is outlined here. Given any rule combination  $L^A(0)$  and  $L^B(0)$  for the players, the outcome is identical in both games (counting from the last stage) (see Figure 1). Specifically, the game outcome  $z = \max\{L^A(0), L^B(0)\}$  and the number of actions that are inconsistent with backward induction is  $S - z$ . As a consequence,  $G_6$  has a larger deviation than  $G_4$  (see equation (I.1)). Since the initial distribution of beliefs is the same in both games (i.e., having the same  $\lambda$ ), the initial expected deviation must be higher in  $G_6$ .<sup>10</sup>

Figure 3 shows that the expected deviation from backward induction becomes smaller over time. There is a question whether this trend will persist and eventually unravel to backward induction outcome. The following theorem formally states that this is indeed the case for dynamic level- $k$  model.<sup>11</sup>

**Theorem 2** *In a centipede game with  $S$  stages, the dynamic level- $k$  model implies that the time unraveling property is satisfied. Formally, the deviation from backward induction  $\delta(L^A(t), L^B(t), G)$  converges to zero and all players will choose  $L_S = L_\infty$  as  $t \rightarrow \infty$ . That is, players will eventually take in every subgame.*

*Proof:* See appendix.

Theorem 2 states that the dynamic level- $k$  model satisfies the time unraveling property. The basic idea of the proof is outlined here. No players choose  $L_0$ . As a consequence,  $L_1$  players

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<sup>10</sup>Binmore et al. (2002) report an experiment to show that subjects violate backward induction in alternating-offers bargaining games. They identify 2 behavioral properties implied by the principle of backward induction and show that subjects frequently violate them. The first behavioral property is the so-called truncation consistency, which suggests that behaviors should be invariant to replacing a subgame with its equilibrium payoff. In the centipede game, this implies that behaviors in a 4-stage game should be identical to behaviors in the first four stages in a 6-stage game. The experimental data from McKelvey and Palfrey (1992) also strongly rejected this property. The limited induction property predicts this violation and hence can help explain why truncation consistency is violated in alternating-offers bargaining games. The second behavioral property identified is the so-called subgame consistency, which implies that behaviors should be invariant to whether the same game is played independently or part of a larger game. The dynamic level- $k$  model cannot explain violation of subgame consistency in alternating-offers bargaining games. There was, however, no evidence to suggest that subgame consistency is violated in the experimental centipede game (McKelvey and Palfrey, 1992).

<sup>11</sup>Theorem 2 shows that all players will choose  $L_S = L_\infty$  as  $t \rightarrow \infty$ . However, during the transient phase, it is possible for players to switch from  $L_k$  to  $L_{k'}$  ( $k' < k$ ) if they repeatedly encounter opponents of lower level rules. This phenomenon is occasionally observed in the data and can be accommodated by our model.

will learn that other players are  $L_1$  or higher. Using our notation, this means that  $B_0^i(t)$  will decline over time and the speed of decline depends on the initial belief weight  $\beta$ . For a specific  $\beta$ , there is a corresponding number of rounds after which all  $L_1$  players will move up to  $L_2$  or higher. No players will then choose  $L_0$  and  $L_1$ . In the same way,  $L_2$  players will learn that other players are  $L_2$  or higher and will eventually move to  $L_3$  or higher. Consequently, we will see a ‘domino’ phenomenon whereby lower level players will successively disappear from the population. In this regard, players believe that others become more sophisticated over time and correspondingly do so themselves. In the limit, all players converge to  $L_S$  (and learning stops).

The proof also reveals an interesting insight. The number of rounds it takes for  $L_k$  to disappear is increasing in  $k$ . For example, when  $\beta = 1$ , it takes 6 rounds for  $L_0$  to disappear and another 42 rounds for  $L_1$  to disappear. In general, for an initial belief weight  $\beta$ , it takes a total of  $(7^k - 1)\beta$  rounds for rule  $L_k$  to disappear. Note that each higher level rule takes an exponentially longer time to be eliminated from the population.<sup>12</sup> This result suggests that time unraveling occurs rather slowly in the centipede game. In the experiments conducted by McKelvey and Palfrey (1992), there are only 10 game rounds and no substantial learning is observed. In Section III, we fit the dynamic level- $k$  model and show that only  $L_1$  disappears in their data set.

## B. Finitely Repeated Prisoner’s Dilemma

In the finitely repeated prisoner’s dilemma, backward induction prescribes that all players should defect in every repetition or subgame (see Figure 4 for a game with 10 repetitions). This is because defection is a dominant strategy in every subgame. To explain cooperative behavior, Kreps et al. (1982) assume there is a proportion of players who always adopts Tit-For-Tat (TFT), which starts with cooperation and reciprocates by choosing the opponent’s action in the previous round. To avoid over-fitting, we choose  $L_0$  to be TFT. A player who believes her opponent uses  $L_0$  will play TFT until the very last round when it is optimal to

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<sup>12</sup>In our model, the cumulative rule counts  $N_k^i(t)$  do not decay over time. If decay is allowed, that is,  $N_k^i(t) = \delta \cdot N_k^i(t-1) + I(k, t) \cdot 1$ , then unraveling can occur at a much faster rate. For example, if  $\delta = 0$ , it takes only one round for each successively lower level rule to disappear.



defect. We denote this rule by  $L_1 = \{TFT, D\}$ .  $L_1$  is better than TFT because defection yields a payoff of 5 in the last round while cooperation yields a payoff of 3 (see Figure 4).  $\{TFT, D, D\}$  is inferior to  $\{TFT, D\}$  because the former yields a payoff of  $5+1=6$  in the last 2 rounds whereas the latter yields a payoff of  $3+5=8$ . In general,  $L_k$  best-responds to  $L_{k-1}$  so that  $L_k = \{TFT, \overbrace{D, \dots, D}^k\}$ .<sup>13</sup> Therefore, in a repeated prisoner's dilemma game with  $S$  repetitions or subgames,  $L_S = L_{S+1} = \dots = L_\infty$ . Put differently, all rules  $L_S$  or higher prescribe the same actions as backward induction. Consequently, we pool all these higher level rules together and collectively call them  $L_S$ .

	Cooperate	Defect		Cooperate	Defect		Cooperate	Defect	
Cooperate	3,3	0,5	Cooperate	3,3	0,5	.....	Cooperate	3,3	0,5
Defect	5,0	1,1	Defect	5,0	1,1		Defect	5,0	1,1
	Repetition 1			Repetition 2			Repetition 10		

Figure 4: Finitely repeated prisoner's dilemma games (10 repetitions).

**Theorem 3** *In the finitely repeated prisoner's dilemma, the dynamic level- $k$  model implies that the limited induction property is satisfied. Formally, if the initial distribution of beliefs is the same in both games (i.e., having the same  $\lambda$ ), then the expected deviation from backward induction is smaller in a game with a smaller number of repetitions.*

*Proof:* See appendix.

The basic idea of the proof is similar to the one in the centipede game. For any combination of initial rules  $L^A(0)$  and  $L^B(0)$ , let  $z = \max\{L^A(0), L^B(0)\}$ . Then defection begins after

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<sup>13</sup>Choosing cooperation initially followed by defection in the last  $k$  repetitions is also a best-response to  $L_{k-1}$ . However, this simpler rule will perform poorly when played against a higher level rule. Since players are fully aware of the entire rule hierarchy and may believe that others can use a higher level rule, it is in their best interest to adopt a more robust TFT based rule.

$S - z$  repetitions, and both players will choose defection in the remaining  $z - 1$  repetitions. As a consequence, a repeated prisoner's dilemma with a larger number of repetitions will have a longer string of sustained cooperation before defection begins. Thus the expected deviation of backward induction is smaller for a game with fewer repetitions.

**Theorem 4** *In the finitely repeated prisoner's dilemma with  $S$  repetitions, the dynamic level- $k$  model implies that the time unraveling property is satisfied. Formally, the deviation from backward induction converges to zero and all players will choose  $L_S = L_\infty$  as  $t \rightarrow \infty$ . That is, players will eventually defect in every repetition.*

*Proof: See appendix.*

Like before no players choose  $L_0$  (i.e., TFT) so all players will defect in the last stage.  $L_1$  players will learn over time that other players are  $L_1$  or higher and eventually move to  $L_2$  or higher (i.e.,  $L_1$  disappears). After this transition, all players will defect in the second last repetition. The same logic continues and eventually all players will defect in every single repetition. Like before, the number of rounds for  $L_k$  to disappear is increasing in  $k$ . In general, for an initial belief weight  $\beta$ , it takes a total of  $(2^k - 1)\beta$  rounds for rule  $L_k$  to disappear. Note that the rate of convergence is faster for repeated prisoner's dilemma than centipede game. In general, the rate of convergence depends on the payoff structure of the game.

## C. Chain Store Paradox

In the chain store game, an incumbent (e.g., Safeway) faces potential competition from numerous entrants in separate locations and must interact with each one of them sequentially (Selten, 1978). Entrants in each location can either choose to enter the market or stay out. In Figure 5, there are 10 entrants. If an entrant chooses to stay out, the chain store receives a payoff of \$5 m and the entrant receives a payoff of \$1 m. If the entrant enters the market, the chain store must choose to either stage a price war or share the market with the entrant. If the chain store fights, both players receive \$0 m. If the chain store shares, both players receive \$2 m.

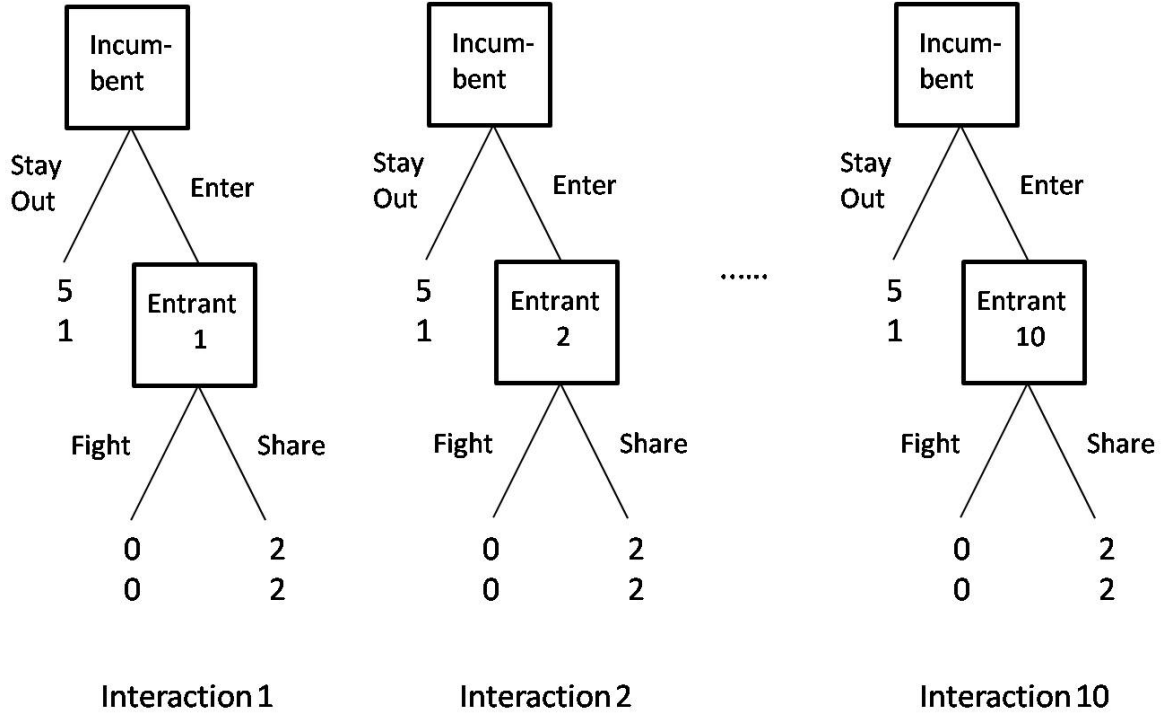


Figure 5: Chain store game with 10 entrants over 10 interactions).

Focusing on the very last entrant, the chain store will choose to share the market with the entrant if it enters. Repeating this logic, backward induction predicts that each entrant will enter at its respective location and the chain store will share the market with each one of them. This seems implausible because introspection suggests that the chain store is likely to fight at least in some of the markets and not all entrants will choose to enter in their locations. Specifically, Selten (1978) wrote:

*The disturbing disagreement between plausible game behavior and game theoretical reasoning constitutes the chain store paradox.*

To explain entry and sharing, Jung et al. (1994) and Camerer et al. (2002) assume that there is a probability that the incumbent is a fighter type, who always fights upon entry. To avoid over-fitting, we choose  $L_0^I$  for the incumbent to be always fight conditional on entrance.  $L_1^E$  entrants who believe the incumbent uses  $L_0^I$  will maximize their payoff by adopting the so-called grim trigger rule (GTR). The GTR rule prescribes that an entrant contemplating entry will choose to stay out unless the chain store is observed to share the market in the past. Given the sequential nature of the game,  $L_1^I$  incumbent will best-respond to  $L_1^E$

entrants by always fighting until the very last interaction when it is optimal to share the market condition on entrance, i.e.,  $\{F, F, \dots, F, S\}$ .

In general,  $L_k^E$  entrants best-respond to a  $L_{k-1}^I$  incumbent and this implies that  $L_k^E = \{GTR, \overbrace{E, \dots, E}^{k-1}\}$ . Similarly, a  $L_k^I$  incumbent best-responds to  $L_{k-1}^E$  entrants and this implies that  $L_k^I = \{F, F, F, \overbrace{S, \dots, S}^k\}$ . Therefore, in a chain store game with  $S$  interactions,  $L_{S+1}^E = \dots = L_\infty$  and  $L_S^I = L_{S+1}^I = \dots = L_\infty$ . Put differently, all rules with level  $S$  or higher prescribe the same actions as the rule of backward induction. Consequently, we pool all these higher level rules together and collectively call them  $L_S^E$  and  $L_S^I$ .

Note that we have defined two separate rule hierarchies, one for each role, because the incumbent and entrants have a different strategy space. Further, in this game, each interaction is a sequential game where the incumbent chooses only after observing the action of the entrant. Consequently, our rule hierarchies are defined in a zig-zag structure where  $L_k^E$  entrants best-respond to  $L_{k-1}^I$  incumbent and  $L_k^I$  incumbent always best-responds to  $L_k^E$  entrants. Note that the zig-zag structure is for notational convenience and it does not change the main results.

**Theorem 5** *In the chain store game, the dynamic level- $k$  model implies that the limited induction property is satisfied. Formally, if the initial distribution of beliefs is the same in both games (i.e., having the same  $\lambda$ ) and all entrants have the same initial belief (i.e. they assign  $\beta$  to the same  $k$ ), then the expected deviation from backward induction is smaller in a game with a smaller number of interactions.*

*Proof:* See appendix.

Let  $L^I(0)$  and  $L^E(0)$  be the initial rules for incumbent and all entrants respectively. For any combination of rule  $L^I(0)$  and  $L^E(0)$ , the outcome in the last  $z = \max\{L^I(0), L^E(0) - 1\}$  interactions is the same regardless of the total number of interactions. As a consequence, a chain store game with a larger number of repetitions will have a longer string of entrants staying out and the incumbent fighting before entrants begin to enter. Thus the expected deviation of backward induction is smaller for a game with fewer repetitions.

**Theorem 6** *In the chain store game with  $S$  interactions, the dynamic level- $k$  model implies that the time unraveling property is satisfied. Formally, the deviation from backward induction converges to zero and players will choose  $L_S = L_\infty$  as  $t \rightarrow \infty$ . That is, every entrant will eventually enter and incumbent will always share in every interaction.*

*Proof: See appendix.*

The idea of the proof is similar to that in the previous two games. No incumbent will always fight. Given this fact,  $L_1^E$  entrants will learn that the incumbent is  $L_1^I$  or higher and eventually will share in the last interaction. This implies that  $L_1^E$  will disappear. After this happens, the  $L_1^I$  incumbent will learn that entrants are  $L_2^E$  or higher. As a consequence,  $L_1^I$  will eventually disappear. The same logic continues and eventually all entrants will enter and the incumbent will share the market in every single interaction. This is consistent with backward induction.

## D. Properties of Level-0 Rule

The above rule hierarchies are generated from iterated best responses to a specific level-0 rule. To avoid over-fitting, we have used the level-0 rules from the existing literature. For instance, following McKelvey and Palfrey (1992), we use a decision rule that passes in every subgame as level-0 in the centipede game. Similarly, as in Kreps et al. (1982), the Tit-For-Tat rule is used as level-0 in the repeated prisoner's dilemma. Finally, in chain store game, we define always fighting as level-0 for the incumbent (following Jung et al., 1994 and Camerer et al. 2002).

There are 2 ways to formally determine the level-0 rule. First, we can structurally estimate the level-0 rule using the data. One can parameterize the space of possible decision rules and empirically determine the best-fitting level-0 rule. Second, one can develop a theory for choosing a level-0 rule. We explore the latter approach below. Further research can provide a more comprehensive analysis.

A good level-0 rule may satisfy the following 2 attractive properties:

1. Maximize group payoff: A level-0 player always chooses a decision rule that if others do the same will lead to the largest total payoff for the group. For instance, in the centipede game, if every player chooses a decision rule that passes in every stage, the group will achieve the largest possible payoff.
2. Protect individual payoff: While maximizing group payoff, a level-0 player also ensures that the chosen decision rule is robust against continued exploitation by others. For instance, in the repeated prisoner’s dilemma, a level-0 who attempts to maximize group payoff by always cooperating may be exploited by an individual who always chooses to defect. A robust rule like Tit-For-Tat will prevent this kind of exploitation and promote cooperative behavior.

The above two properties do not define a unique level-0 rule in all games. However, they do narrow down the possible candidates for consideration. For the above 3 games that we consider, the chosen level-0 rules satisfy the two properties.

## IV. An Empirical Application to Centipede Game

### A. Dynamic Level- $k$ Model

We use the dynamic level- $k$  model to explain violations of backward induction in experimental centipede game conducted by McKelvey and Palfrey (1992). The authors ran experimental sessions on the 4-stage and 6-stage centipede games shown in Figure 1 using subjects from Caltech (2 sessions) and Pasadena Community College (4 sessions). In each group of subjects, half the sessions were run on the 4-stage game and the other half on the 6-stage game. Each subject played the game either 9 or 10 times. Contrary to backward induction, players did not always take immediately in both 4-stage and 6-stage games. In fact, a large majority passed in the first stage. The distribution of game outcomes is unimodal with the mode occurring at the intermediate outcomes (Outcomes 2 and 3 in 4-stage games, Outcomes 3 and 4 in 6-stage games). Table 1 suggests that Caltech subjects take one stage earlier than PCC subjects in both 4 and 6-stage games. Specifically, the modal outcome is Outcome 3 in 4-stage game and Outcome 4 in 6-stage game in Caltech subject pool while it is Outcome 2 in 4-stage game and Outcome 3 in 6-stage game in PCC subject pool. These

results present considerable challenge to backward induction. We estimate the dynamic level- $k$  model using the data from both subject pools with a goal of explaining their main features.

Outcome	4	3	2	1	0
Caltech ( $N=100$ )	0.06	0.43	0.28	0.14	0.09
PCC ( $N=181$ )	0.08	0.31	0.42	0.16	0.03

Table 1a: A Comparison of Outcomes in 4-stage Games Between Caltech and PCC subjects

Outcome	6	5	4	3	2	1	0
Caltech ( $N=100$ )	0.02	0.09	0.39	0.28	0.20	0.01	0.01
PCC ( $N=181$ )	0	0.05	0.09	0.44	0.28	0.12	0.02

Table 1b: A Comparison of Outcomes in 6-stage Games Between Caltech and PCC subjects

The prediction of dynamic level- $k$  model at each game round is completely parameterized by  $\tau$  and  $\beta$ . Conditional on the two parameters, the model generates players' choice probabilities for each rule  $L_k$  for all game rounds. These choice probabilities in turn determine the distribution of outcomes over time. Let  $P_t^i(k|\tau, \beta)$  denote the model's predicted probability that any player in population  $i$  will choose a rule  $L_k$  in round  $t$ . To generate  $P_t^i(k|\tau, \beta)$ , we consider all possible matches in a large population and determine the fraction of players in population  $i$  choosing each rule  $L_k$  in round  $t$  ( $i = A, B$ ). Put differently, we use a large sample approach such that every player, regardless of her level, is matched repeatedly with many opponents that are drawn from the true distribution of the population. As a consequence, all players of the same level in each population are assumed to choose the same rule in each round. In this regard, this approach does not allow for heterogeneity beyond the initial chosen rules and presents a tough test for our model.<sup>14</sup>

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<sup>14</sup>It is possible to use an alternative approach where every player is matched with one opponent in each round and updates her belief based only on the rule chosen by the opponent. This approach gives rise to a prediction that depends on the combination of the initial types of players. Consequently, subjects with the same initial rules might choose differently across time depending on their matches over game rounds (i.e., we allow heterogeneity across both initial rules and time). This estimation approach may provide a better fit. However, there are a very large number of combinations, which make model estimation challenging.

The dynamic level- $k$  model specifies  $P_t^i(k|\tau, \beta)$  in the following way. In the first round,  $P_1^i(k|\tau, \beta) = \phi(k-1)$ ,  $i = A, B$ . At the end of round  $t$ , players in each population update their prior belief  $B^i(t-1)$  based on their opponents' chosen rule. This updating process gives rise to  $B^i(t)$ . Each player then chooses a rule  $L_k$  that maximizes her expected payoff. If two rules yield the same expected payoffs, players are assumed to always choose the simpler rule (i.e. a lower level rule).<sup>15</sup> We can then compute the choice probabilities  $P_{t+1}^i(k|\tau, \beta)$  in round  $t+1$ .

Let the outcome of a game at time  $t$  be  $O(t)$  which denotes the subgame at which the game ends. For example, if Player A takes immediately,  $O(t) = 6$  and if everyone passes to the end,  $O(t) = 0$  in 6-stage game. Let  $P(O(t))$  be the probability of observing outcome  $O(t)$ . One can then compute the probability of observing a data point  $O(t)$  conditional on  $\tau$  and  $\beta$ . The probability is then

$$P(O(t)) = \left[ \sum_k \sum_{k'} P_t^A(k|\tau, \beta) \cdot P_t^B(k'|\tau, \beta) \cdot I(\max\{k, k'\}, O(t)) \right], \quad (\text{IV.1})$$

where  $I(\max\{k, k'\}, O(t))$  is 1 if  $\max\{k, k'\} = O(t)$  and 0 otherwise.

The dynamic level- $k$  model makes the following sharp predictions in the centipede game depending on the value of  $\beta$  parameter:

1. Players' choice probabilities  $P_t^i(k|\tau, \beta)$  do not change at every round. In fact, they change at most twice within 10 game rounds.
2. For  $\beta \geq \frac{5}{3}$ ,  $P_t^i(k|\tau, \beta) = \phi(k-1)$ ,  $\forall t$ . For  $\frac{5}{3} > \beta \geq \frac{1}{6}$ ,  $P_t^i(k|\tau, \beta)$  changes only once within 10 game rounds. For  $\beta < \frac{1}{6}$ ,  $P_t^i(k|\tau, \beta)$  changes twice within 10 game rounds.
3. The model starts by predicting that outcome 0 (i.e., players pass all the way) occurs with zero probability. At every change of the choice probabilities, the next higher outcome (outcome 1 at the first change and outcome 2 at the second change) will also occur with zero probability.

The dynamic level- $k$  predicts that some game outcomes will occur with zero probability. To facilitate empirical estimation, we need to incorporate an error structure. To avoid

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<sup>15</sup>For example, in the centipede game, for player B,  $L_1 = L_2 = \{-, P, -, T\}$ . We assume that player B always chooses  $L_1$  over  $L_2$ .



specification bias, we use the simplest possible error structure (see Crawford and Iriberri 2007a). We assume a probability  $(1 - \epsilon)$  that the data matches our model prediction and a probability  $\epsilon$  that the observed outcome is uniformly distributed over all possible outcomes. Hence, the likelihood function is given by:

$$L = \prod_t \left[ (1 - \epsilon) \cdot P(O(t)) + \epsilon \cdot \frac{1}{S + 1} \right]. \quad (\text{IV.2})$$

We fit the dynamic level- $k$  model to the data using maximum likelihood estimation. We separately estimate the dynamic level- $k$  model for Caltech and PCC subject pools because of the apparent differences in their behaviors (see Tables 1a-b). However we use the same set of parameters to fit both the 4-stage and 6-stage games.

Table 2 shows the estimation results for Caltech subjects. As shown, the estimated parameters are  $\hat{\lambda} = 1.32, \hat{\beta} = 0.25, \hat{\epsilon} = 0.29$ , and the log-likelihood is  $-305.8$ . The estimated  $\hat{\lambda} = 1.32$  suggests that 27% of the players are level 1, 35% are level 2, 23% are level 3, and the remaining 15% are level 4 or higher. The estimated  $\hat{\beta}$  lies between  $\frac{1}{6}$  and  $\frac{5}{3}$  suggesting that players' choice probabilities change only once. The estimated  $\hat{\beta} = 0.25$  suggests a small initial belief weight, which implies that time unraveling can begin to occur very quickly and in fact after only 2 rounds.<sup>16</sup>

Parameter	Backward Induction	Naive Belief	Static Level $k$	Full Model
$\lambda$	$\infty$	0	1.76	1.32
$\beta$	$\infty$	0.08	$\infty$	0.25
$\epsilon$	1.00	0.97	0.21	0.29
Log Likelihood	-357.1	-355.0	-312.3	-305.8

Table 2: Maximum Likelihood Estimates of Dynamic Level- $k$  Model and Its Special Cases (Caltech Subjects)

Table 3a-b compare the model prediction and the actual data for the first 2 and last 8 rounds in the 4-stage and 6-stage games for Caltech subjects. The best fitted dynamic level- $k$  model

<sup>16</sup>Any value of  $\beta$  between  $\frac{1}{6}$  and  $\frac{1}{3}$  will predict a change after round 2 and we simply report the midpoint of these 2 values.

makes two predictions. First, in both games, Outcome 0 (i.e., passes all the way) should not occur in all rounds. Second, the model prescribes that unraveling occurs, and as a consequence Outcome 1 should not occur after round 2. These predictions were roughly consistent with the data in the following ways:

1. In both games, Outcome 0 occurs infrequently. For example, Outcome 0 occurs less than 1% of the time in 6-stage game. The model however fails to capture a nontrivial proportion of occurrence of outcome 0 in 4-stage game (even though the proportion declines from 15% to 8% after round 2).
2. In 4-stage game, the proportions of Outcomes 0 and 1 decline after 2 rounds. For example, the proportion of Outcome 1 decreases from 25% to 11%. In 6-stage game, Outcomes 0 and 1 rarely occur after round 2 (about 1% of the time).

Outcome	4	3	2	1	0
Rounds 1-2 Data ( $N=20$ )	0	0.35	0.25	0.25	0.15
Prediction ( $\epsilon = 0$ )	0.15	0.32	0.36	0.17	0
Rounds 3-10 Data ( $N=80$ )	0.08	0.45	0.29	0.11	0.08
Prediction ( $\epsilon = 0$ )	0.27	0.52	0.21	0	0

Table 3a: A Comparison of Data and Dynamic Level- $k$  Model Prediction in 4-stage games (Caltech Subjects)

Outcome	6	5	4	3	2	1	0
Rounds 1-2 Data ( $N=20$ )	0	0	0.30	0.25	0.45	0	0
Prediction ( $\epsilon = 0$ )	0.01	0.04	0.13	0.29	0.36	0.17	0
Rounds 8-10 Data ( $N=80$ )	0.03	0.11	0.41	0.29	0.14	0.01	0.01
Prediction ( $\epsilon = 0$ )	0.01	0.06	0.24	0.47	0.21	0	0

Table 3b: A Comparison of Data and Dynamic Level- $k$  Model Prediction in 6-stage games (Caltech Subjects)

Finally, the error rate is around 29% which is not negligible. We believe the value is attributed to the sharp prediction of the dynamic level- $k$  model that both outcome 0 and 1

should not occur after round 2.

We also fit three nested cases of the dynamic level- $k$  model. They are all rejected by the likelihood ratio test. The backward induction model (i.e.,  $\lambda = \beta = \infty$ ) yields a log-likelihood of -357.11, which is strongly rejected in favor of the full model ( $\chi^2 = 102.7$ ). The naive belief model (i.e.,  $\lambda = 0$ ) (which assumes that all players believe that their opponents will always pass) yields an estimated  $\hat{\beta} = 0.08$  and a log-likelihood of -355.00. The model is again strongly rejected with a  $\chi^2 = 98.5$ . The static level- $k$  model corresponds to  $\beta = \infty$ . Since players' initial belief will persist throughout the game plays, they will always choose the same rule  $L_k$  across rounds. This restriction yields a parameter estimate of  $\hat{\lambda} = 1.76$  and provides a reasonable fit with a log-likelihood of -312.29. However, the static model is also rejected by the likelihood ratio test ( $\chi^2 = 13.1, p < 0.01$ ).

Given the MLE estimates of  $\hat{\lambda} = 1.32, \hat{\beta} = 0.25$ , we can generate the dynamic level- $k$  model's predicted frequencies for each of the outcomes. Figure 6 shows the actual and predicted frequencies of each outcome. The top panel shows the results for the 4-stage game and the bottom panel shows the results for the 6-stage game. Backward induction predicts that only Outcome 4 in 4-stage game and Outcome 6 in 6-stage game can occur (i.e., Player A takes immediately). This backward induction prediction is strongly rejected by the data. As shown, the dynamic- $k$  model does a reasonable job in capturing the uni-modal distribution of the outcomes. In addition, the dynamic level- $k$  model is able to capture the two most frequently occurring outcomes. Specifically, the dynamic level- $k$  model correctly predicts that the two most frequently played outcomes are 2 and 3 in 4-stage game and 3 and 4 in 6-stage game.

We also fit our model to the data obtained from PCC subjects. Table 4 shows the estimation results. The dynamic level- $k$  model yields parameter estimates of  $\hat{\lambda} = 1.30$  and  $\hat{\beta} = 1.67$ . This estimated initial belief weight  $\hat{\beta} > \frac{5}{3}$  suggests that unraveling never occurs (i.e., there is no change in the choice probabilities). Therefore, for PCC subjects, the dynamic level- $k$  model delivers identical predictions as the static level- $k$  model (i.e.,  $\beta = \infty$ ), which also yields an estimated  $\hat{\lambda} = 1.30$ . Both the static and dynamic level- $k$  models give a log-likelihood score of -514.75. In contrast, the backward induction prediction ( $\lambda = \beta = \infty$ ) gives a log-likelihood score of -646.47 and the naive belief model ( $\lambda = 0$ ) gives a log-likelihood

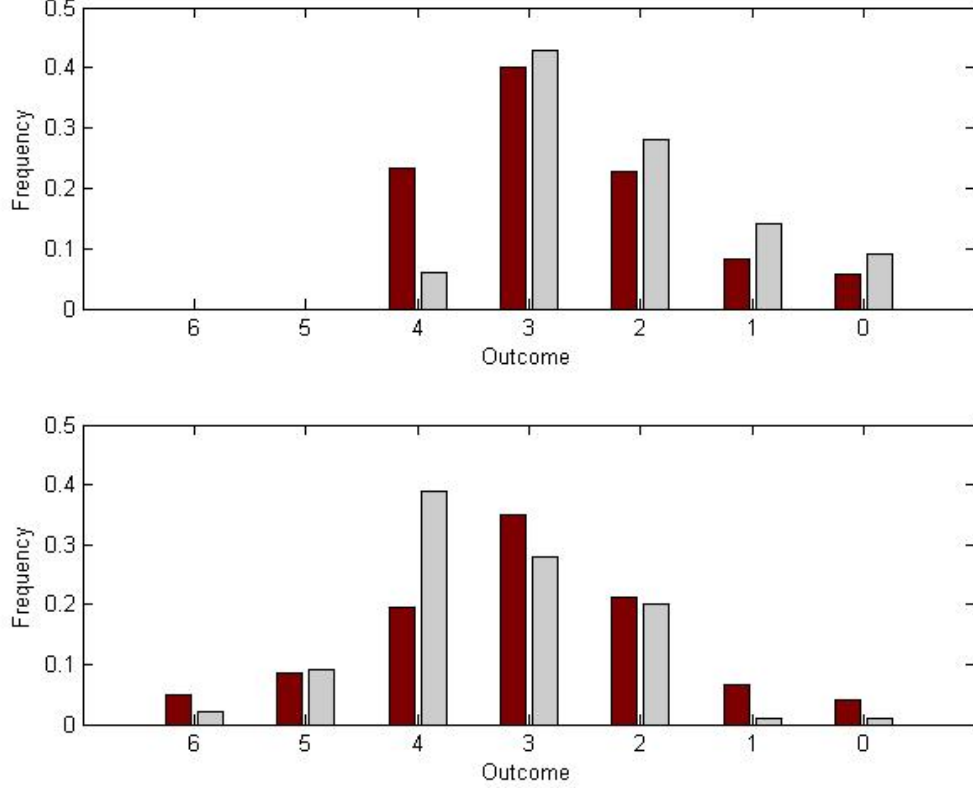


Figure 6: Dynamic Level- $k$  Model Fit (dark bars) and Data (light bars) (Caltech subjects).

score of -644.29, so both models are strongly rejected by the likelihood ratio test. Since the dynamic level- $k$  model predicts no unraveling, it is interesting to examine the actual data to see whether this is indeed the case. Tables 5a-b compare the predictions of the dynamic level- $k$  model with the actual data. Note that PCC subjects exhibit a slower process of time unraveling. For instance, Outcomes 0 and 1 still occur with more than 10% probability in rounds 3-10 in 6-stage game (while the same outcomes occur with only 1% chance in Caltech subjects).

Parameter	Backward Induction	Naive Belief	Static Level $k$	Full Model
$\lambda$	$\infty$	0	1.30	1.30
$\beta$	$\infty$	1.67	$\infty$	1.67
$\epsilon$	1.00	0.99	0.08	0.08
Log Likelihood	-646.5	-644.3	-514.7	-514.7

Table 4: Maximum Likelihood Estimates of Dynamic Level- $k$  Model and Its Special Cases (PCC Subjects)

Outcome	4	3	2	1	0
Rounds 1-2 Data ( $N=38$ )	0.08	0.24	0.42	0.24	0.03
Prediction ( $\epsilon = 0$ )	0.26	0.39	0.27	0.08	0
Rounds 3-10 Data ( $N=143$ )	0.08	0.34	0.42	0.14	0.03
Prediction ( $\epsilon = 0$ )	0.26	0.39	0.27	0.08	0

Table 5a: A Comparison of Data and Dynamic Level- $k$  Model Prediction in 4-stage games (PCC Subjects)

Outcome	6	5	4	3	2	1	0
Rounds 1-2 Data ( $N=38$ )	0	0.13	0.05	0.32	0.32	0.16	0.03
Prediction ( $\epsilon = 0$ )	0.03	0.10	0.20	0.31	0.27	0.08	0
Rounds 8-10 Data ( $N=143$ )	0	0.03	0.10	0.48	0.27	0.10	0.01
Prediction ( $\epsilon = 0$ )	0.03	0.10	0.20	0.31	0.27	0.08	0

Table 5b: A Comparison of Data and Dynamic Level- $k$  Model Prediction in 6-stage games (PCC Subjects)

Finally, given the MLE estimates of  $\hat{\lambda} = 1.30, \hat{\beta} = 1.67$  for PCC subjects, we can generate the dynamic level- $k$  model's predicted frequencies for each of the outcomes. Figure 7 shows the actual and predicted frequencies of each outcome. The top panel shows the results for the 4-stage game and the bottom panel shows the results for the 6-stage game. The plot shows that the dynamic level- $k$  model provides a reasonably good fit of the data.

## B. Reputation-based Model

As indicated before, backward induction does not admit passing behavior in the centipede game. To explain why players pass, one can transform the game into a game of incomplete information by introducing some uncertainty over the player type (Kreps et al., 1982). If there is a small fraction  $\theta$  of players that always pass (the so-called altruists), then even the self-interested players may find it in their best interest to pass initially. However, this

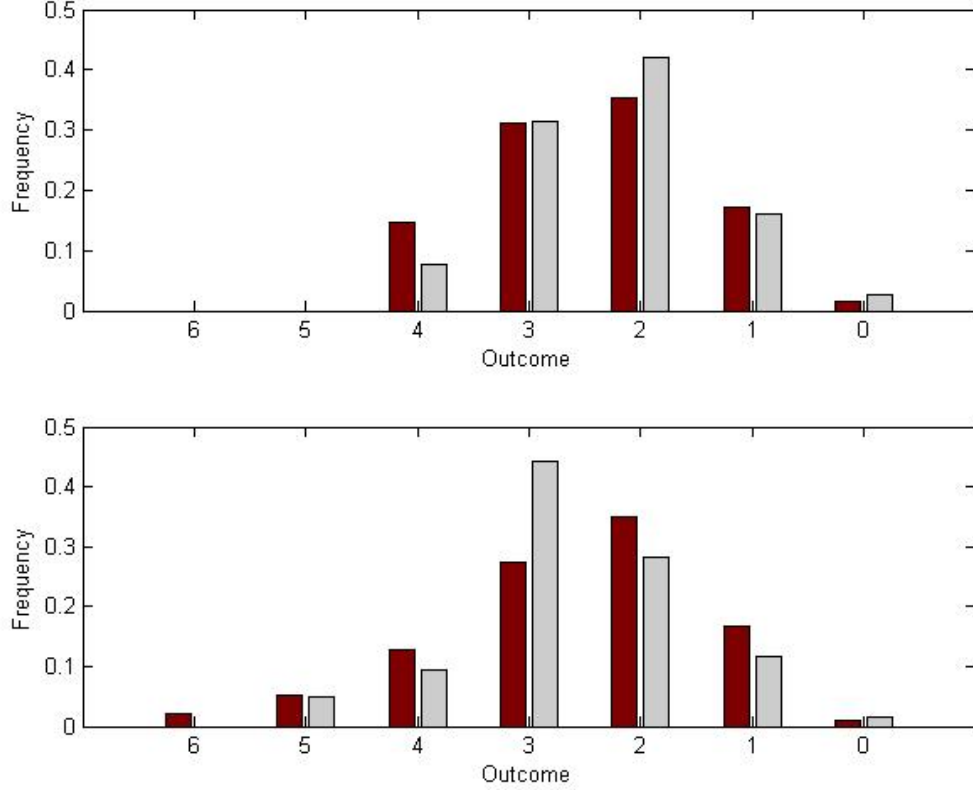


Figure 7: Dynamic Level- $k$  Model Fit (dark bars) and Data (light bars) for PCC subjects.

reputation-based model generates the following empirical implications that run counter to the observed data (see McKelvey and Palfrey, 1992):

1. When  $\theta$  is small ( $0 \leq \theta \leq \frac{1}{49}$ ), the model will not correctly predict the most frequently occurring outcomes. For instance, the model predicts the modal outcome to be Outcome 4 in 4-stage game while the data suggests it should be Outcome 2. Similarly, the model predicts the two most frequently occurring outcomes to be Outcomes 4 and 5 in 6-stage game while the data suggests they should be Outcomes 2 and 3.
2. When  $\theta$  is intermediate ( $\frac{1}{49} < \theta \leq \frac{1}{7}$ ), the model will not admit Outcome 4 and 4-6 in 4-stage and 6-stage respectively.
3. When  $\theta$  is large ( $\theta > \frac{1}{7}$ ), the model predicts players either pass all way (Outcome 0) or take at the last stage (Outcome 1), which is contrary to the data.

The above discussion suggests that there is not a common  $\theta$  value that will explain the unimodal distribution of game outcomes in both the 4-stage and 6-stage games well.

We estimate the reputation-based model using the centipede game data. Like before, we assume that there is a probability  $\epsilon$  that the observed outcome will be evenly distributed over all possible outcomes. The best fitted model yields  $\hat{\theta} = 0.050, \hat{\epsilon} = 0.62$  for Caltech subjects and  $\hat{\theta} = 0.075, \hat{\epsilon} = 0.31$  for PCC subjects. The log-likelihood scores are -329.80 and -518.78 respectively, suggesting that the model did a worse job in fitting the data compared to the dynamic level- $k$  model.

### C. Models of Social Preferences

Could the unimodal distribution of game outcomes in centipede game be attributed to social preferences? The answer is no. To see this we apply the inequity aversion model of Fehr and Schmidt (1999). Let  $x_i$  and  $x_{-i}$  denote the material payoffs of player  $i$  and opponent  $-i$  respectively. Then, the payoff of player  $i$ ,  $U_i(x_i, x_{-i})$ , is given by:

$$U_i(x_i, x_{-i}) = x_i - \alpha \cdot [x_i - x_{-i}]^+ - \beta \cdot [x_{-i} - x_i]^+ \quad (\text{IV.3})$$

where  $0 < \alpha < 1$  captures a player's aversion to being ahead and  $\beta > 0$  captures a player's aversion to being behind. The parameter  $\alpha$  is between 0 and 1 because a player who is ahead will not give up a \$1 to benefit her opponent less than \$1. One can then solve the game by backward induction given these revised payoff functions.

**Theorem 7** *In the centipede game, social preferences lead to either taking immediately or passing all the way in both the 4-stage and 6-stage games. Specifically, social preferences predicts that if  $3 \cdot \alpha - 6 \cdot \beta < 2$ , players will take immediately; otherwise they will pass all the way.*

*Proof:* See Appendix.

The above theorem states that a model of social preferences only admits either Outcome 0 or 4 in 4-stage game and either Outcome 0 or 6 in 6-stage game. The occurrence of either outcome depends on the fairness parameters  $\alpha$  and  $\beta$ . If players are sufficiently averse to being behind (high  $\beta$ ), they will take immediately, which is consistent to backward induction. On the other hand, if players are sufficiently averse to being ahead (high  $\alpha$ ), they will

pass all the way. Both predictions are inconsistent with the unimodal distribution of actual game outcomes.

We estimate the social preference model using the centipede game data. Note that the model fit is identical to the backward induction model. If  $3 \cdot \alpha - 6 \cdot \beta \leq 2$ , the model reduces to backward induction, so the best fit yields a log-likelihood of -357.11 for Caltech subjects and -646.47 for PCC subjects. If  $3 \cdot \alpha - 6 \cdot \beta \geq 2$ , the model predicts that players pass all the way, so the best fit yields again a log-likelihood of -357.11 and -646.47. Hence the model perform poorly compared to the dynamic level- $k$  model.

## V. Conclusions

In economic experiments, backward induction is frequently violated. We develop a dynamic level- $k$  model to explain two systematic violations of backward induction. First, players tend to deviate more from backward induction in games with a greater number of stages or sub-games. Our model captures this limited induction by allowing players to have heterogeneous initial beliefs about others' rationality and hence adopt different rules from a rule hierarchy. Second, players move closer to backward induction over time. Our model captures this time unraveling by allowing players to update beliefs of their opponents' rules and hence adjust their own rules over time. We show that this adjustment process leads to convergence towards backward induction.

We have applied the dynamic level- $k$  model to 3 canonical games (centipede, finitely repeated prisoner's dilemma, and chain store games) that demonstrate the limitations of backward induction. In all three games, we prove that the limited induction and time unraveling properties hold. Limited induction holds because the same rule always has a higher deviation from backward induction in a game with more stages. Time unraveling occurs because the dynamic level- $k$  model implies a domino effect over time as lower level rules are successively eliminated.

We fit our model to experimental centipede games of McKelvey and Palfrey (1992). Our estimation results show that the dynamic level- $k$  model captures the uni-modal distribution of game outcomes in both 4-stage and 6-stage games reasonably well. Special cases including



backward induction and static level- $k$  model are strongly rejected by the data. Interestingly, Caltech subjects, who are arguably more sophisticated, tend to learn faster than PCC subjects.

We rule out 2 alternative explanations for the observed phenomena in the centipede game. First, we show that the reputation-based model of Kreps et al. (1982) cannot capture the uni-modal distribution of game outcomes in the manner consistent with the data. A limitation of the reputation-based model is that there is no common fraction of the altruistic players that will simultaneously account for the degree of passing in both the 4-stage and 6-stage games. Second, we show that social preferences in the form of inequity aversion (Fehr and Schmidt 1999) lead to either taking immediately or passing all the way. Consequently, it cannot explain the uni-modal distribution of game outcomes in both games. Also, a model of social preferences cannot capture both the limited induction and time unraveling properties.

Our model is a generalization of backward induction. Since the model converges to backward induction in the limit, it can be conceptualized as a tracing procedure for backward induction, hence providing a dynamic foundation for backward induction. Framing it this way, the violations of backward induction in the experiments are simply “transient” behaviors and our model explicitly characterizes that trajectory.

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## Appendix A: Proofs

**Proof of Theorem 1** Consider 2 centipede games with  $s$  and  $S$  subgames, where  $s < S$ . Suppose that the proportion of players who choose each rule level is the same in both games. Let  $\pi_k$  denote the proportion of players who choose rule  $L_k$ . Let  $E\delta(S)$  denote the expected deviation in a game with  $S$  subgames.

Suppose Player A adopts a level- $k_A$  and Player B adopts a level- $k_B$  rule in a game with  $S$  stages. Then, the deviation is

$$\delta(L_{k_A}, L_{k_B}, S) = \frac{1}{S} \left[ S - \left( \frac{\min(\lfloor k_A/2 \rfloor, S)}{2} + \frac{\min(\lceil k_B/2 \rceil, S)}{2} \right) \right],$$

which is weakly increasing in  $S$  for every  $k_A$  and  $k_B$ .

Now, comparing the expected deviations in both games, we have

$$E\delta(S) - E\delta(s) = \sum_{k_A=1}^S \sum_{k_B=1}^S \pi_{k_A} \pi_{k_B} [\delta(L_{k_A}, L_{k_B}, S) - \delta(L_{k_A}, L_{k_B}, s)] \geq 0,$$

which gives the required result. ■

**Proof of Theorem 2** Consider any player in the centipede game. For concreteness, we focus on the 4-stage game (see Figure 1) but the proof proceeds in the same way for any number of stages. Suppose this player holds beliefs  $B(t) = (b_0, b_1, \dots, b_4)$  at time  $t$ , where  $b_k$  denotes this player's belief of the probability that the opponent will use rule  $L_k$ . Let  $\mathbf{b}_j^k \equiv \sum_{i=j}^k b_i$ . Given these beliefs, the player chooses a best-response rule to maximize expected payoffs.

Let  $V_s^i$  denote Player  $i$ 's expected payoff, when there are  $s$  stages remaining, from following an optimal strategy henceforth. In the last stage, Player B always takes regardless of beliefs since taking yields 32 a payoff of but passing yields only a payoff of 16. Therefore,  $V_1^B = 32$ . In the second-last stage, Player A's expected payoff is 16 from taking and  $64b_0 + 8(1 - b_0)$  from passing (because after passing, Player B will pass again if and only if level-0 rule is used, which occurs with probability  $b_0$ ). Thus,  $V_2^A = \max\{16, 64b_0 + 8(1 - b_0)\}$  and Player A takes if and only if  $b_0 \leq 1/7$ . Similarly, in the third-last stage, Player B's expected payoff is 8 from taking and  $V_1^B \mathbf{b}_0^1 + 4(1 - \mathbf{b}_0^1) = 32\mathbf{b}_0^1 + 4(1 - \mathbf{b}_0^1)$  from passing, so  $V_3^B = \max\{8, 32\mathbf{b}_0^1 + 4(1 - \mathbf{b}_0^1)\}$  and Player B takes if and only if  $\mathbf{b}_0^1 \leq 1/7$ . Finally,

in the first stage of the 4-stage game, Player A's expected payoff is 4 from taking and  $V_2^A \mathbf{b}_0^2 + 2(1 - \mathbf{b}_0^2)$  from passing. Note that if  $\mathbf{b}_0^2 \leq 1/7$ , we must have  $b_0 \leq \mathbf{b}_0^2 \leq 1/7$  and  $V_2^A = 16$ , so  $V_2^A \mathbf{b}_0^2 + 2(1 - \mathbf{b}_0^2) \leq 4$  and Player A takes; but if  $\mathbf{b}_0^2 \geq 1/7$ , we must have  $V_2^A \mathbf{b}_0^2 + 2(1 - \mathbf{b}_0^2) \geq 16\mathbf{b}_0^2 + 2(1 - \mathbf{b}_0^2) \geq 4$  so Player A passes. Therefore, Player A takes if and only if  $\mathbf{b}_0^2 \leq 1/7$ . Applying this logic, observe that in the  $s$ -th last stage of the centipede game (where  $s \geq 2$ ), the player will take if and only if  $\mathbf{b}_0^{s-2} \leq 1/7$ . Therefore, given beliefs  $B(t) = (b_0, b_1, \dots, b_4)$ , the best response rule  $L_{k^*}$  must be such that  $k^*$  is the largest integer satisfying  $\mathbf{b}_0^{k^*-2} \leq 1/7$ .

In the model, each player starts at  $t = 0$  with a belief weight of  $\beta$  on some  $k$  and adds a unit weight at each round  $t$ . Note that no player will choose  $L_0$ , so the weights will be added to levels  $k \geq 1$  in each round. Therefore, at every  $t \geq 6\beta$ , we must have  $b_0 \leq 1/7$ . In other words, after round  $t = 6\beta$ , no player will choose  $L_0$  or  $L_1$ , so the weights will be added to levels  $k \geq 2$  in subsequent rounds. Similarly, after another  $6 \cdot 7\beta$  rounds, at every  $t \geq (6 + 6 \cdot 7)\beta$ , we must have  $\mathbf{b}_0^1 \leq 1/7$ . Continuing with this logic, note that at every  $t \geq (7^k - 1)\beta$ , we must have  $\mathbf{b}_0^{k-1} \leq 1/7$ . Therefore, in a  $S$ -stage game, at every  $t \geq (7^{S-1} - 1)\beta$ , we have  $\mathbf{b}_0^{S-2} \leq 1/7$ , so all players use  $L_S = L_\infty$ .

■

**Proof of Theorem 3** Consider 2 repeated prisoner's dilemma games with  $s$  and  $S$  repetitions, where  $\Delta \equiv S - s > 0$ . Suppose that the proportion of players who choose each rule level is the same in both games. Let  $\pi_k$  denote the proportion of players who choose rule  $L_k$ . Let  $E\delta(S)$  denote the expected deviation in a game with  $S$  repetitions.

Suppose the players adopt rules of levels  $k_1$  and  $k_2$ . Denote  $\bar{k} = \max(k_1, k_2)$  and  $\underline{k} = \min(k_1, k_2)$ . Then, the deviation is

$$\delta(L_{k_1}, L_{k_2}, S) = \frac{1}{S} \left[ S - \min(\bar{k}, S) + \frac{\mathbf{1}_{\{k_1 \neq k_2\}}}{2} \right]$$

because mutual cooperation is sustained in the first  $S - \min(\bar{k}, S)$  repetitions and one of the players continues to cooperate in the next round if  $k_1 \neq k_2$ .

Next, we consider the above deviation in several special cases. Denote the sets  $A = \{0, 1, \dots, s-1\}$ ,  $B = \{s, s+1, \dots, S-1\}$ ,  $C = \{S, S+1, \dots\}$ . If  $k_1 = k_2 \equiv k$ , we have the

following cases:

$$\begin{aligned}
k \in A &\Rightarrow \delta(S) = \frac{s - k + \Delta}{s + \Delta}, \quad \delta(s) = \frac{s - k}{s} \\
k \in B &\Rightarrow \delta(S) = \frac{s - k + \Delta}{s + \Delta}, \quad \delta(s) = 0 \\
k \in C &\Rightarrow \delta(S) = \delta(s) = 0
\end{aligned}$$

If  $k_1 \neq k_2$ , we have the following cases:

$$\begin{aligned}
\underline{k} \in A, \bar{k} \in A &\Rightarrow \delta(S) = \frac{s - \bar{k} + 1/2 + \Delta}{s + \Delta}, \quad \delta(s) = \frac{s - \bar{k} + 1/2}{s} \\
\underline{k} \in A, \bar{k} \in B &\Rightarrow \delta(S) = \frac{s - \bar{k} + 1/2}{S} \geq \frac{1/2}{S}, \quad \delta(s) = \frac{1/2}{s} \\
\underline{k} \in A, \bar{k} \in C &\Rightarrow \delta(S) = \frac{1/2}{S}, \quad \delta(s) = \frac{1/2}{s} \\
\underline{k} \in B \cup C &\Rightarrow \delta(S) \geq \delta(s) = 0
\end{aligned}$$

Finally, we compute the expected deviations in both games. We obtain:

$$\begin{aligned}
E\delta(S) - E\delta(s) &= \sum_{k_1=1}^S \sum_{k_2=1}^S \pi_{k_1} \pi_{k_2} [\delta(L_{k_1}, L_{k_2}, S) - \delta(L_{k_1}, L_{k_2}, s)] \\
&\geq \sum_{k_1, k_2 \in A} \pi_{k_1} \pi_{k_2} [\delta(L_{k_1}, L_{k_2}, S) - \delta(L_{k_1}, L_{k_2}, s)] \\
&\quad + \sum_{\underline{k} \in A, \bar{k} \in B \cup C} \pi_{k_1} \pi_{k_2} [\delta(L_{k_1}, L_{k_2}, S) - \delta(L_{k_1}, L_{k_2}, s)] \\
&\geq \sum_{k_1, k_2 \in A} \pi_{k_1} \pi_{k_2} \left[ \frac{s - \bar{k} + \mathbf{1}_{\{k_1 \neq k_2\}}/2 + \Delta}{s + \Delta} - \frac{s - \bar{k} + \mathbf{1}_{\{k_1 \neq k_2\}}/2}{s} \right] \\
&\quad + \sum_{\underline{k} \in A, \bar{k} \in B \cup C} \pi_{k_1} \pi_{k_2} \left[ \frac{1/2}{S} - \frac{1/2}{s} \right] \\
&\geq \sum_{k_1, k_2 \in A} \pi_{k_1} \pi_{k_2} \left[ \frac{s - \bar{k} + \mathbf{1}_{\{k_1 \neq k_2\}}/2 + \Delta}{s + \Delta} - \frac{s - \bar{k} + \mathbf{1}_{\{k_1 \neq k_2\}}/2}{s} \right] \\
&\quad + \sum_{k_1, k_2 \in A} \pi_{k_1} \pi_{k_2} \left[ \frac{1}{S} - \frac{1}{s} \right] \\
&= \sum_{k_1, k_2 \in A} \pi_{k_1} \pi_{k_2} \left[ \frac{s - \bar{k} + \mathbf{1}_{\{k_1 \neq k_2\}}/2 + \Delta + 1}{s + \Delta} - \frac{s - \bar{k} + \mathbf{1}_{\{k_1 \neq k_2\}}/2 + 1}{s} \right] \\
&\geq 0.
\end{aligned}$$

The second last inequality holds because  $\text{Prob}\{\underline{k} \in A, \bar{k} \in B \cup C\} \geq 2 \cdot \text{Prob}\{k_1, k_2 \in A\}$ , and the last inequality holds because  $\frac{s - \bar{k} + \mathbf{1}_{\{k_1 \neq k_2\}}/2 + 1}{s} \leq 1$  whenever  $k_1, k_2 \in A$ . Therefore, the expected deviation is higher in a game with more repetitions.

■

**Proof of Theorem 4** Consider any player in the repeated prisoner's dilemma with  $S$  repetitions. Suppose this player holds beliefs  $B(t) = (b_0, b_1, \dots, b_S)$  at time  $t$ , where  $b_k$  denotes this player's belief of the probability that the opponent will use rule  $L_k$ . Let  $\mathbf{b}_j^k \equiv \sum_{i=j}^k b_i$ . Given these beliefs, the player chooses a best-response rule to maximize expected payoffs.

Let  $V_s$  denote the player's expected continuation payoff from following an optimal strategy henceforth, when there are  $s$  stages remaining and mutual cooperation was sustained for all previous  $S - s$  stages.

In the last repetition, the player always defects (since this is a dominant strategy), so  $V_1 = 5b_0 + 1 \cdot (1 - b_0)$ , since there is a probability of  $b_0$  that the opponent will use rule  $L_0$  and cooperate in the last repetition. In general, in the  $s$ -th last repetition, the player's expected continuation payoff from defecting is  $D_s \equiv (5 + (s - 1))\mathbf{b}_0^{s-1} + s(1 - \mathbf{b}_0^{s-1})$  because the opponent cooperates in this repetition with probability  $\mathbf{b}_0^{s-1}$  and the payoff in all subsequent repetitions will be 1; similarly, the player's expected continuation payoff from cooperating is  $C_s \equiv (3 + V_{s-1})\mathbf{b}_0^{s-1} + (s - 1)(1 - \mathbf{b}_0^{s-1})$ . Therefore, for every value of  $s$ , we obtain  $V_s = \max\{D_s, C_s\} = \max\{(5 + (s - 1))\mathbf{b}_0^{s-1} + s(1 - \mathbf{b}_0^{s-1}), (3 + V_{s-1})\mathbf{b}_0^{s-1} + (s - 1)(1 - \mathbf{b}_0^{s-1})\}$ .

Next, we show that the intermediate result that  $V_{s+1} - V_s \geq 1$  for every  $s \geq 1$ . To see this, first, note that  $V_2 - 1 \geq V_1$  because  $V_2 - 1 \geq D_2 - 1 = [5\mathbf{b}_0^1 + (1 - \mathbf{b}_0^1)] \geq V_1$ , so the statement above holds for  $s = 1$ . Second, note that if  $V_s - V_{s-1} \geq 1$ , it then follows that

$$\begin{aligned} C_{s+1} - C_s &= [(3 + V_s)\mathbf{b}_0^s + s(1 - \mathbf{b}_0^s)] - [(3 + V_{s-1})\mathbf{b}_0^{s-1} + (s - 1)(1 - \mathbf{b}_0^{s-1})] \\ &\geq [(3 + V_s)\mathbf{b}_0^{s-1} + s(1 - \mathbf{b}_0^{s-1})] - [(3 + V_{s-1})\mathbf{b}_0^{s-1} + (s - 1)(1 - \mathbf{b}_0^{s-1})] \\ &= [(V_s - V_{s-1})\mathbf{b}_0^{s-1} + (1 - \mathbf{b}_0^{s-1})] \\ &\geq 1, \end{aligned}$$

where the first inequality above holds because  $3 + V_s \geq s$ . The above result, together with the fact that  $D_{s+1} - D_s \geq 1$ , implies that  $V_{s+1} - V_s \geq 1$ . Therefore, by induction, we have  $V_{s+1} - V_s \geq 1$  for every  $s \geq 1$ . Note that this result implies that  $V_s - s$  is increasing in  $s$ .

When there are  $s$  repetitions remaining, the player cooperates if and only if

$$\begin{aligned}
C_s &\geq D_s \\
(3 + V_{s-1})\mathbf{b}_0^{s-1} + (s-1)(1 - \mathbf{b}_0^{s-1}) &\geq (5 + (s-1))\mathbf{b}_0^{s-1} + s(1 - \mathbf{b}_0^{s-1}) \\
V_{s-1}\mathbf{b}_0^{s-1} &\geq (2 + (s-1))\mathbf{b}_0^{s-1} + (1 - \mathbf{b}_0^{s-1}) \\
(V_{s-1} - s)\mathbf{b}_0^{s-1} &\geq 1.
\end{aligned}$$

Since both  $V_s - s$  and  $\mathbf{b}_0^s$  are increasing in  $s$ , the player's optimal rule given beliefs  $B(t) = (b_0, b_1, \dots, b_S)$  must be  $L_{k^*}$ , where  $k^* + 1$  is the smallest integer  $s$  satisfying  $(V_{s-1} - s)\mathbf{b}_0^{s-1} \geq 1$ .

In the model, each player starts at  $t = 0$  with a belief weight of  $\beta$  on some  $k$  and adds a unit weight at each round  $t$ . Note that no player will choose  $L_0$ , so the weights will be added to levels  $k \geq 1$  in each round. Therefore, at every  $t \geq \beta$ , we must have  $b_0 \leq 1/2$ , which implies  $V_1 = 5b_0 + 1 \cdot (1 - b_0) \leq 3$  and hence  $(V_1 - 2)\mathbf{b}_0^1 \leq V_1 - 2 \leq 1$ , so  $C_2 \leq D_2$ . In other words, after round  $t = \beta$ , no player will choose  $L_0$  or  $L_1$ , so the weights will be added to levels  $k \geq 2$  in subsequent rounds. Similarly, after another  $2\beta$  rounds, at every  $t \geq 3\beta$ , we must have  $\mathbf{b}_0^1 \leq 1/2$ . This then implies that  $V_2 = D_2 = 6\mathbf{b}_0^1 + 2(1 - \mathbf{b}_0^1) \leq 4$  and hence  $(V_2 - 3)\mathbf{b}_0^2 \leq V_2 - 3 \leq 1$ , so  $C_3 \leq D_3$  and the player will choose rules with level 3 or higher. Continuing with this logic, note that at every  $t \geq (2^k - 1)\beta$ , we must have  $\mathbf{b}_0^{k-1} \leq 1/2$  and the player will choose only rules of level  $k + 1$  or higher. Therefore, in a prisoner's dilemma with  $S$  repetitions, at every  $t \geq (2^{S-1} - 1)\beta$ , we have  $\mathbf{b}_0^{S-2} \leq 1/2$ , so all players use  $L_S = L_\infty$ . ■

**Proof of Theorem 5** Consider 2 chain store games with  $s$  and  $S$  interactions, where  $s < S$ . In each game, assume that the entrants share the same beliefs about the incumbent. In addition, suppose that the proportion of players who choose each rule level is the same in both games. Let  $\pi_k$  denote the proportion of players who choose rule  $L_k$ . Let  $E\delta(S)$  denote the expected deviation in a game with  $S$  subgames.

Suppose the incumbent uses rule level  $k_I$  and the entrants use rule level  $k_E$ . Then the deviation is

$$\delta(L_{k_I}, L_{k_E}, S) = \frac{1}{S} \left\{ [S - k_E]^+ + \frac{[\min(k_E, S) - \min(k_I, S)]^+}{2} \right\}.$$

This is because given the above rules, the entrant chooses to stay out in the first  $[S - k_E]^+$  interactions, and if  $k_E > k_I$  the incumbent will fight and the entrant will enter for the next  $[\min(k_E, S) - \min(k_I, S)]$  rounds.



Consider two cases. If  $k_E \leq k_I$ , the above reduces to

$$\delta(L_{k_I}, L_{k_E}, S) = \begin{cases} 0, & S \leq k_E, \\ \frac{1}{S}(S - k_E), & S > k_E, \end{cases}$$

which is increasing in  $S$ . Similarly, if  $k_E > k_I$ , we have

$$\delta(L_{k_I}, L_{k_E}, S) = \begin{cases} 0, & S \leq k_I, \\ \frac{1}{S}[(S - k_I)/2], & k_I < S \leq k_E, \\ \frac{1}{S}[(S - k_E) + (k_E - k_I)/2], & S > k_E, \end{cases}$$

which is also increasing in  $S$ . Therefore, comparing the expected deviations in both games, we have

$$E\delta(S) - E\delta(s) = \sum_{k_I=1}^S \sum_{k_E=1}^S \pi_{k_I} \pi_{k_E} [\delta(L_{k_I}, L_{k_E}, S) - \delta(L_{k_I}, L_{k_E}, s)] \geq 0,$$

which gives the required result. ■

**Proof of Theorem 6** Consider any player in the chain store game with  $S$  interactions. Suppose this player holds beliefs  $B(t) = (b_0, b_1, \dots, b_S)$  at time  $t$ , where  $b_k$  denotes this player's belief of the probability that the opponent will use rule  $L_k$ . Let  $\mathbf{b}_j^k \equiv \sum_{i=j}^k b_i$ . Given these beliefs, the player chooses a best-response rule to maximize expected payoffs.

We first show that the incumbent will eventually share in every interaction.

Let  $V_s^E$  denote the incumbent's expected continuation payoff from following an optimal strategy henceforth, when: (i) there are  $s$  stages remaining, (ii) the incumbent has not shared during any of the previous  $S - s$  interactions, and (iii) the entrant has just entered. Similarly,  $V_s^O$  denote the incumbent's expected continuation payoff from following an optimal strategy henceforth, when (i) and (ii) above hold but the incumbent has not observed the decision of the current entrant.

In the last interaction, the incumbent always shares if the entrant enters, so  $V_1^E = 2$ . Further,  $V_1^O = 5b_0 + V_1^E(1 - b_0) = 5b_0 + 2(1 - b_0)$ , since there is a probability of  $b_0$  that the entrant will use rule  $L_0$  and stay out in the last interaction. In general, in the  $s$ -th last repetition, after the entrant enters, the incumbent's expected continuation payoff is  $2s$  from sharing and  $V_{s-1}^O$  from fighting. Therefore, we have  $V_s^E = \max\{2s, V_{s-1}^O\}$  and the

incumbent fights with  $s$  interactions remaining if and only if  $V_{s-1}^O \geq 2s$ . It also follows that  $V_s^O = (5 + V_{s-1}^O)\mathbf{b}_0^{s-1} + V_s^E(1 - \mathbf{b}_0^{s-1})$ .

Next, we show that  $V_{s-1}^O \geq 2s$  implies  $V_s^O \geq 2(s+1)$  for every  $s \geq 2$ . We first check this for  $s = 2$ : when  $V_1^O - 4 = b_0 - 2(1 - b_0) \geq 0$ , we have

$$\begin{aligned} V_2^O - 6 &= [3 + (V_1^O - 4)]\mathbf{b}_0^1 + \{[V_1^O - 4]^+ - 2\}(1 - \mathbf{b}_0^1) \\ &\geq (V_1^O - 4) + 3\mathbf{b}_0^1 - 2(1 - \mathbf{b}_0^1) \\ &\geq (V_1^O - 4) + 3b_0 - 2(1 - b_0) \geq 0, \end{aligned}$$

as stated. In general, suppose  $V_{s-1}^O \geq 2s$ . Further, suppose  $V_{t-1}^O \geq 2t$  holds for  $t = s-1, s-2, \dots, s-r+1$  but not for  $t = s-r$ . Since the above inequality holds for  $t = s-r+1$  but not for  $t = s-r$ , we know that  $V_{s-r-1}^O - 2(s-r) < 0$  but

$$V_{s-r}^O - 2(s-r+1) = [3 + (V_{s-r-1}^O - 2(s-r))]\mathbf{b}_0^{s-r-1} + \{[V_{s-r-1}^O - 2(s-r)]^+ - 2\}(1 - \mathbf{b}_0^{s-r-1}) \geq 0,$$

so we must have  $3\mathbf{b}_0^{s-r-1} - 2(1 - \mathbf{b}_0^{s-r-1}) \geq 0$  and hence  $3\mathbf{b}_0^{s-1} - 2(1 - \mathbf{b}_0^{s-1}) \geq 0$ . Note that if  $r = s-1$  above (i.e.,  $V_{t-1}^O \geq 2t$  holds for all  $t$  from 2 to  $s$ ), then we also have  $3\mathbf{b}_0^{s-1} - 2(1 - \mathbf{b}_0^{s-1}) \geq 3b_0 - 2(1 - b_0) \geq b_0 - 2(1 - b_0) = V_1^O - 4 \geq 0$ . Therefore, we obtain

$$\begin{aligned} V_s^O - (s+1) &= [3 + (V_{s-1}^O - s)]\mathbf{b}_0^{s-1} + \{[V_{s-1}^O - s]^+ - 2\}(1 - \mathbf{b}_0^{s-1}) \\ &\geq (V_{s-1}^O - s) + 3\mathbf{b}_0^{s-1} - 2(1 - \mathbf{b}_0^{s-1}) \geq 0, \end{aligned}$$

as stated.

When there are  $s$  interactions remaining, the incumbent fights if and only if  $V_{s-1}^O \geq 2s$ . From the result in the previous paragraph, we know that the incumbent will fight in all previous interactions if the same applies to the current interaction. Therefore, given beliefs  $B(t) = (b_0, b_1, \dots, b_s)$ , the incumbent's best-response rule must be  $L_{k^*}$ , where  $k^* + 1$  is the smallest integer  $s$  satisfying  $V_{s-1}^O \geq 2s$ .

In the model, each incumbent starts at  $t = 0$  with a belief weight of  $\beta$  on some  $k$  and adds a unit weight at each round  $t$ . Note that no incumbent will choose  $L_0$ , so the weights will be added to levels  $k \geq 1$  in each round. Therefore, after a finite number of rounds, we must have  $b_0 \leq 2/3$ , which implies  $V_1^O - 4 = b_0 - 2 \cdot (1 - b_0) \leq 0$ . In other words, after a finite number of rounds, the incumbent will never choose  $L_0$  or  $L_1$ , so the weights will be added to levels  $k \geq 2$  in all subsequent rounds. Similarly, after another finite number of rounds, we must have  $\mathbf{b}_0^1 \leq 2/5$ . This then implies that  $V_2^O \leq 3\mathbf{b}_0^1 - 2(1 - \mathbf{b}_0^1) \leq 0$ , so the

incumbent will choose rules with level 3 or higher subsequently. Continuing with this logic, note that for every  $k$ , after a finite number of rounds, we must have  $\mathbf{b}_0^{k-1} \leq 2/5$  and the incumbent will choose only rules of level  $k + 1$  or higher. Therefore, in a chain store game with  $S$  interactions, after a finite number of rounds, we have  $\mathbf{b}_0^{S-2} \leq 2/5$ , so the incumbent will use  $L_S = L_\infty$  subsequently.

Finally, we show that the entrant will eventually share in every interaction. Consider an entrant that is followed by  $s$  subsequent entrants. This entrant's expected payoff is 1 from staying out and  $2(1 - \mathbf{b}_0^s)$  from entering. Thus, this entrant will enter if and only if  $\mathbf{b}_0^s \leq 1/2$ . Following the same logic above, note that for every  $s$ , we will eventually have  $\mathbf{b}_0^s \leq 1/2$  after a finite number of rounds. Therefore, in a chain store game with  $S$  interactions, the entrant will eventually enter at every interaction. ■

**Proof of Theorem 7** We will prove the result for a 4-stage game. The proof for a 6-stage game proceeds similarly. First, consider the case where  $3\alpha - 6\beta \leq 2$ . In the last stage, the payoffs to Players A and B respectively are (8,32) if Player B takes and (64,16) if Player B passes. The corresponding utilities to Player B are  $32 - 24\alpha$  and  $16 - 48\beta$  respectively. Since the condition  $3\alpha - 6\beta \leq 2$  holds, Player B prefers to take. Reasoning backwards, in the second last stage, the payoffs to both players are (16,4) if Player A takes and (8,32) if Player A passes (since Player B will then take in the last stage). The corresponding utilities to Player A are  $16 - 12\alpha$  and  $8 - 24\beta$ , so by the same condition above, Player A takes. Using the same reasoning, players will take in every stage.

Next, consider the other case where  $3\alpha - 6\beta \geq 2$ . By the calculations above, we know that Player B will now pass in the last stage. Now, in the second last stage, the payoffs to Players A and B are (16,4) if Player A takes and (64,16) if Player A passes (since Player B will also pass in the last stage). The utilities are  $16 - 12\alpha$  and  $64 - 48\alpha$ . Since  $\alpha \leq 1$ , the latter is greater and thus Player A passes. Proceeding backwards, in the third last stage (i.e., Stage II), Player B's payoff from taking is  $8 - 6\alpha$ , which is less than his payoff  $32 - 24\alpha$  from taking in the last stage, because  $\alpha \leq 1$ . Since Player B prefers passing in the last stage, the same applies in this stage as all players will pass subsequently. Using the same logic, players will pass in every stage. ■