# Identification and Estimation of Dynamic Games when Players' Beliefs Are Not in Equilibrium 

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#### Abstract

This paper deals with the identification and estimation of dynamic games when players' beliefs about other players' actions may not be in equilibrium, i.e., they are not unbiased expectations of the actual behavior of other players. For instance, our model applies to dynamic competition in oligopoly industries when firms may have biased beliefs about competitors' strategies. First, we show that a exclusion restriction, typically used to identify empirical games, provides testable nonparametric restrictions of the null hypothesis of equilibrium beliefs. Second, we prove that an additional assumption, that we call no strategic uncertainty at two 'extreme' points, is sufficient for nonparametric point-identification of players' payoff and belief functions. Third, we propose a simple two-step estimation method and a sequential generalization of the method that improves its asymptotic and finite sample properties. Finally, we illustrate our model and methods using both Monte Carlo experiments and an empirical application of a dynamic game of store location by retail chains.


Keywords: Dynamic games; Rational behavior; Rationalizability; Identification; Estimation; Market entry-exit.

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## 1 Introduction

The principle of revealed preference (Samuelson, 1938) is a cornerstone in the empirical analysis of decision models, either static or dynamic, single-agent problems or games. Under the principle of revealed preference, agents maximize expected payoffs and their actions reveal information on the structure of payoff functions. This simple but powerful concept has allowed econometricians to use data on agents' decisions to identify important structural parameters for which there is very limited information from other sources. Examples of parameters and functions that have been estimated using the principle of revealed preference are agents' degree of risk aversion, intertemporal rates of substitution, market entry costs, adjustment costs and switching costs, consumer willingness to pay, preference for a political party, or the benefits of a merger. In the context of empirical games, where players' expected payoffs depend on beliefs about behavior of other players, most applications have combined the principle of revealed preference with the assumption that players' beliefs are in equilibrium. The assumption of equilibrium beliefs is very useful in the estimation of games. Equilibrium restrictions have identification power even in models with multiple equilibria (Tamer, 2003, and Aradillas-Lopez and Tamer, 2008). Imposing these restrictions contributes to improve asymptotic and finite sample properties of estimators. Furthermore, models where agents' beliefs are endogenously determined in equilibrium are attractive for the evaluation of counterfactual policy experiments because they take into account how the new policy can affect agents' behavior also through an endogenous change in agents' beliefs.

Despite these attractive implications of the assumption of equilibrium beliefs, there are empirical applications of games where the assumption is not realistic and it is of interest to relax it. ${ }^{1}$ For instance, competition in oligopoly industries is often characterized by strategic uncertainty (Besanko et al., 2010). Firm managers are very secretive about their own strategies and face significant uncertainty about the strategies of their competitors. In fact, it is often the case that firms have incentives to misrepresent their own strategies. ${ }^{2}$ In this context, it can be difficult for firms to construct unbiased beliefs about the behavior of competitors. Another example of applications where the assumption of equilibrium beliefs seems unrealistic is in the evaluation of the effect of a policy change in a strategic environment. Suppose that to evaluate the policy change we estimate an empirical game using data before and after the new policy. It seems reasonable to think that it will take time for players (e.g., firms) to learn about the new strategies of other players after the policy change. For a while firms' beliefs will be out of equilibrium, and imposing the restriction of equilibrium beliefs may bias our estimates of the effects of the new policy. Another example comes from the structural estimation of games using data from laboratory experiments. It is well

[^1]established in the experimental economics literature on games that there is significant heterogeneity in players' elicited beliefs, and that this heterogeneity is often one of the most important factors in explaining heterogeneity in observed behavior in laboratory experiments. ${ }^{3}$ Imposing the assumption of equilibrium beliefs in these applications does not seem reasonable. However, interestingly, recent empirical papers also show that there is a significant mismatch between stated or elicited beliefs and the beliefs inferred from players' actions (see Costa-Gomes and Weizsäcker, 2008, and Rutström and Wilcox, 2009). The results in our paper can be applied to estimate beliefs and payoffs, using either observational or laboratory data, when the researcher does not have data on stated or elicited beliefs makes minimum assumptions on belief and payoff functions. ${ }^{4}$

In this paper we study nonparametric identification, estimation, and inference in dynamic discrete games of incomplete information when we assume that players are rational, in the sense that each player maximizes expected payoff given some beliefs, but we relax the assumption that these beliefs are in equilibrium, and impose minimum restrictions on them. In the general class of econometric models that we consider, players' beliefs are probability distributions over the set of other players' actions. These distributions are nonparametrically specified and they are treated as incidental parameters that, together with the structural parameters of the game, determine the stochastic process followed by players' actions. Our framework includes as particular case games with multiple equilibria where every player has beliefs that correspond to an equilibrium but their beliefs are not 'synchronized', i.e., some players believe that the game is in an equilibrium, say A, and other players think that the game is in a different equilibrium, say B. We illustrate this case in our numerical experiments in section 5.

When players beliefs are not in equilibrium they are different from the actual distribution of players' actions in the population. Therefore, without other restrictions, beliefs cannot be identified and estimated by simply using a nonparametric estimator of the distribution of players' actions. First, we show that a exclusion restriction, that is typically used to identify payoffs in empirical games, provides testable nonparametric restrictions of the null hypothesis of equilibrium beliefs. Second, we show that, together with the exclusion restriction, a large-support condition on one of the explanatory variables is sufficient for point-identification. While the type of exclusion restriction that we need is quite plausible in dynamic games of oligopoly competition, the large support condition is not satisfied in many applications. However, we show that this condition can be replaced with a more general restriction that we call no strategic uncertainty at two 'extreme' points. We also present identification results from a semiparametric model where the biased in players' beliefs is parameterized using a flexible functional form. Third, we propose a simple two-step estimation method of structural parameters and beliefs, and a sequential extension of

[^2]this method that provides estimators with better statistical properties. We also present several procedures for testing the null hypothesis of equilibrium beliefs. Finally, we illustrate our model and methods using both Monte Carlo experiments and an empirical application of a dynamic game of store location by retail chains.

This paper builds on the recent literature on estimation of dynamic games of incomplete information (see Aguirregabiria and Mira, 2007, Bajari, Benkard and Levin, 2007, Pakes, Ostrovsky and Berry, 2007, and Pesendorfer and Schmidt-Dengler, 2008). All the papers in this literature assume that the data come from a Markov Perfect Equilibrium. We relax that assumption. Our paper also builds and extends the work of Aradillas-Lopez and Tamer (2008) who study the identification power of the assumption of equilibrium beliefs in simple static games. We extend their work in several ways. First, we study dynamic games, including static games as a particular case. The implications of dropping the assumption of equilibrium beliefs, and the associated identification issues, are very different between static and dynamic games. As we show in this paper, the characterization and derivation of bounds on choice probabilities is significantly more complicated in dynamic than in static games, and some results in Aradillas-Lopez and Tamer cannot be extended to dynamic games. Therefore, we follow a different approach to the one considered by AradillasLopez and Tamer. Second, they concentrate on identification while we also propose and implement new tests and estimators. And third, they deal with models with a parametric specification of the payoff function while we consider nonparametric payoffs.

Our approach also differs from Aradillas-Tamer in one key aspect. In relaxing the assumption of Nash equilibrium, they consider a very specific departure from equilibrium beliefs. They assume that players are level-k rational with respect to their beliefs about their opponents' behavior, a concept which derives from the notion of rationalizability (Bernheim, 1984, and Pearce, 1984). Their approach is especially useful in the context of static games with binary or ordered decision variables, as, under the condition that players' payoffs are monotone in the decision of their opponents, it yields a sequence of closed form bounds on players' beliefs that grow tighter as the level of rationality $k$ gets larger. Unfortunately, in the case of dynamic games, the assumptions of AradillasLopez and Tamer do not yield a representation of bounds on players' beliefs that is practical to implement, even for simple dynamic games. We describe this issue at the end of section 2. As such we do not use a bound-approach and level-k rationalizability. Instead, we concentrate on level- 1 rationalizability and study conditions and methods for nonparametric point identification and estimation of preferences and beliefs.

To illustrate our model and methods, we consider an empirical application of a dynamic game of store location between McDonalds and Burger King. Most empirical studies on bounded rationality have concentrated on individual behavior, and there is very little empirical work on bounded
rationality of firms. ${ }^{5}$ The estimation of reduced form models using panel data of McDonalds' and Burger King's store location decisions show that the probability that these firms open a new store in a local market does not respond to the number of stores of the competing firm, or even responds positively (Toivanen and Waterson, 2005). This evidence is robust to controlling for unobserved market heterogeneity, and it cannot be explained by a standard static model of store location. We propose and estimate a structural dynamic game of entry in local markets that incorporates three alternative explanations for this puzzle: positive spillover effects; firms' forward looking behavior; and biased beliefs about the behavior of the competitor.

The rest of the paper includes the following sections. Section 2 presents the model and basic assumptions. In section 3, we present our identification results. Section 4 describes estimation methods and testing procedures. Section 5 presents our Monte Carlo experiments. The empirical application is described in section 6 . We summarize and conclude in section 7 .

## 2 Model

### 2.1 Basic framework

This section presents a dynamic game of incomplete information where two players make binary choices over $T$ periods. ${ }^{6}$ The time horizon $T$ can be either finite or infinite. We use the indexes $i \in\{1,2\}$ and $j \in\{1,2\}$ to represent a player and his opponent, respectively. Time is discrete and indexed by $t \in\{1,2, \ldots, T\}$. Every period $t$, players choose simultaneously and non-cooperatively between alternatives 0 and 1 . Let $Y_{i t} \in\{0,1\}$ represent the choice of player $i$ at period $t$. Each player makes this decision to maximize his expected and discounted payoff, $E_{t}\left(\sum_{s=0}^{T} \beta^{s} \Pi_{i, t+s}\right)$, where $\beta \in(0,1)$ is the discount factor, and $\Pi_{i t}$ is his payoff at period $t$. The one-period payoff function has the following structure:

$$
\begin{equation*}
\Pi_{i t}=\pi_{i t}\left(Y_{i t}, Y_{j t}, \mathbf{X}_{t}\right)+\varepsilon_{i t}\left(Y_{i t}\right) \tag{1}
\end{equation*}
$$

$Y_{j t}$ represents the current action of the other player; $\mathbf{X}_{t}$ is a vector of state variables which are common knowledge for both players; $\varepsilon_{i t} \equiv\left(\varepsilon_{i t}(0), \varepsilon_{i t}(1)\right)$ is a pair of private information variables for firm $i$ at period $t$; and $\pi_{i t}($.$) is a real valued function.$

The vector of common knowledge state variables $\mathbf{X}_{t}$ has three different components: $\mathbf{X}_{t}=$ $\left(W_{t}, S_{i t}, S_{j t}\right) . W_{t} \in \mathcal{W}$ is a vector of common state variables that evolves exogenously according to a Markov process with transition probability function $f_{t}^{W}\left(W_{t+1} \mid W_{t}\right)$. For each player, $S_{i t} \in \mathcal{S}$ is

[^3]a vector that contains player-specific state variables. Some of the variables in $S_{i t}$ are endogenous (i.e., their evolution over time depends on players' actions), but the vector may also contain playerspecific exogenous state variables. The vector $\mathbf{S}_{t} \equiv\left(S_{i t}, S_{j t}\right)$ evolves over time according to the transition probability function $f_{t}^{S}\left(\mathbf{S}_{t+1} \mid Y_{i t}, Y_{j t}, \mathbf{X}_{t}\right)$. The private information shocks $\varepsilon_{i t}(0)$ and $\varepsilon_{i t}(1)$ are independent of $\mathbf{X}_{t}$ and independently distributed over time, players, and actions. Without loss of generality, these private information shocks have zero mean. The distribution function of $\varepsilon_{i t}$ is $G$ that is absolutely continuous and strictly increasing with respect to the Lebesgue measure on $\mathbb{R}^{2}$. When the game has infinite horizon $(T=\infty)$, we assume that all the primitive functions, $\pi_{i t}, G, f_{t}^{W}$, and $f_{t}^{S}$, are constant over time such that the dynamic game has a stationary Markov structure.

EXAMPLE: Dynamic game of market entry and exit. Consider two firms competing in a market. Each firm sells a differentiated product. Every period, firms decide whether or not to be active in the market. Then, incumbent firms compete in prices. Let $Y_{i t} \in\{0,1\}$ represent the decision of firm $i$ to be active in the market at period $t$. The profit of firm $i$ at period $t$ has the structure of equation (1), $\Pi_{i t}=\pi_{i t}\left(Y_{i t}, Y_{j t}, \mathbf{X}_{t}\right)+\varepsilon_{i t}\left(Y_{i t}\right)$. We now describe the specific form of the payoff function $\pi_{i t}$ and the state variables $\mathbf{X}_{t}$ and $\varepsilon_{i t}$. The profit of an inactive firm $\left(Y_{i t}=0\right)$ is $\Pi_{i t}=\varepsilon_{i t}(0)$, i.e., the average profit of an inactive firm, $\pi_{i t}\left(0, Y_{j t}, \mathbf{X}_{t}\right)$, is normalized to zero. The profit of an active firm $\left(Y_{i t}=1\right)$ is $\pi_{i t}\left(1, Y_{j t}, \mathbf{X}_{t}\right)+\varepsilon_{i t}(1)$ where:

$$
\begin{equation*}
\pi_{i t}\left(1, Y_{j t}, \mathbf{X}_{t}\right)=W_{t}\left(\left(1-Y_{j t}\right) \theta_{i}^{M}+Y_{j t} \theta_{i}^{D}\right)-\theta_{i 0}^{F C}-\theta_{i 1}^{F C} \exp \left\{-S_{i t}\right\}-1\left\{S_{i t}=0\right\} \theta_{i}^{E C} \tag{2}
\end{equation*}
$$

The term $W_{t}\left(\left(1-Y_{j t}\right) \theta_{i}^{M}+Y_{j t} \theta_{i}^{D}\right)$ represents the variable profit of firm $i . W_{t}$ represents market size (e.g., market population) and it is an exogenous state variable. $\theta_{i}^{M}$ and $\theta_{i}^{D}$ are parameters that represent the per capita variable profit of firm $i$ when the firm is a monopolist and when it is a duopolist, respectively. The term $\theta_{i 0}^{F C}+\theta_{i 1}^{F C} \exp \left\{-S_{i t}\right\}$ is the fixed cost of firm $i$, where $\theta_{i 0}^{F C}$ and $\theta_{i 1}^{F C}$ are parameters, and $S_{i t}$ is an endogenous state variable that represents the number of consecutive periods that the firm has been actively operating in the market, i.e., a firm's experience. $S_{i t}=0$ means that the firm was not active at $t-1$, and $S_{i t}=k>0$ means that the firm entered the market at period $t-k-1$ and has remained active every period until $t-1$. The transition rule of firm experience is deterministic, $S_{i t+1}=Y_{i t}\left(S_{i t}+Y_{i t}\right)$. If $\theta_{i 1}^{F C}>0$, fixed costs decline with firm's experience in the market, e.g., passive learning (by being active in the market) or other forms of learning. The term $1\left\{S_{i t}=0\right\} \theta_{i}^{E C}$ represents sunk entry costs, where $1\{$.$\} is the binary indicator$ and $\theta_{i}^{E C}$ is a parameter. The vector of state variables in this model is $\mathbf{X}_{t}=\left(W_{t}, S_{i t}, S_{j t}\right)$.

Most previous literature on estimation of dynamic discrete games assumes that the data comes from a Markov Perfect Equilibrium (MPE). This equilibrium concept incorporates four main assumptions.

ASSUMPTION MOD-1 (Payoff relevant state variables): Players' strategy functions depend only on payoff relevant state variables: $\mathbf{X}_{t}$ and $\varepsilon_{i t}$. Similarly, a player's belief about the strategy of other player is a function only of the payoff relevant state variables of the other player.

ASSUMPTION MOD-2 (Maximization of expected payoffs): Players are forward looking and maximize expected intertemporal payoffs.

ASSUMPTION MOD-3 (Unbiased beliefs on own future behavior): A player's beliefs about his own actions in the future are unbiased expectations of his actual actions in the future.

ASSUMPTION 'EQUIL' (Unbiased or equilibrium beliefs on other players' behavior): Strategy functions are common knowledge, and players' have rational expectations on the current and future behavior of other players. That is, players' beliefs about other players' actions are unbiased expectations of the actual actions of other players.

First, let us examine the implications of imposing only Assumption MOD-1. The payoff-relevant information set of player $i$ is $\left\{\mathbf{X}_{t}, \varepsilon_{i t}\right\}$. The space of $\mathbf{X}_{t}$ is $\mathcal{X} \equiv \mathcal{W} \times \mathcal{S}^{2}$. At period $t$, players observe $\mathbf{X}_{t}$ and choose their respective actions. Let $\sigma_{i t}\left(\mathbf{X}_{t}, \varepsilon_{i t}\right)$ be a strategy function for player $i$ at period $t$. This is a function from the support of $\left(\mathbf{X}_{t}, \varepsilon_{i t}\right)$ into the binary set $\{0,1\}$, i.e., $\sigma_{i t}: \mathcal{X} \times \mathbb{R} \rightarrow\{0,1\}$. Given any strategy function $\sigma_{i t}$, we can define a choice probability function $P_{i t}\left(\mathbf{X}_{t}\right)$ that represents the probability of $Y_{i t}=1$ conditional on $\mathbf{X}_{t}$ given that player $i$ follows strategy $\sigma_{i t}$. That is,

$$
\begin{equation*}
P_{i t}\left(\mathbf{X}_{t}\right) \equiv \int 1\left\{\sigma_{i t}\left(\mathbf{X}_{t}, \varepsilon_{i t}\right)=1\right\} d G\left(\varepsilon_{i t}\right) \tag{3}
\end{equation*}
$$

It is convenient to represent players' behavior using these Conditional Choice Probability (CCP) functions. When the variables in $\mathbf{X}_{t}$ have a discrete support, we can represent the CCP function $P_{i t}($.$) using a finite dimension vector \mathbf{P}_{i t} \equiv\left\{P_{i t}\left(\mathbf{X}_{t}\right): \mathbf{X}_{t} \in \mathcal{X}\right\}$. Throughout the paper we use either the function $P_{i t}($.$) or the vector \mathbf{P}_{i t}$ to represent the actual behavior of player $i$ at period $t$.

Without imposing Assumption 'Equil' ('Equilibrium Beliefs'), a player's beliefs about the behavior of other players do not necessarily represent the actual behavior of the other players. Therefore, we need functions other than $\sigma_{j t}($.$) and P_{j t}($. $)$ to represent players $i$ ' beliefs about the strategy of player $j$. Let $b_{j t}^{\left(t_{0}\right)}\left(\mathbf{X}_{t}, \varepsilon_{j t}\right)$ be player $i$ 's belief at period $t_{0}$ about the strategy function of player $j$ at period $t$. In principle, this function may vary with $t_{0}$ due to players' learning and forgetting, or to other factors that make players to change their beliefs over time. Let $B_{j t}^{\left(t_{0}\right)}\left(\mathbf{X}_{t}\right)$ be the choice probability associated with $b_{j t}^{\left(t_{0}\right)}\left(\mathbf{X}_{t}, \varepsilon_{j t}\right)$, i.e., $B_{j t}^{\left(t_{0}\right)}\left(\mathbf{X}_{t}\right) \equiv \int 1\left\{b_{j t}^{\left(t_{0}\right)}\left(\mathbf{X}_{t}, \varepsilon_{j t}\right)=1\right\} d G_{j t}\left(\varepsilon_{j t}\right)$. When $\mathcal{X}$ is a discrete and finite space, we can represent function $B_{j t}^{\left(t_{0}\right)}($.$) using a finite-dimensional vec-$ tor $\mathbf{B}_{j t}^{\left(t_{0}\right)} \equiv\left\{B_{j t}\left(\mathbf{X}_{t}\right): \mathbf{X}_{t} \in \mathcal{X}\right\}$. Using this notation, Assumption 'Equil' can be represented as $B_{j t}^{\left(t_{0}\right)}\left(\mathbf{X}_{t}\right)=P_{j t}\left(\mathbf{X}_{t}\right)$ for every $t_{0}, t \geq t_{0}$, and $\mathbf{X}_{t} \in \mathcal{X}$.

| Table 1Sequence of Beliefs $B_{j t}^{\left(t_{0}\right)}$ |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Period when | Period of the opponents' behavior ( $t$ ) |  |  |  |  |  |
| beliefs are formed ( $t_{0}$ ) | $t=1$ | $t=2$ | $t=3$ | ... | $t=T-1$ | $t=T$ |
| $t_{0}=1$ | $B_{j 1}^{(1)}$ | $B_{j 2}^{(1)}$ | $B_{j 3}^{(1)}$ | $\ldots$ | $B_{j, T-1}^{(1)}$ | $B_{j T}^{(1)}$ |
| $t_{0}=2$ | - | $B_{j 2}^{(2)}$ | $B_{j 3}^{(2)}$ | ... | $B_{j, T-1}^{(2)}$ | $B_{j T}^{(2)}$ |
| $t_{0}=3$ | - | - | $B_{j 3}^{(3)}$ | $\ldots$ | $B_{j, T-1}^{(3)}$ | $B_{j T}^{(3)}$ |
| : | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $t_{0}=T-1$ | - | - | - | $\ldots$ | $B_{j, T-1}^{(T-1)}$ | $B_{j T}^{(T-1)}$ |
| $t_{0}=T$ | - | - | - | ... | - | $B_{j T}^{(T)}$ |

The following assumption replaces the assumption of 'Equilibrium Beliefs' and summarized our minimum conditions on players' beliefs.

ASSUMPTION MOD-4: If the dynamic game has finite horizon ( $T<\infty$ ), then players' beliefs functions $B_{i t}^{\left(t_{0}\right)}$ may vary over the time period of the opponent's behavior, $t$, but they are not 'revised' over the time period $t_{0}$, i.e., $B_{i t}^{\left(t_{0}\right)}=B_{i t}$ for any period $t_{0}$. If the dynamic game has infinite horizon $(T=\infty)$, then players' beliefs functions $B_{i t}^{\left(t_{0}\right)}$ may be revised over the time period $t_{0}$, but they do not vary over time period of the opponent's behavior, $t$, because the decision problem is stationary, i.e., $B_{i t}^{\left(t_{0}\right)}=B_{i}^{\left(t_{0}\right)}$ for any period $t$.

Assumption MOD-4 imposes restrictions on the time pattern of beliefs. Using table 1, we can describe this assumption by saying that beliefs are constant either across columns and across rows. For finite horizon dynamic games, we assume that beliefs are constant across rows. This implies that each player considers that the opponent can change his behavior over time, but beliefs for the opponent's behavior at a given period are constant over the entire game and they are not revised as time goes by. Therefore, for finite horizon games we do not allow for any form of learning. For infinite horizon games, we assume that players' know that the game is stationary and their beliefs satisfy this stationarity condition. However, players can revise their beliefs over time.

For the rest of the paper, we maintain Assumptions MOD-1 to MOD-4 but we do not impose the restriction of 'Equilibrium Beliefs'. We assume that players are 'rational', in the sense that they maximize expected and discounted payoff given their beliefs on other players' behavior. Our approach is agnostic about the formation of players' beliefs.

For the sake of simplicity in the presentation of out results, the main text of the paper deals with finite horizon games, but we show in the Appendix that all our results apply to infinite horizon dynamic games. To illustrate both cases, we consider a finite horizon game in the Monte Carlo experiments in section 5, and an infinite horizon game in the empirical application in section 6 .

### 2.2 Best response mappings

We say that a strategy function $\sigma_{i t}($.$) (and the associated CCP function P_{i t}$ ) is rational if for every possible value of $\left(\mathbf{X}_{t}, \varepsilon_{i t}\right) \in \mathcal{X} \times \mathbb{R}$ the action $\sigma_{i t}\left(\mathbf{X}_{t}, \varepsilon_{i t}\right)$ maximizes player $i$ 's expected and discounted value given his beliefs on the opponent's strategy. Given his beliefs, player $i$ 's best response at period $t$ is the optimal solution of a single-agent dynamic programming (DP) problem. This DP problem can be described in terms of: (i) a discount factor, $\beta$; (ii) a sequence of expected one-period payoff functions, $\left\{\pi_{i t}^{\mathbf{B}}\left(Y_{i t}, \mathbf{X}_{t}\right)+\varepsilon_{i t}\left(Y_{i t}\right): t=1,2, \ldots, T\right\}$, where

$$
\begin{equation*}
\pi_{i t}^{\mathbf{B}}\left(Y_{i t}, \mathbf{X}_{t}\right)=\left(1-B_{j t}\left(\mathbf{X}_{t}\right)\right) \pi_{i t}\left(Y_{i t}, 0, \mathbf{X}_{t}\right)+B_{j t}\left(\mathbf{X}_{t}\right) \pi_{i t}\left(Y_{i t}, 1, \mathbf{X}_{t}\right) ; \tag{4}
\end{equation*}
$$

and (iii) a sequence of transition probability functions $\left\{f_{i t}^{\mathbf{B}}\left(\mathbf{X}_{t+1} \mid Y_{i t}, \mathbf{X}_{t}\right): t=1,2, \ldots, T\right\}$, where

$$
\begin{align*}
f_{i t}^{\mathbf{B}}\left(\mathbf{X}_{t+1} \mid Y_{i t}, \mathbf{X}_{t}\right)= & f_{t}^{W}\left(W_{t+1} \mid W_{t}\right)  \tag{5}\\
& {\left[\left(1-B_{j t}\left(\mathbf{X}_{t}\right)\right) f_{t}^{S}\left(\mathbf{S}_{t+1} \mid Y_{i t}, 0, \mathbf{X}_{t}\right)+B_{j t}\left(\mathbf{X}_{t}\right) f_{t}^{S}\left(\mathbf{S}_{t+1} \mid Y_{i t}, 1, \mathbf{X}_{t}\right)\right] }
\end{align*}
$$

Let $V_{i t}^{\mathbf{B}}\left(\mathbf{X}_{t}, \varepsilon_{i t}\right)$ be the value function for player $i$ 's DP problem given his beliefs. By Bellman's principle, the sequence of value functions $\left\{V_{i t}^{\mathbf{B}}: t=1,2, \ldots, T\right\}$ can be obtained recursively using backwards induction in the following Bellman equation:

$$
\begin{equation*}
V_{i t}^{\mathbf{B}}\left(\mathbf{X}_{t}, \varepsilon_{i f t}\right)=\max _{Y_{i t} \in\{0,1\}}\left\{v_{i t}^{\mathbf{B}}\left(Y_{i t}, \mathbf{X}_{t}\right)+\varepsilon_{i t}\left(Y_{i t}\right)\right\} \tag{6}
\end{equation*}
$$

where $v_{i t}^{\mathbf{B}}\left(Y_{i t}, \mathbf{X}_{t}\right)$ is the conditional choice value function

$$
\begin{equation*}
v_{i t}^{\mathbf{B}}\left(Y_{i t}, \mathbf{X}_{t}\right) \equiv \pi_{i t}^{\mathbf{B}}\left(Y_{i t}, \mathbf{X}_{t}\right)+\beta \int V_{i t+1}^{\mathbf{B}}\left(\mathbf{X}_{t+1}, \varepsilon_{i t+1}\right) d G\left(\varepsilon_{i t+1}\right) f_{i t}^{\mathbf{B}}\left(\mathbf{X}_{t+1} \mid Y_{i t}, \mathbf{X}_{t}\right) \tag{7}
\end{equation*}
$$

The best response function of player $i$ at period $t$ given beliefs $\mathbf{B}_{j}$ is the optimal decision rule of this DP problem. This best response function can be represented using the following threshold condition:

$$
\begin{equation*}
\left\{Y_{i t}=1\right\} \text { iff }\left\{\varepsilon_{i t}(0)-\varepsilon_{i t}(1) \leq v_{i t}^{\mathbf{B}}\left(1, \mathbf{X}_{t}\right)-v_{i t}^{\mathbf{B}}\left(0, \mathbf{X}_{t}\right)\right\} \tag{8}
\end{equation*}
$$

The best response probability function $(B R P F)$ is a probabilistic representation of the best response function. More precisely, it is the best response function integrated over the distribution of $\varepsilon_{i t}$. In this model, the BRPF is:

$$
\begin{aligned}
\operatorname{Pr}\left(Y_{i t}=1 \mid \mathbf{X}_{t}\right) & =\int 1\left\{\varepsilon_{i t}(0)-\varepsilon_{i t}(1) \leq v_{i t}^{\mathbf{B}}\left(1, \mathbf{X}_{t}\right)-v_{i t}^{\mathbf{B}}\left(0, \mathbf{X}_{t}\right)\right\} d G\left(\varepsilon_{i t}\right) \\
& =\Lambda\left(v_{i t}^{\mathbf{B}}\left(1, \mathbf{X}_{t}\right)-v_{i t}^{\mathbf{B}}\left(0, \mathbf{X}_{t}\right)\right)
\end{aligned}
$$

where $\Lambda$ is the $\operatorname{CDF}$ of $\varepsilon_{i t}(0)-\varepsilon_{i t}(1)$. Therefore, under Assumptions MOD-1 to MOD-3 the actual behavior of player $i$, represented by the CCP function $P_{i t}($.$) , satisfies the following condition:$

$$
\begin{equation*}
P_{i t}\left(\mathbf{X}_{t}\right)=\Lambda\left(v_{i t}^{\mathbf{B}}\left(1, \mathbf{X}_{t}\right)-v_{i t}^{\mathbf{B}}\left(0, \mathbf{X}_{t}\right)\right) \tag{9}
\end{equation*}
$$

This equation summarizes all the restrictions that Assumptions MOD-1 to MOD-3 impose on players' choice probabilities. The right hand side of equation (9) is the best response function of a rational player. We use $\Psi_{i t}\left(\mathbf{B}_{j}\right)$ to represent the vector-value function $\left\{\Lambda\left(v_{i t}^{\mathbf{B}}(1, \mathbf{X})-v_{i t}^{\mathbf{B}}(0, \mathbf{X})\right)\right.$ : $\mathbf{X} \in \mathcal{X}\}$.

The concept of Markov Perfect Equilibrium (MPE) is completed with assumption 'Equil' ('Equilibrium Beliefs'). Under this assumption, players' beliefs are in equilibrium, i.e., $B_{j t}\left(\mathbf{X}_{t}\right)=P_{j t}\left(\mathbf{X}_{t}\right)$ for every pair period $t$ and every state $\mathbf{X}_{t}$. A MPE can be described as a sequence of CCP vectors, $\left\{\mathbf{P}_{i t}, \mathbf{P}_{j t}: t=1,2, \ldots, T\right\}$ such that for every player $i$ and time period $t$, we have that

$$
\begin{equation*}
\mathbf{P}_{i t}=\Psi_{i t}\left(\mathbf{P}_{j}\right) \tag{10}
\end{equation*}
$$

For the sake of notational simplicity, and with some abuse of notation, we omit in some expressions the vector of state variables $\mathbf{X}_{t}$ as an argument of payoff or belief functions.

### 2.3 Aradillas-Lopez and Tamer's approach in dynamic games

The purpose of this section is twofold. First, we want to describe the relationship between our paper and Aradillas-Lopez and Tamer (2008). Second, we explain in some detail why their approach, while useful for identification and estimation of static games, has very limited applicability to dynamic games.

The static game of incomplete information in Aradillas-Tamer can be seen as specific case of our framework. To see this, consider the final period of the game $T$ in our model. At this last period the decision problem facing the players is equivalent to that of a static game. At period $T$ there is no future and the difference between the conditional choice value functions is simply the difference between the conditional choice current profits, $\pi_{i T}^{\mathbf{B}}(1)-\pi_{i T}^{\mathbf{B}}(0)$, that is equal to ( $1-B_{j T}$ ) $\left[\pi_{i T}(1,0)-\pi_{i T}(0,0)\right]+B_{j T}\left[\pi_{i T}(1,1)-\pi_{i T}(0,1)\right]$. Therefore, the BRPF is:

$$
\begin{equation*}
P_{i T}=\Lambda\left(\left(1-B_{j T}\right)\left[\pi_{i T}(1,0)-\pi_{i T}(0,0)\right]+B_{j T}\left[\pi_{i T}(1,1)-\pi_{i T}(0,1)\right]\right) \tag{11}
\end{equation*}
$$

Aradillas and Tamer assume that players' payoffs are submodular in players' decisions $\left(Y_{i}, Y_{j}\right)$, i.e., for every value of the state variables $\mathbf{X}$,

$$
\begin{equation*}
\left[\pi_{i t}(1,0)-\pi_{i t}(0,0)\right]>\left[\pi_{i t}(1,1)-\pi_{i t}(0,1)\right] \tag{12}
\end{equation*}
$$

Under this assumption, and in the context of a parametrically specified model, they derive informative bounds around players' conditional choice probabilities when players are level-k rational,
and show that the bounds become tighter as $k$ increases. For instance, without further restrictions on beliefs (i.e., rationality of level 1), the threshold function (and thus player $i$ 's conditional choice probability) takes its largest possible value when beliefs are $B_{j T}=0$, and it takes its smallest possible value when beliefs are $B_{j T}=1$. This result yields informative bounds on the period $T$ choice probabilities of player $i$ :

$$
\begin{equation*}
\Lambda\left(\pi_{i T}(1,0)-\pi_{i T}(0,0)\right) \geq P_{i T} \geq \Lambda\left(\pi_{i T}(1,1)-\pi_{i T}(0,1)\right) \tag{13}
\end{equation*}
$$

These bounds on conditional choice probabilities can be used to set identify the structural parameters in players' preferences.

In their setup, the monotonicity of players' payoffs in the decisions of other players implies monotonicity of players' best response probability functions (BRPF) in the beliefs about other players actions. This type of monotonicity is very convenient in their approach, not only from the perspective of identification, but also because it yields a very simple approach to calculate upper and lower bounds on conditional choice probabilities. However, this property does not extend to dynamic games, even the simpler ones. We now illustrate this issue.

Consider the dynamic game at some period $t$ smaller than $T$. To obtain bounds on players' choice probabilities analogous to the ones obtained at the last period, we need to find, for every value of the state variables $\mathbf{X}$, the smallest and largest feasible values of the best response $\Lambda\left(v_{i t}^{\mathbf{B}}(1, \mathbf{X})-\right.$ $\left.v_{i t}^{\mathbf{B}}(0, \mathbf{X})\right)$. That is, we need to minimize (and maximize) this best response with respect to beliefs $\left\{B_{j t}, B_{j t+1}, \ldots, B_{j T}\right\}$. Without making further assumptions, this best response function is not monotonic in beliefs at every possible state. In fact, this monotonicity is only achieved under very strong conditions not only on the payoff function but also on the transition probability of the state variables and on belief functions themselves.

Therefore, in a dynamic game, to find the largest and smallest value of a best response (and ultimately the bounds on choice probabilities) at periods $t<T$, one needs to explicitly solve a nontrivial optimization problem. In fact, the maximization (minimization) of the BRPF with respect to beliefs is a extremely complex task. The main reason is that the best response probability evaluated at a value of the state variables depends on beliefs at every period in the future and at every possible value of the state variables in the future. Therefore, to find bounds on best responses we must solve an optimization problem with a dimension equal to the number of values in the space of state variables times the number of future periods. This is because, in general, the maximization (minimization) of a best response with respect to beliefs does not have a timerecursive structure except under very special assumptions (see Aguirregabiria, 2008). For instance, though $B_{j T}\left(\mathbf{X}_{T}\right)=0$ maximizes the best response at the last period $T$, in general the maximization of a best response at period $T-1$ is not achieved setting $B_{j T}\left(\mathbf{X}_{T}\right)=0$ for any value of $\mathbf{X}_{T}$. More generally, the beliefs from period $t$ to $T$ that optimize best responses at $t$ are not equal to the
beliefs from period $t$ to $T$ that optimize best responses at $t-1$. So at each point in time we need to re-optimize with respect to beliefs about strategies at every period in the future. That is, while the optimization of expected and discounted payoffs has the well-known time-recursive structure, the maximization (minimization) of the BRPFs does not.

In summary, the extension of Aradillas-Lopez and Tamer's bounds approach to dynamic games suffers of substantial computational problems. Here we propose an alternative approach.

## 3 Identification

### 3.1 Conditions on Data Generating Process

Suppose that the researcher has panel data with realizations of the game over multiple locations and time periods. Using the terminology in empirical applications of games in Industrial Organization, we employ the term local market to refer to a location. We use the letter $m$ to index local markets. The researcher observes a random sample of $M$ local markets with information on $\left\{Y_{i m t}, S_{i m t}\right.$, $\left.W_{m t}\right\}$ for every player $i \in\{1,2\}$ and every period $t \in\left\{1,2, \ldots, T^{\text {data }}\right\}$. Note that $T^{\text {data }}$ represents the number of periods in the data, while $T$ is the time horizon of the dynamic game. If the game has a finite horizon $(T<\infty)$, then we assume that the dataset includes all the periods in the game such that $T^{\text {data }}=T$. For an infinite horizon game, obviously we have that $T^{\text {data }}<T=\infty$. We assume that $T^{d a t a}$ is small and the number of local markets, $M$, is large and for the identification results in this section we assume that $M$ is infinite. Since the main text deals with the finite horizon game, we use $T$ for the rest of the paper to represent both the horizon of the game and the number of periods in the data. We deal with the infinite horizon game in the Appendix.

In our basic framework, we assume that the only unobservable variables for the researcher are the private information shocks $\left\{\varepsilon_{i m t}\right\}$ which are assumed to be independently and identically distributed across markets and over time. However, in section 3.5 we extend this basic framework to incorporate time-invariant, local market-specific unobservables for the econometrician which are common knowledge to players.

We want to use this sample to estimate the model structural 'parameters' or functions: i.e., payoffs $\left\{\pi_{i t}, \beta\right\}$; transition probabilities $\left\{f_{t}^{S}, f_{t}^{W}\right\}$; distribution of unobservables $\Lambda$; and beliefs $\left\{B_{i t}\right\}$. For primitives other than players' beliefs, we make some assumptions that are standard in previous research on identification of static games and of dynamic structural models with rational or equilibrium beliefs. ${ }^{7}$ We assume that the distribution of the unobservables, $\Lambda$, is known to the researcher up to a scale parameter. We study identification of the payoff functions $\pi_{i t}$ up to scale, but for notational convenience we omit the scale parameter. ${ }^{8}$ Following the standard approach

[^4]in dynamic decision models, we assume that the discount factors, $\beta$, is known to the researcher. Finally, note that the transition probability functions $f_{t}^{S}$ and $f_{t}^{W}$ are nonparametrically identified. ${ }^{9}$ Therefore, we concentrate on the identification of the payoff functions $\pi_{i t}$ and belief function $B_{i t}$ and assume that $\left\{f_{t}^{S}, f_{t}^{W}, \Lambda, \beta\right\}$ are known.

Let $P_{i m t}^{0}(\mathbf{X})$ be the true conditional probability function $\operatorname{Pr}\left(Y_{i m t}=1 \mid \mathbf{X}_{m t}=\mathbf{X}\right)$ that represents the actual behavior of player $i$ in market $m$ at period $t$. Let $B_{j m t}^{0}(\mathbf{X})$ be the probability function with player $i$ 's 'true' beliefs in market $m$ at period $t$. And let $\boldsymbol{\pi}^{0} \equiv\left\{\pi_{i t}^{0}: i=1,2 ; t=1,2, \ldots, T\right\}$ be the true payoff functions in the population. Assumption $I D-1$ summarizes our conditions on the Data Generating Process.

ASSUMPTION ID-1. (A) For every player $i, P_{i m t}^{0}$ is the best response of player $i$ given his beliefs $B_{j m t}^{0}$ and the payoff functions $\boldsymbol{\pi}^{0}$. (B) Players have the same beliefs in markets with the same observable characteristics $\mathbf{X}$, i.e., for every market $m$ with $\mathbf{X}_{m t}=\mathbf{X}, B_{j m t}(\mathbf{X})=B_{j t}(\mathbf{X})$.

Assumption $I D-1(\mathrm{~A})$ establishes that players are rational in the sense that their actual behavior is the best response given their beliefs. Assumption $I D-1(\mathrm{~B})$ establishes that a player should have the same beliefs in two markets with the same state variables and at the same period of time. This assumption is common in the literature of estimation of games under the restriction of equilibrium beliefs (e.g., Bajari, Benkard, and Levin, 2007, or Bajari et al, 2010). Note that beliefs can vary across markets according to the state variables in $\mathbf{X}_{m t}$. Note that this assumption allows players having different belief functions in different markets as long as these markets have different values of time-invariant observable exogenous characteristics. That is, we can distinguish in our sample different "market types" according to some time-invariant observable characteristics. If the number of market types is small (more precisely, if it does not increase with $M$ ), then we can allow players' beliefs to be completely different in each market type. It is also important to note that when we incorporate time-invariant unobserved market heterogeneity in our model, in section 3.5., we can allow for different belief functions for each market type, where now market types can be defined in terms of unobservables.

In dynamic games where beliefs are in equilibrium, Assumption $I D$ - 1 effectively allows the researcher to identify player beliefs. Under this assumption, conditional choice probabilities are identified, and if beliefs are in equilibrium, these beliefs are equal to the conditional choice probabilities. When beliefs are not in equilibrium we can not identify beliefs in this way. However, assumption $I D-1$ still implies that CCPs are identified from the data. This assumption implies that for any player $i$, any period $t$, and any value of $\mathbf{X} \in \mathcal{X}$, we have that $P_{i m t}^{0}(\mathbf{X})=P_{i t}^{0}(\mathbf{X})=E\left(Y_{i m t} \mid \mathbf{X}_{m t}=\mathbf{X}\right)$, and this conditional expectation can be estimated consistently using data on $\left\{Y_{i m t}, \mathbf{X}_{m t}\right\}$. This in

[^5]turn, as we will show, is important for the identification of beliefs themselves.

### 3.2 Identification with equilibrium beliefs

For notational simplicity, we omit the market subindex $m$ for the rest of this section. The model restrictions are summarized in the best response conditions $P_{i t}^{0}(\mathbf{X})=\Lambda\left(v_{i t}^{\mathbf{B}^{0}}(1, \mathbf{X})-v_{i t}^{\mathbf{B}^{0}}(0, \mathbf{X})\right)$. Given these conditions, we want to identify payoffs $\pi^{0}$ and beliefs $B_{j}^{0}$. It is simple to verify that, without further restriction, the order condition for identification is not satisfied. Suppose that the state space $\mathcal{X}$ is finite, and $|\mathcal{X}|$ represents its dimension or number of elements in the set. For each player $i$ and period $t$, the model imposes $|\mathcal{X}|$ restrictions but it has $5|\mathcal{X}|$ parameters or unknowns, i.e., $4|\mathcal{X}|$ unknowns in the payoff function $\pi_{i t}\left(Y_{i t}, Y_{j t}, \mathbf{X}_{t}\right)$, and $|\mathcal{X}|$ unknowns in the beliefs function $B_{j t}\left(\mathbf{X}_{t}\right)$.

It is important to note that the order condition of identification does not hold even if we assume that beliefs are in equilibrium and make the standard "normalization" assumption in the payoff function (i.e., $\pi_{i t}\left(0, Y_{j t}, \mathbf{X}_{t}\right)=0$ for every $\left(Y_{j t}, \mathbf{X}_{t}\right)$ ). These assumptions imply $3|\mathcal{X}|$ additional restrictions, which are not enough to identify the payoff function. We need to impose at least $|\mathcal{X}|$ additional restrictions to obtain identification. Therefore, even if we are willing to assume equilibrium beliefs, we still have to impose restrictions on the payoff function in order to get identification.

In this paper, we show that some nonparametric restrictions commonly used to get identification of payoffs under the assumption of equilibrium beliefs, provide over-identifying restrictions that can be used to test the hypothesis of equilibrium beliefs and to relax this assumption.

Assumptions ID-2 and ID-2' present two alternative sets of nonparametric restrictions on the payoff function that have been commonly used for identification in games with equilibrium beliefs. ${ }^{10}$ ASSUMPTION ID-2 ('Normalization' and Exclusion Restrictions): ${ }^{11}$ The one-period payoff function $\pi_{i t}$ : (i) is 'normalized' to zero for $Y_{i t}=0$, i.e., $\pi_{i t}\left(0, Y_{j t}, \mathbf{X}_{t}\right)=0$ for any value of $\left(Y_{j t}, \mathbf{X}_{t}\right)$; and (ii) it does not depend on the stock variable of the other player, $S_{j t}$.

$$
\pi_{i t}\left(Y_{i t}, Y_{j t}, S_{i t}, S_{j t}, W_{t}\right)=\left\{\begin{array}{ccc}
\tilde{\pi}_{i t}\left(Y_{j t}, S_{i t}, W_{t}\right) & \text { if } & Y_{i t}=1  \tag{14}\\
0 & \text { if } & Y_{i t}=0
\end{array}\right.
$$

where $\tilde{\pi}_{i t}$ is a real-valued function.
ASSUMPTION ID-2' (Additivity): The one-period payoff function is:

$$
\begin{equation*}
\pi_{i t}\left(Y_{i t}, Y_{j t}, \mathbf{X}_{t}\right)=r_{i t}\left(\left[Y_{i t}+S_{i t}\right]-\left[Y_{j t}+S_{j t}\right], W_{t}\right)-c_{i t}\left(Y_{i t}, S_{i t}, W_{t}\right) \tag{15}
\end{equation*}
$$

[^6]where $r_{i t}$ and $c_{i t}$ are real-valued functions with the 'normalization' conditions $c_{i t}\left(0, S_{i t}, W_{t}\right)=0$, and $r_{i t}\left(\min , W_{t}\right)=0$, where $\min$ is the minimum possible value of the argument $\left[Y_{i t}+S_{i t}\right]-$ $\left[Y_{j t}+S_{j t}\right]$.

Either the exclusion restriction in assumption $I D-2$ or the additivity condition in $I D-2$, are common in empirical applications of dynamic games. For instance, in the dynamic game of market entry and exit of our Example in section 2, assumption ID-2 implies that the profit of a non active firm is zero, and it depends on the own experience but not on the experience of the competitor. An example of a model that satisfies assumption ID-2' is a dynamic game of quality competition and Bertrand price competition between differentiated product firms (e.g., Pakes and McGuire, 1994) where a firm's revenue depends on the difference between the own quality and the quality of the competitor, and production costs depend on the own quality but not on the quality of the competitor. We have also a similar structure in dynamic games of Cournot competition with capacity constraints (e.g., Besanko and Doraszelski, 2004), or in the dynamic game of retail store competition that we present and estimate in section 5.

The restrictions in assumptions $I D-2$ or $I D-2$ ' are sufficient to identify the payoff function in a dynamic game where beliefs are assumed in equilibrium and the support of the vector of state variables $S_{i t}$ contains at least three points, $|\mathcal{S}| \geq 3$. To see this, remember that the number of unknowns in the model is $5|\mathcal{X}|$. Under the assumption of equilibrium beliefs, the number of restrictions is $2|\mathcal{X}|$, i.e., $|\mathcal{X}|$ restrictions from the best response conditions, $P_{i t}=\Psi_{i t}\left(B_{j t}\right)$, and $|\mathcal{X}|$ additional restrictions from the assumption of equilibrium beliefs, $B_{j t}=P_{j t}$. Therefore, identification requires $3|\mathcal{X}|$ restrictions on the payoff function. Let us check the number of restrictions imposed by assumption ID-2. The 'normalization' condition (i) imposes the following $2|\mathcal{X}|$ restrictions: $\pi_{i t}\left(0, Y_{j}, \mathbf{X}\right)=0$ for any value of $\left(Y_{j}, \mathbf{X}\right)$. And the 'exclusion restriction' in condition (ii) implies the $2|\mathcal{W}||\mathcal{S}|(|\mathcal{S}|-1)$ restrictions: $\pi_{i t}\left(1, Y_{j}, S_{i}, S_{j}, W\right)=\pi_{i t}\left(1, Y_{j t}, S_{i}, S_{j}^{\prime}, W\right)$ for any $\left(Y_{j}, S_{i}, W\right)$ and any $S_{j} \neq S_{j}$. In total, assumption $I D$-2 imposes $2|\mathcal{X}|+2|\mathcal{W}||\mathcal{S}|(|\mathcal{S}|-1)$ restrictions on the payoff function. The order condition is satisfied if the number of restrictions is greater or equal than $3|\mathcal{X}|$, and it is simple to verify that this condition is satisfied if $|\mathcal{S}| \geq 2$. Now, let us check for identification under assumption ID-2'. Under this assumption the number of unknowns in the payoff function is $2|\mathcal{W}||\mathcal{S}|$ from the 'revenue' function $r_{i t}$, and $|\mathcal{W}||\mathcal{S}|$ from the cost function $c_{i t}$. Since the unrestricted payoff function $\pi_{i t}\left(Y_{i}, Y_{j}, \mathbf{X}\right)$ has $4|\mathcal{X}|$ unknowns, this assumption imposes $4|\mathcal{X}|-3|\mathcal{W}||\mathcal{S}|$ restrictions on the payoff function. It is simple verify that $|\mathcal{S}| \geq 3$ implies that the number of restrictions is greater than $3|\mathcal{X}|$ and the order condition is satisfied.

However, assumptions ID-2 or ID-2' are not enough for the identification of our model where beliefs may be biased. Before we introduce additional restrictions that provide identification of payoffs and beliefs in our model, we want to show that without further restrictions we can test the null hypothesis of equilibrium beliefs.

For the sake of notational simplicity, we omit the superindex 0 that indicates the "true" value of functions in the population under study.

### 3.3 Test of unbiased beliefs

PROPOSITION 1: Suppose that: (i) the transition probability of the endogenous state variables, $S_{i t}$ and $S_{j t}$, is such that $f_{t}^{S}\left(S_{t+1} \mid Y_{i t}, Y_{j t}, S_{t}, W_{t}\right)=f_{t}^{S}\left(S_{t+1} \mid Y_{i t}, Y_{j t}, W_{t}\right)$; and (ii) a player's expected payoffs at any period $t^{\prime}>t$ are unbiased. Then, under assumptions ID-1 and ID-2(A) (or under assumptions ID-1 and ID-2(B)) the null hypothesis of unbiased beliefs at period $t$ is testable.

Proof: In the Appendix.
The test of unbiased beliefs implied by Proposition 1 is the following. Let $\mathbf{X}^{a}, \mathbf{X}^{b}, \mathbf{X}^{c}$, and $\mathbf{X}^{d}$ be four different values of the vector $\mathbf{X}$ such that they have exactly the same value of the component $\left(S_{i}, W\right)$, but they can have different values for the element $S_{j}$, say $S_{j}^{a}, S_{j}^{b}$, $S_{j}^{c}$, and $S_{j}^{d}$. Under the conditions of Proposition 1, we have that the following equation should hold:

$$
\begin{equation*}
\frac{q_{i t}\left(\mathbf{X}^{a}\right)-q_{i t}\left(\mathbf{X}^{b}\right)}{q_{i t}\left(\mathbf{X}^{c}\right)-q_{i t}\left(\mathbf{X}^{d}\right)}=\frac{B_{j t}\left(\mathbf{X}^{a}\right)-B_{j t}\left(\mathbf{X}^{b}\right)}{B_{j t}\left(\mathbf{X}^{c}\right)-B_{j t}\left(\mathbf{X}^{d}\right)} \tag{16}
\end{equation*}
$$

where $q_{i t}(\mathbf{X}) \equiv \Lambda^{-1}\left(P_{i t}(\mathbf{X})\right)$. Given that the distribution function $\Lambda$ is invertible and it is known (up to scale) to the researcher, the function $q_{i t}($.$) is identified everywhere in the support of \mathbf{X}$. Therefore, the left-hand-side of equation (A.2.2) is identified. This expression shows that under the conditions of Proposition 1 there is a function of beliefs that is identified, without having to impose the assumption of equilibrium beliefs. The right-hand-side of equation (A.2.2) is a function of beliefs only, not of preferences, and it is identified.

This result provides a nonparametric test for the null hypothesis of equilibrium beliefs. Define the function:

$$
\begin{equation*}
\delta_{i t}\left(\mathbf{X}^{a}, \mathbf{X}^{b}, \mathbf{X}^{c}, \mathbf{X}^{d}\right) \equiv\left\{\frac{q_{i t}\left(\mathbf{X}^{a}\right)-q_{i t}\left(\mathbf{X}^{b}\right)}{q_{i t}\left(\mathbf{X}^{c}\right)-q_{i t}\left(\mathbf{X}^{d}\right)}\right\}-\left\{\frac{P_{j t}\left(\mathbf{X}^{a}\right)-P_{j t}\left(\mathbf{X}^{b}\right)}{P_{j t}\left(\mathbf{X}^{c}\right)-P_{j t}\left(\mathbf{X}^{d}\right)}\right\} \tag{17}
\end{equation*}
$$

It is clear that we can nonparametrically identify $\delta_{i t}$. If beliefs are in equilibrium, $\delta_{i t}$ should be equal to zero. In section 4, we describe a test of the null hypothesis of unbiased beliefs based on this result.

### 3.4 Identification of payoff and belief functions

We now present a restriction on beliefs that, together with assumptions ID-1 and ID-2 (or ID-2'), is sufficient to separately identify payoffs and beliefs in the model.

ASSUMPTION ID-3 (No strategic uncertainty at two 'extreme' points): There are two values in the support of the distribution of $S_{j t}$, say $S_{j}^{L}$ and $S_{j}^{H}$, such that: (i) conditional on either of these values the probability distribution of $\left(W_{t}, S_{i t}\right)$ has positive probability on its whole support; and (ii)
for every value of $\left(S_{i}, W\right)$ beliefs are in equilibrium, i.e., $B_{j t}\left(S_{j}^{L}, S_{i}, W\right)=P_{j t}\left(S_{j}^{L}, S_{i}, W\right)$, and $B_{j t}\left(S_{j}^{H}, S_{i}, W\right)=P_{j t}\left(S_{j}^{H}, S_{i}, W\right)$.

The intuition behind this assumption is simple. For some values of the opponent's stock variable (e.g., very large or very small values) strategic uncertainty disappears and beliefs about the opponent's choice probabilities become unbiased. For instance, this might be just because for these values of the stock variable the opponent's choice becomes certain, i.e., his choice probability gets arbitrarily close to zero or one. But the conditions for the absence of strategic uncertainty in assumption ID-3 are more general.

The following Proposition summarizes our main identification result.
PROPOSITION 2: Under assumptions ID-1, ID-2, and ID-3 the payoff functions $\left\{\pi_{i t}\right.$ for any $i, t\}$ and the beliefs functions $\left\{B_{i t}\right.$ for any $\left.i, t\right\}$ are nonparametrically identified.

Proof. In the Appendix.
Our proof of Proposition 2 is constructive and it provides closed-form expressions of the unknown parameters (payoffs and beliefs) in terms of the identified CCP functions. For the description of these formulas, it is useful to introduce the concepts of integrated value function and continuation value function. ${ }^{12}$ The integrated value function is defined as $\bar{V}_{i t}^{\mathbf{B}}\left(\mathbf{X}_{t}\right) \equiv \int V_{i t}^{\mathbf{B}}\left(\mathbf{X}_{t}, \varepsilon_{i t}\right) d G\left(\varepsilon_{i t}\right)$. The continuation value function provides the expected and discounted value of future payoffs given future beliefs, current state, and current choices of both players. It is defined as $d_{i t}\left(Y_{i t}, Y_{j t}, \mathbf{X}_{t}\right) \equiv$ $\beta \sum_{\mathbf{X}_{t+1}} \bar{V}_{i t+1}^{\mathbf{B}}\left(\mathbf{X}_{t+1}\right) f_{t}\left(\mathbf{X}_{t+1} \mid Y_{i t}, Y_{j t}, \mathbf{X}_{t}\right)$. The integrated Bellman equation implies the following recursive relationship between integrated and continuation value functions:

$$
\begin{equation*}
\bar{V}_{i t}^{\mathbf{B}}\left(\mathbf{X}_{t}\right)=\int \max _{Y_{i t} \in\{0,1\}}\left\{v_{i t}^{\mathbf{B}}\left(Y_{i t}, \mathbf{X}_{t}\right)+\varepsilon_{i t}\left(Y_{i t}\right)\right\} d G\left(\varepsilon_{i t}\right) \tag{18}
\end{equation*}
$$

where $v_{i t}^{\mathbf{B}}\left(Y_{i t}, \mathbf{X}_{t}\right)=\left(1-B_{j t}\left(\mathbf{X}_{t}\right)\right)\left[\pi_{i t}\left(Y_{i t}, 0, \mathbf{X}_{t}\right)+d_{i t}\left(Y_{i t}, 0, \mathbf{X}_{t}\right)\right]+B_{j t}\left(\mathbf{X}_{t}\right)\left[\pi_{i t}\left(Y_{i t}, 1, \mathbf{X}_{t}\right)+d_{i t}\left(Y_{i t}, 1, \mathbf{X}_{t}\right)\right]$. If $\left\{\varepsilon_{i t}(0), \varepsilon_{i t}(1)\right\}$ have an independent double exponential distribution, this integrated Bellman equation becomes:

$$
\begin{equation*}
\bar{V}_{i t}^{\mathbf{B}}\left(\mathbf{X}_{t}\right)=v_{i t}^{\mathbf{B}}\left(0, \mathbf{X}_{t}\right)-\ln \left(1-P_{i t}\left(\mathbf{X}_{t}\right)\right) \tag{19}
\end{equation*}
$$

and $P_{i t}\left(\mathbf{X}_{t}\right)=\exp \left\{v_{i t}^{\mathbf{B}}\left(1, \mathbf{X}_{t}\right)-v_{i t}^{\mathbf{B}}\left(0, \mathbf{X}_{t}\right)\right\} /\left[1+\exp \left\{v_{i t}^{\mathbf{B}}\left(1, \mathbf{X}_{t}\right)-v_{i t}^{\mathbf{B}}\left(0, \mathbf{X}_{t}\right)\right\}\right]$.
Proposition 2 shows that we can identify payoff and belief functions using the following recursive procedure. We omit the the vector of state variables $\mathbf{X}$ as an argument, and use superindexes $H$ and $L$ to indicate functions evaluated at $\mathbf{X}^{H} \equiv\left(S_{i}, S_{j}^{H}, W\right)$ and $\mathbf{X}^{L} \equiv\left(S_{i}, S_{j}^{L}, W\right)$, respectively. To identify payoff and beliefs functions at every period $t$, we start at the last period $t=T$ and

[^7]apply the following recursive formulas, in this order: (i) the payoff function,
\[

$$
\begin{align*}
\Delta_{i t} \equiv \widetilde{\pi}_{i t}(1)-\widetilde{\pi}_{i t}(0) & =\frac{\left[q_{i t}^{H}-\left(1-P_{j t}^{H}\right) \widetilde{d}_{i t}^{H}(0)-P_{j t}^{H} \widetilde{d}_{i t}^{H}(1)\right]-\left[q_{i t}^{L}-\left(1-P_{j t}^{L}\right) \widetilde{d}_{i t}^{L}(0)+P_{j t}^{L} \widetilde{d}_{i t}^{L}(1)\right]}{P_{j t}^{H}-P_{j t}^{L}} \\
\widetilde{\pi}_{i t}(0) & =q_{i t}^{L}-\widetilde{d}_{i t}^{L}(0)-P_{j t}^{L}\left[\Delta_{i t}+\widetilde{d}_{i t}^{L}(1)-\widetilde{d}_{i t}^{L}(0)\right] \\
\widetilde{\pi}_{i t}(1) & =\Delta_{i t}+q_{i t}^{L}-\widetilde{d}_{i t}^{L}(0)-P_{j t}^{L}\left[\Delta_{i t}+\widetilde{d}_{i t}^{L}(1)-\widetilde{d}_{i t}^{L}(0)\right] \tag{20}
\end{align*}
$$
\]

where $\widetilde{d}_{i t}\left(Y_{j}, \mathbf{X}\right) \equiv d_{i t}\left(1, Y_{j}, \mathbf{X}\right)-d_{i t}\left(0, Y_{j}, \mathbf{X}\right)$; (ii) the beliefs function,

$$
\begin{equation*}
B_{j t}=\frac{q_{i t}-\widetilde{\pi}_{i t}(0)-\widetilde{d}_{i t}(0)}{\widetilde{\pi}_{i t}(1)-\widetilde{\pi}_{i t}(0)+\widetilde{d}_{i t}(1)-\widetilde{d}_{i t}(0)} ; \tag{21}
\end{equation*}
$$

(iii) integrated value function,

$$
\begin{align*}
\bar{V}_{i t}^{\mathbf{B}} & =v_{i t}^{\mathbf{B}}(0)+\ln \left(1-P_{i t}\right) \\
& =\left[\left(1-B_{j t}\right) d_{i t}(0,0)+B_{j t} d_{i t}(0,1)\right]-\ln \left(1-P_{i t}\right) \tag{22}
\end{align*}
$$

and (iv) previous period continuation value function, $d_{i t-1}\left(Y_{i}, Y_{j}, \mathbf{X}\right) \equiv \beta \sum_{\mathbf{X}^{\prime}} \bar{V}_{i t}^{\mathbf{B}}\left(\mathbf{X}^{\prime}\right) f_{t-1}\left(\mathbf{X}^{\prime} \mid Y_{i}, Y_{j}, \mathbf{X}\right)$. To initilize the backwards induction procedure at period $T$, we take into account that the continuation values at the last period are zero, i.e., $d_{i T}=0$.
**** SOME DISCUSSION / INTUITION OF THESE FORMULAS ****

### 3.5 Extensions

PROPOSITION 3 (Infinite horizon dynamic game): If the game has infinite horizon and all the functions (payoff, transition probability, and beliefs) are constant over time, then under assumptions ID-1, ID-2, and ID-3 the payoff function $\pi_{i}$ and the beliefs function $B_{j}$ are nonparametrically identified.

Proof. In the Appendix.
PROPOSITION 4 (Finite mixture, time-invariant, unobserved heterogeneity).

## 4 Estimation and Inference

We begin this section by presenting a simple nonparametric test of the null hypothesis of equilibrium beliefs. Sections 4.2 and 4.3 deal with estimation. Our proof of identification above suggests a method for the estimation of the just-identified nonparametric model. Section 4.2 provides a detailed description of that estimation method. In most empirical applications, the specification of the payoff function involves parametric restrictions. Therefore, we extend the estimation method to deal with parametric models. Finally, for parametric models, we describe how the two-step method can be extended recursively to generate a sequence of estimators with better statistical properties.

### 4.1 Test of Equilibrium Beliefs

In principle, we could use a standard Lagrange Multiplier (LM) or Score test of the null hypothesis of equilibrium beliefs. That test is based on the constrained maximum likelihood estimation (MLE) of structural parameters and beliefs. Let $\boldsymbol{\theta}$ be the vector of structural parameters in the payoff function. Define the log-likelihood function:

$$
\begin{equation*}
l(\boldsymbol{\theta}, \mathbf{P}) \equiv \sum_{m=1}^{M} \sum_{t=1}^{T} \sum_{i=1}^{2} Y_{i m t} \log \Lambda\left(v_{i t}^{\mathbf{P}}\left(\mathbf{X}_{m t}, \boldsymbol{\theta}\right)\right)+\left(1-Y_{i m t}\right) \log \left(1-\Lambda\left(v_{i t}^{\mathbf{P}}\left(\mathbf{X}_{m t}, \boldsymbol{\theta}\right)\right)\right) \tag{23}
\end{equation*}
$$

The constrained MLE is defined as a vector ( $\hat{\boldsymbol{\theta}}_{M L E}, \hat{\mathbf{P}}_{M L E}$ ) such that:

$$
\begin{align*}
\left(\hat{\boldsymbol{\theta}}_{M L E}, \hat{\mathbf{P}}_{M L E}\right)= & \arg \max _{(\theta, \mathbf{P})} l(\boldsymbol{\theta}, \mathbf{P})  \tag{24}\\
& \text { subject to: } \mathbf{P}=\Lambda\left(v^{\mathbf{P}}(\boldsymbol{\theta})\right)
\end{align*}
$$

We want to test the null hypothesis $\mathbf{P}=\Lambda\left(v^{\mathbf{P}}(\boldsymbol{\theta})\right)$, that consists of $2|\mathcal{X}|$ constrains on $(\boldsymbol{\theta}, \mathbf{P})$. The standard LM statistic for testing this null hypothesis is:

$$
\begin{equation*}
L M=\frac{\partial l\left(\hat{\boldsymbol{\theta}}_{M L E}, \hat{\mathbf{P}}_{M L E}\right)}{\partial(\boldsymbol{\theta}, \mathbf{P})^{\prime}}\left[\frac{\partial^{2} l\left(\hat{\boldsymbol{\theta}}_{M L E}, \hat{\mathbf{P}}_{M L E}\right)}{\partial(\boldsymbol{\theta}, \mathbf{P}) \partial(\boldsymbol{\theta}, \mathbf{P})^{\prime}}\right]^{-1} \frac{\partial l\left(\hat{\boldsymbol{\theta}}_{M L E}, \hat{\mathbf{P}}_{M L E}\right)}{\partial(\boldsymbol{\theta}, \mathbf{P})} \tag{25}
\end{equation*}
$$

Under the null hypothesis, this statistic is asymptotically distributed as a chi-square with $2|\mathcal{X}|$ degrees of freedom.

This LM test has at least two important limitations. A first limitation is its implementation. Maximum likelihood estimation of dynamic games is computationally very demanding both because the high dimension of the state space and because of the existence of multiple equilibria. Second, this is a general specification test. The null hypothesis is not only that beliefs are in equilibrium but also that the parametric specification of preferences and the distribution of unobservables is correct. We would like to have a procedure that specifically tests for the equilibrium beliefs and not for other specification assumptions of the model.

The test that we propose is the following. Let $\mathbf{X}^{a}, \mathbf{X}^{b}, \mathbf{X}^{c}$, and $\mathbf{X}^{d}$ be four different values of the vector $\mathbf{X}$ that have the same value of the component $\left(S_{i}, W\right)$ and different values for the element $S_{j}$, say $S_{j}^{a}, S_{j}^{b}, S_{j}^{c}$, and $S_{j}^{d}$, such that $S_{j}^{a} \neq S_{j}^{b}$ and $S_{j}^{c} \neq S_{j}^{d}$. Define the function $\delta_{i}^{0}$ :

$$
\begin{equation*}
\delta_{i}\left(\mathbf{X}^{a}, \mathbf{X}^{b}, \mathbf{X}^{c}, \mathbf{X}^{d}\right) \equiv\left\{\frac{q_{i}\left(\mathbf{X}^{a}\right)-q_{i}\left(\mathbf{X}^{b}\right)}{q_{i}\left(\mathbf{X}^{c}\right)-q_{i}\left(\mathbf{X}^{d}\right)}\right\}-\left\{\frac{P_{j}\left(\mathbf{X}^{a}\right)-P_{j}\left(\mathbf{X}^{b}\right)}{P_{j}\left(\mathbf{X}^{c}\right)-P_{j}\left(\mathbf{X}^{d}\right)}\right\} \tag{26}
\end{equation*}
$$

As shown in section 3, under Assumptions 1-4, if player $i$ has rational beliefs then $\delta_{i}\left(\mathbf{X}^{a}, \mathbf{X}^{b}, \mathbf{X}^{c}\right.$, $\left.\mathbf{X}^{d}\right)=0$ for every value of $\left(\mathbf{X}^{a}, \mathbf{X}^{b}, \mathbf{X}^{c}, \mathbf{X}^{d}\right)$. Therefore, testing $\delta_{i}\left(\mathbf{X}^{a}, \mathbf{X}^{b}, \mathbf{X}^{c}, \mathbf{X}^{d}\right)=0$ implies testing the null hypothesis of rational beliefs (and Assumptions 1 to 4).

Let $H$ be the number of all possible combinations of four different values of $S_{j}$ with $S_{j}^{a} \neq S_{j}^{b}$ and $S_{j}^{c} \neq S_{j}^{d}$. We index these values by $h$. Let $\tilde{\mathbf{X}}_{m t}^{(h)}$ be the quadruplet $h$ when the values of $S_{i}$ and
$W$ are the ones in observation $(m, t)$ : i.e., $\tilde{\mathbf{X}}_{m t}^{(h)}=\left(\mathbf{X}_{m t}^{(h) a}, \mathbf{X}_{m t}^{(h) b}, \mathbf{X}_{m t}^{(h) c}, \mathbf{X}_{m t}^{(h) d}\right)=\left(\left[S_{j}^{(h) a}, S_{i m t}, W_{m t}\right]\right.$, $\left.\left[S_{j}^{(h) b}, S_{i m t}, W_{m t}\right],\left[S_{j}^{(h) c}, S_{i m t}, W_{m t}\right],\left[S_{j}^{(h) d}, S_{i m t}, W_{m t}\right]\right)$. Under the hypothesis of equilibrium beliefs, we have that $E\left(\delta_{i}\left(\tilde{\mathbf{X}}_{m t}^{(h)}\right)\right)=0$ for every $h$, where the expectation $E($.$) is taken over the distribution$ of $\left(S_{i m t}, W_{m t}\right)$. This is exactly the null hypothesis that we test:

$$
\begin{equation*}
H_{0}: E\left(\delta_{i}\left(\tilde{\mathbf{X}}_{m t}^{(h)}\right)\right)=0 \quad \text { for every quadruple } h \tag{27}
\end{equation*}
$$

Let $\hat{\delta}_{i}($.$) be the estimator of \delta_{i}($.$) that we obtain when we replace P_{i}$ and $P_{j}$ by nonparametric estimates of these CCP functions. Define the statistic $\bar{\delta}_{i}^{(h)}$ as the sample mean of $\hat{\delta}_{i}\left(\tilde{\mathbf{X}}_{m t}^{(h)}\right)$, i.e., $\bar{\delta}_{i}^{(h)}=(M T)^{-1} \sum_{m=1}^{M} \sum_{t=1}^{T} \hat{\delta}_{i}\left(\tilde{\mathbf{X}}_{m t}^{(h)}\right)$. Then, define the statistic:

$$
\begin{equation*}
\hat{D}=\sum_{h=1}^{H}\left(\frac{\bar{\delta}_{i}^{(h)}}{s e\left(\bar{\delta}_{i}^{(h)}\right)}\right)^{2} \tag{28}
\end{equation*}
$$

where $\operatorname{se}\left(\bar{\delta}_{i}^{(h)}\right)$ is the standard error of $\bar{\delta}_{i}^{(h)}$, that we can obtain using nonparametric bootstrap. Under the null hypothesis, $\hat{S}$ is asymptotically distributed as a Chi-square with $H$ degrees of freedom.

### 4.2 Estimation with nonparametric payoff function

For notational simplicity, unless it is strictly necessary, I omit the vector of state variables $\mathbf{X}$ as an argument in functions.

Step 1: Nonparametric estimation of $C C P s, \widehat{P}_{i t}$, for every player, time period, and state X, and (if needed) of the transition probabilities $f_{t}^{S}$ and $f_{t}^{W}$. We also construct $\widehat{q}_{i t}=\Lambda^{-1}\left(\widehat{P}_{i t}\right)$.

Step 2: Recursive estimation of preferences and beliefs. We start at the last period $T$, where the continuation function is zero, and apply recursively the following formulas to estimate payoff and beliefs functions t every period $t$ : (i) the payoff function,

$$
\begin{align*}
& \widehat{\Delta}_{i t}=\frac{\left[\widehat{q}_{i t}^{H}-\left(1-\widehat{P}_{j t}^{H}\right) \hat{\tilde{d}}_{i t}^{H}(0)-\widehat{P}_{j t}^{H}\right.}{\left.\hat{d}_{i t}^{H}(1)\right]-\left[\widehat{q}_{i t}^{L}-\left(1-\widehat{P}_{j t}^{L}\right) \widehat{\tilde{d}}_{i t}^{L}(0)+\widehat{P}_{j t}^{L} \widehat{\widetilde{d}}_{i t}^{L}(1)\right]} \\
& \widehat{P}_{j t}^{H}-\widehat{P}_{j t}^{L}  \tag{29}\\
& \widehat{\tilde{\pi}}_{i t}(0)=\widehat{q}_{i t}^{L}-\widehat{\tilde{d}}_{i t}^{L}(0)-\widehat{P}_{j t}^{L}\left[\Delta_{i t}+\widehat{\tilde{d}}_{i t}^{L}(1)-\widehat{\tilde{d}}_{i t}^{L}(0)\right] \\
& \widehat{\tilde{\pi}}_{i t}(1)=\widehat{\Delta}_{i t}+\widehat{\tilde{\pi}}_{i t}(0)
\end{align*}
$$

(ii) the beliefs function,

$$
\begin{equation*}
\widehat{B}_{j t}=\frac{\widehat{q}_{i t}-\widehat{\widetilde{\pi}}_{i t}(0)-\widehat{\widetilde{d}}_{i t}(0)}{\widehat{\widetilde{\pi}}_{i t}(1)-\widehat{\widetilde{\pi}}_{i t}(0)+\widehat{\widetilde{d}}_{i t}(1)-\widetilde{\widetilde{d}}_{i t}(0)} ; \tag{30}
\end{equation*}
$$

(iii) integrated value function,

$$
\begin{equation*}
\widehat{\bar{V}}_{i t}^{\mathbf{B}}=\left[\left(1-\widehat{B}_{j t}\right) \widehat{d}_{i t}^{\mathbf{B}}(0,0)+\widehat{B}_{j t} \widehat{d}_{i t}^{\mathbf{B}}(0,1)\right]-\ln \left(1-\widehat{P}_{i t}\right) ; \tag{31}
\end{equation*}
$$

and (iv) previous period continuation value function, $\widehat{d}_{i t-1}^{\mathbf{B}}\left(Y_{i}, Y_{j}, \mathbf{X}\right) \equiv \beta \sum_{\mathbf{X}^{\prime}} \widehat{\bar{V}}_{i t}^{\mathbf{B}}\left(\mathbf{X}^{\prime}\right) \widehat{f}_{t-1}\left(\mathbf{X}^{\prime} \mid Y_{i}, Y_{j}, \mathbf{X}\right)$.
This estimator is consistent and asymptotically normal. The derivation of the asymptotic variance is cumbersome. In our empirical application we use the bootstrap method to obtain standard errors and confidence intervals for the estimates.

### 4.3 Estimation with parametric payoff function

In most applications, we assume a parametric specification of the payoff function. A very common class of parametric specifications is the linear in parameters model:

$$
\begin{equation*}
\widetilde{\pi}_{i t}\left(Y_{j t}, \mathbf{X}_{t}\right)=z_{i t}\left(Y_{j t}, \mathbf{X}_{t}\right) \boldsymbol{\theta}_{i} \tag{32}
\end{equation*}
$$

where $z_{i t}\left(Y_{j t}, \mathbf{X}_{t}\right)$ is a $1 \times K$ vector of known functions, and $\boldsymbol{\theta}_{i}$ is a $K \times 1$ vector of unknown structural parameters in player $i$ 's payoff function. Let $\boldsymbol{\theta}$ be $\left\{\boldsymbol{\theta}_{i}: i=1,2\right\}$. For instance, in the dynamic game of market entry and exit in the Example of section 2, the profit function in equation (2) can be written as $\pi_{i}\left(Y_{j t}, S_{i t}, W_{t}\right)=z_{i t}\left(Y_{j t}, S_{i t}, W_{t}\right) \boldsymbol{\theta}_{i}$, where the vector of parameters $\boldsymbol{\theta}_{i}$ is $\left(\theta_{i}^{M}, \theta_{i}^{D}, \theta_{i 0}^{F C}, \theta_{i 1}^{F C}, \theta_{i}^{E C}\right)^{\prime}$ and

$$
\begin{equation*}
z_{i t}\left(Y_{j t}, S_{i t}, W_{t}\right)=\left\{W_{t}\left(1-Y_{j t}\right), W_{t} Y_{j t},-1,-\exp \left\{-S_{i t}\right\},-1\left\{S_{i t}=0\right\}\right\} \tag{33}
\end{equation*}
$$

Given this specification, the model implies the following relationship:

$$
\begin{equation*}
q_{i t}(\mathbf{X})=z_{i t}^{\mathbf{B}}(\mathbf{X}) \boldsymbol{\theta}_{i} \tag{34}
\end{equation*}
$$

where $z_{i t}^{\mathbf{B}}(\mathbf{X}) \equiv\left(1-B_{j t}(\mathbf{X})\right) z_{i t}(0, \mathbf{X})+B_{j t}(\mathbf{X}) z_{i t}(1, \mathbf{X})$, and $\boldsymbol{\theta}_{i}$ is the true vector of parameters in the population.

To estimate $\boldsymbol{\theta}_{i}$ we propose a simple three steps method. The first two-steps are the same as for the nonparametric model.

Step 3: Given the estimates from step 2, we can apply a pseudo maximum likelihood method in the spirit of Aguirregabiria and Mira $(2002,2007)$ to estimate the structural parameters $\boldsymbol{\theta}^{0}$. Define the pseudo likelihood function:

$$
Q(\boldsymbol{\theta}, \mathbf{B}, \mathbf{P}) \equiv \sum_{m=1}^{M} \sum_{t=1}^{T} \sum_{i=1}^{2} Y_{i m t} \log \Lambda\left(\tilde{z}_{i m t}^{\mathbf{B}, \mathbf{P}} \boldsymbol{\theta}_{i}+\tilde{e}_{i m t}^{\mathbf{B}, \mathbf{P}}\right)+\left(1-Y_{i m t}\right) \log \left(1-\Lambda\left(\tilde{z}_{i m t}^{\mathbf{B}, \mathbf{P}} \boldsymbol{\theta}_{i}+\tilde{e}_{i m t}^{\mathbf{B}, \mathbf{P}}\right)\right)
$$

$\tilde{z}_{i m t}^{\mathbf{B}, \mathbf{P}}$ is the sum of expected and discounted stream of $\left\{z_{i t^{\prime}}\left(Y_{j t^{\prime}}, \mathbf{X}_{t^{\prime}}\right): t^{\prime}=t, t+1, \ldots, T\right\}$ given that player $i$ behaves according to the choice probabilities $P_{i t^{\prime}}($.$) in \mathbf{P}$, and player $j$ behaves according
to the probabilities $B_{j t^{\prime}}($.$) in \mathbf{B}$. Similarly, $\tilde{e}_{i m t}^{\mathbf{B}, \mathbf{P}}$ is the sum of expected and discounted stream of $\left\{e\left(P_{i t^{\prime}}\left(\mathbf{X}_{t^{\prime}}\right)\right): t^{\prime}=t, t+1, \ldots, T\right\}$, and for the logit model $e\left(P_{i t^{\prime}}\left(\mathbf{X}_{t^{\prime}}\right)\right)=\gamma-\ln P_{i t^{\prime}}\left(\mathbf{X}_{t^{\prime}}\right)$ where $\gamma$ is Euler's constant. From steps 1 and 2, we have consistent estimates of CCPs, $\hat{\mathbf{P}}^{0}$, and beliefs, $\hat{\mathbf{B}}^{0}$. Then, a consistent pseudo maximum likelihood estimator of $\boldsymbol{\theta}$ is defined as the value $\hat{\boldsymbol{\theta}}^{(1)}$ that maximizes $Q\left(\boldsymbol{\theta}, \hat{\mathbf{B}}^{0}, \hat{\mathbf{P}}^{0}\right)$. Note that the sample criterion function $Q\left(\boldsymbol{\theta}, \hat{\mathbf{B}}^{0}, \hat{\mathbf{P}}^{0}\right)$ is just the log likelihood function of a standard logit model with the restriction that the parameter of variable $\tilde{e}_{i m t}^{\mathbf{B}, \mathbf{P}}$ is equal to 1 . The estimator is root-M consistent and asymptotically normal.

In fact, steps 1 to 3 can be applied recursively to improve the statistical properties of our estimators. Given the parametric estimator of the payoff function in step $3, \hat{\pi}_{i t}^{(1)}\left(Y_{j t}, \mathbf{X}_{t}\right)=z_{i t}\left(Y_{j t}, \mathbf{X}_{t}\right)$ $\hat{\boldsymbol{\theta}}_{i}^{(1)}$, that is more precise than the original nonparametric estimator, we can obtain a more precise estimator of players' CCPs. The new estimator of CCPs is based on the best response condition $P_{i t}(\mathbf{X})=\Lambda\left(\tilde{z}_{i t}^{\mathbf{B}, \mathbf{P}}(\mathbf{X}) \boldsymbol{\theta}_{i}^{0}+\tilde{e}_{i t}^{\mathbf{B}, \mathbf{P}}(\mathbf{X})\right)$ and it is equal to:

$$
\begin{equation*}
\hat{P}_{i t}^{(1)}(\mathbf{X})=\Lambda\left(\tilde{z}_{i t}^{\hat{\mathbf{B}}^{0}, \hat{\mathbf{P}}^{0}}(\mathbf{X}) \hat{\boldsymbol{\theta}}_{i}^{(1)}+\tilde{e}_{i t}^{\hat{\mathbf{B}}^{0}, \hat{\mathbf{P}}^{0}}(\mathbf{X})\right) \tag{35}
\end{equation*}
$$

where $\hat{\mathbf{B}}^{0}$ and $\hat{\mathbf{P}}^{0}$ represent the initial nonparametric estimators of beliefs and CCPs, respectively, and $\hat{\boldsymbol{\theta}}_{i}^{1}$ is the pseudo maximum likelihood estimator of the parameters in the payoff function. And given this new estimator of CCPs we can also obtain a new estimator of beliefs. The new estimator is:

$$
\begin{equation*}
\widehat{B}_{j t}^{(1)}=\frac{\widehat{q}_{i t}^{(1)}-\widehat{\widetilde{\pi}}_{i t}^{(1)}(0)-\widehat{\widetilde{d}}_{i t}^{(1)}(0)}{\left[\widehat{\widetilde{\pi}}_{i t}^{(1)}(1)-\widehat{\widetilde{\pi}}_{i t}^{(1)}(0)\right]+\left[\widehat{\widehat{d}}_{i t}^{(1)}(1)-\widehat{\widetilde{d}}_{i t}^{(1)}(0)\right]} ; \tag{36}
\end{equation*}
$$

where $\hat{P}_{j t}^{(1)}, \hat{q}_{i t}^{(1)}, \widehat{\widetilde{\pi}}_{i t}^{(1)}$, and $\hat{\widetilde{d}}_{i t}^{(1)}$, are the new estimates of CCPs, payoffs and continuation values. Given $\hat{\mathbf{B}}^{(1)}$ and $\hat{\mathbf{P}}^{(1)}$ we can apply step 3 again to obtain a new vector of estimates of the structural parameters, $\widehat{\boldsymbol{\theta}}^{(2)}$. We can apply this procedure recursively to update CCPs, beliefs, and structural parameters and to obtain a sequence of estimators $\left\{\hat{\boldsymbol{\theta}}^{(K)}, \hat{\mathbf{B}}^{(K)}, \hat{\mathbf{P}}^{(K)}: K \geq 1\right\}$.

### 4.4 Estimation with semiparametric beliefs

So far we have considered a fully nonparametric specification of players beliefs. In some applications, the researcher may be willing to consider a more restrictive class of players' beliefs as long as it is flexible enough and it includes equilibrium beliefs as a specific case. Here we present that type of specification and describe the estimation of this model.

We assume that player $i$ 's beliefs about the behavior of player $j$ has the following form:

$$
\begin{equation*}
B_{j t}(\mathbf{X})=\frac{\sum_{\mathbf{X}^{\prime} \in \mathcal{X}} P_{j t}\left(\mathbf{X}^{\prime}\right) K\left(\frac{\mathbf{X}^{\prime}-\mathbf{X}}{\sigma_{j t}(\mathbf{X})}\right)}{\sum_{\mathbf{X}^{\prime} \in \mathcal{X}} K\left(\frac{\mathbf{X}^{\prime}-\mathbf{X}}{\sigma_{j t}(\mathbf{X})}\right)} \tag{37}
\end{equation*}
$$

where $K($.$) is a Kernel function (e.g., the density of the standard normal), and \sigma_{j t}(\mathbf{X}) \geq 0$ is a bandwidth function. The main idea behind this specification is that the potential biases in players' beliefs come from the aggregation across different states. This specification includes equilibrium beliefs as a particular case. When $\sigma_{j t}(\mathbf{X})=0$ for any value of $\mathbf{X}$, we have that $B_{j t}(\mathbf{X})=P_{j t}(\mathbf{X})$ at every state $\mathbf{X}$. Otherwise, when $\sigma_{j t}(\mathbf{X})>0$, we have that $B_{j t}(\mathbf{X}) \neq P_{j t}(\mathbf{X})$ and players' beliefs are biased. The bandwidth is a measure of how noisy or biased a player's beliefs are.

If the bandwidth function $\sigma_{j t}($.$) is nonparametrically specified, then the previous specification$ of beliefs is fully nonparametric. We impose some restrictions on this bandwidth function. The main idea behind our assumptions is that the bias in the belief $B_{j t}(\mathbf{X})$ increases with the variance of the binary variable $Y_{j t}$ conditional on $\mathbf{X}_{t}$, i.e., with $P_{j t}(\mathbf{X})\left(1-P_{j t}(\mathbf{X})\right)$. This variance goes to zero as $P_{j t}(\mathbf{X})$ goes to zero or goes to one, and it is concave with respect to $P_{j t}(\mathbf{X})$. We consider the following three-parameters function:

$$
\begin{equation*}
\sigma_{j t}(\mathbf{X})=\lambda_{j t}^{0} P_{j t}(\mathbf{X})^{\lambda_{j t}^{1}}\left(1-P_{j t}(\mathbf{X})\right)^{\lambda_{j t}^{2}} \tag{38}
\end{equation*}
$$

where $\lambda_{j t}^{0}>0, \lambda_{j t}^{1} \in[0,1]$, and $\lambda_{j t}^{2} \in[0,1]$, are unknown parameters for the researcher.

## 5 Monte Carlo Experiments

## 6 Empirical Application

We illustrate our model and methods with an application of a dynamic game of store location. There has been recently a significant interest in the estimation of game theoretic models of market entry and store location by retail firms. Most studies have assumed static games: see Mazzeo (2002), Seim (2006), Jia (2008), Nishida (2008), and Zhu and Singh (2009), among others. Holmes (2010) estimates a single-agent dynamic model of store location by Wal-Mart. Beresteanu and Ellickson (2005), Walrath (2008), and Suzuki (2010) propose and estimate dynamic games of store location.

We study store location of McDonalds (MD) and Burger King (BK) using data for the United Kingdom during the period 1991-1995. We divide the UK in local markets (districts) and study these companies' decision of how many stores, if any, to operate in each local market. The profits of a store in a market depends on local demand and cost conditions and on the degree of competition from other firms' stores and from stores of the same chain. There are sunk costs associated with opening a new store, and therefore this decision has implications for future profits. Firms are forward-looking and maximize the value of expected and discounted profits. Each firm has uncertainty about future demand and cost conditions in local markets. Firms also have uncertainty about the current and future behavior of the competitor. In this context, the standard assumption
is that firms have rational expectations about other firms' strategies, and that these strategies constitute a Markov Perfect Equilibrium. Here we relax this assumption. The main question that we want to analyze in this empirical application is whether the beliefs of each of these companies about the store location strategy of the competitor are consistent with the actual behavior of the competitor.

### 6.1 Data and descriptive evidence

The dataset was collected by Otto Toivanen and Michael Waterson, who have used it in their paper Toivanen and Waterson (2005). ${ }^{13}$ Our working sample is a five year panel that tracks 422 local authority districts (local markets), including the information on the stock and flow of MD and BK stores into each district. It also contains socio-economic variables at the district level such as population, density, age distribution, average rent, income per capita, local retail taxes, and distance to the UK headquarters of each of the firms. The local authority district is the smallest unit of local government in the UK, and generally consists of a city or a town sometimes with a surrounding rural area. There are almost 500 local authority districts in Great Britain. Our working sample of 422 districts does not include those that belong to Greater London. ${ }^{14}$ The median district in our sample has an area of 300 square kilometers and a population of 95,000 people. ${ }^{15}$ Table 2 presents descriptive statistics for socio-economic and geographic characteristics of our sample of local authority districts.

[^8]Table 2
Descriptive Statistics on Local Markets (Year 1991)
422 local authority districts (excluding Greater London districts)

| Variable | Median | Std. Dev. | Pctile 5\% | Pctile 95\% |
| ---: | :---: | :---: | :---: | :---: |
| Area (thousand square km) | 0.30 | 0.73 | 0.03 | 1.67 |
| Population (thousands) | 94.85 | 93.04 | 37.10 | 280.50 |
| Children: Age 5-14 (\%) | 12.43 | 1.00 | 10.74 | 14.07 |
| Young: 15-29 (\%) | 21.24 | 2.46 | 17.80 | 25.17 |
| Pensioners: 65-74 (\%) | 9.01 | 1.50 | 6.89 | 11.82 |
| GDP per capita (thousand £) | 92.00 | 12.14 | 74.40 | 112.70 |
| Claimants of UB / Population ratio (\%) | 2.75 | 1.27 | 1.24 | 5.11 |
| Avg. Weekly Rent per dwelling (£) | 25.31 | 10.61 | 19.11 | 35.07 |
| Council tax (thousand $£)$ | 0.24 | 0.05 | 0.11 | 0.31 |
| Number of BK stores | 0.00 | 0.62 | 0.00 | 1.00 |
| Number of MD stores | 1.00 | 1.16 | 0.00 | 3.00 |

Table 3 presents descriptive statistics on the evolution of the number of stores for the two firms. Toivanen and Waterson present a detailed discussion of why the retail chain fast food hamburger industry in the UK during this period can be assumed as a duopoly of BK and MD. In 1990, MD had more than three times the number of stores of BK, and it was active in more than twice the number of local markets than BK. Conditional on being active in a local market, MD had also significantly more stores per market than BK. These differences between MD and BK have not declined significantly over the period 1991-1995. While BK has entered in more new local markets than MD ( 69 new markets for BK and 48 new markets for MD), MD has opened more stores (143 new stores for BK and 166 new stores for MD).

\left.| Table 3 |  |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Evolution of the Number of Stores |  |
| 422 local authority districts (excluding Greater London districts) |  |$\right]$

Table 4 presents the annual transition probabilities of market structure in local markets as described by the number of stores of the two firms. According to this transition matrix, opening a new store is an irreversible decision, i.e., no store closings are observed during this sample period. In Britain during our sample period, the fast food hamburger industry was still young and expanding, as shown by the large proportion of observations/local markets without stores (41.6\%). Although there is significant persistence in every state, the less persistent market structures are those where BK is the leader. For instance, if the state is " $B K=1 \& M D=0$ ", there is a $20 \%$ probability that the next year MD opens at least one store in the market. Similarly, when the state is " $B K=2 \&$ $M D=1^{\prime \prime}$, the chances that MD opens one more store the next year are $31 \%$.

## Table 4

Transition Probability Matrix for Market Structure
Annual Transitions. Market structure: $\mathrm{BK}=\mathrm{x} \& \mathrm{MD}=\mathrm{y}$, where x and y are number of stores

|  | Market Structure at $\mathbf{t + 1}$ |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\begin{aligned} & \mathrm{BK}=0 \\ & \mathrm{MD}=0 \end{aligned}$ | $\begin{aligned} & \mathrm{BK}=0 \\ & \mathrm{MD}=1 \end{aligned}$ | $\begin{gathered} \mathrm{BK}=0 \\ \mathrm{MD} \geq 2 \end{gathered}$ | $\begin{aligned} & \mathrm{BK}=1 \\ & \mathrm{MD}=0 \end{aligned}$ | $\begin{aligned} & \mathrm{BK}=1 \\ & \mathrm{MD}=1 \end{aligned}$ | $\begin{gathered} \mathrm{BK}=1 \\ \mathrm{MD} \geq 2 \end{gathered}$ | $\begin{gathered} \mathrm{BK} \geq 2 \\ \mathrm{MD}=0 \end{gathered}$ | $\begin{gathered} \mathrm{BK} \geq 2 \\ \mathrm{MD}=1 \end{gathered}$ | $\begin{aligned} & \mathrm{BK} \geq 2 \\ & \mathrm{MD} \geq 2 \end{aligned}$ |
| $B K=0 \& M D=0$ | 95.1 | 3.6 | 0.2 | 1.0 | - | - | - | 0.1 | - |
| $\mathrm{BK}=0 \& \mathrm{MD}=1$ | - | 87.2 | 4.2 | - | 7.4 | 1.0 | - | - | 1.4 |
| $\mathrm{BK}=0 \& \mathrm{MD} \geq 2$ | - | - | 82.7 | - | - | 15.8 | - | - | 1.4 |
| $\mathrm{BK}=1 \& \mathrm{MD}=0$ | - | - | - | 76.0 | 18.0 | 2.0 | 4.0 | - | - |
| $\mathrm{BK}=1 \& \mathrm{MD}=1$ | - | - | - | - | 87.1 | 8.1 | - | 3.3 | 1.4 |
| $\mathrm{BK}=1 \& \mathrm{MD} \geq 2$ | - | - | - | - | - | 86.5 | - | - | 13.5 |
| $\mathrm{BK} \geq 2 \& \mathrm{MD}=0$ | - | - | - | - | - | - | 84.6 | 15.4 | - |
| $\mathrm{BK} \geq 2 \& \mathrm{MD}=1$ | - | - | - | - | - | - | - | 69.0 | 31.0 |
| $\mathrm{BK} \geq 2 \& \mathrm{MD} \geq 2$ | - | - | - | - | - | - | - | - | 100.0 |
| Frequency | 41.6 | 23.3 | 6.6 | 2.2 | 10.9 | 8.8 | 0.6 | 1.4 | 4.5 |

Table 5 presents estimates of reduced form Probit models for the decision to open a new store. We obtain separate estimates for MD and BK. Our main interest is in the estimation of the effect of the previous year's number of stores (own stores and competitor's stores) on the probability of opening a new store. We include as control variables population, GDP per capita, population density, proportion of population 5-14, proportion population 15-29, average rent, and proportion of claimants of unemployment benefits. To control for unobserved local market heterogeneity we also present two fixed effects estimations, one with county fixed effects and other with local district fixed effects. We only report estimates of the marginal effects associated with the dummy variables that represent previous year number of stores. The main empirical result from table 5 is that, regardless of the set of control variables that we use, the own number of stores has a strong negative effect on the probability of opening a new store but the effect of the competitor's number of stores is either
negligible or even positive. This finding is very robust to different specifications of the reduced form model and it is analogous to the result from the reduced form specifications in Toivanen and Waterson's paper. Controlling for unobserved heterogeneity using fixed effects reveals that the estimation without fixed effects suffers from a significant upward bias in the marginal effect of the number of own stores. However, the estimated marginal effect of the number of competitor's stores barely changes. The estimates show also a certain asymmetry between the two firms: the absence of response to the competitor's number of stores is more clear for BK than for MD. In particular, when BK has three stores in the market there is a significant reduction in MD's probability of opening a new store. That negative effect does not appear in the reduced form probit for BK.

This empirical evidence cannot be explained by a standard static model of store location by firms that sell substitute products. Here we explore three, non-mutually exclusive, explanations: (a) spillover effects; (b) forward looking behavior (dynamic game); and (c) biased beliefs about the behavior of the competitor.
(a) Spillover effects. The competitor's number of stores may have a positive spillover effect on the profit of a firm. There are several possible sources of this spillover effect. For example one firm may infer from another's decision to open a store in a particular market that market conditions are favorable (informational spillovers). Alternatively, one firm may benefit from another firm's entry through cost reductions, or from product expansion through advertising. Since we do not have price and quantity data at the level of local markets, we do not try to identify the source of the spillover effect. We include this effect in our specification of demand such that the natural interpretation in the context of our model is a product expansion coming from the advertising effect of retail stores. However, this should be interpreted as a shortcut or 'reduced form' specification of different possible spillover effects.

## Table 5 <br> Reduced Form Probits for the Decision to Open a Store

| Explanatory Variable | Estimated Marginal Effects ${ }^{1}(\Delta P(x)$ when dummy goes from 0 to 1)Burger King |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | No FE | County FE | District FE | No FE | County FE | District FE |
| Own number of stores at t-1 |  |  |  |  |  |  |
| Dummy: Own \#stores = 1 | $\begin{gathered} -0.021^{* *} \\ (0.005) \end{gathered}$ | $\begin{gathered} -0.036^{* *} \\ (0.007) \end{gathered}$ | $\begin{gathered} -0.885^{* *} \\ (0.063) \end{gathered}$ | $\begin{gathered} -0.035^{* *} \\ (0.010) \end{gathered}$ | $\begin{gathered} -0.045^{* *} \\ (0.012) \end{gathered}$ | $\begin{gathered} -0.550^{* *} \\ (0.056) \end{gathered}$ |
| Dummy: Own \#stores = 2 | $\begin{gathered} -0.023^{* *} \\ (0.004) \end{gathered}$ | $\begin{gathered} -0.030^{* *} \\ (0.005) \end{gathered}$ | $\begin{aligned} & -0.210^{*} \\ & (0.085) \end{aligned}$ | $\begin{gathered} -0.047^{* *} \\ (0.006) \end{gathered}$ | $\begin{aligned} & -0.060^{*} \\ & (0.008) \end{aligned}$ | $\begin{gathered} -0.757^{* *} \\ (0.041) \end{gathered}$ |
| Dummy: Own \#stores $\geq 3$ | $\begin{gathered} -0.019^{* *} \\ (0.005) \end{gathered}$ | $\begin{gathered} -0.027^{* *} \\ (0.005) \end{gathered}$ | $\begin{gathered} -0.056 \\ (0.036) \end{gathered}$ | $\begin{gathered} -0.043^{* *} \\ (0.006) \end{gathered}$ | $\begin{gathered} -0.053^{* *} \\ (0.008) \end{gathered}$ | $\begin{gathered} -0.816^{* *} \\ (0.038) \end{gathered}$ |
| Competitor's number of stores at $\mathbf{t - 1}$ |  |  |  |  |  |  |
| Dummy: Comp.'s \#stores = 1 | $\begin{aligned} & 0.032^{* *} \\ & (0.011) \end{aligned}$ | $\begin{aligned} & 0.037^{*} \\ & (0.014) \end{aligned}$ | $\begin{aligned} & -0.025 \\ & (0.055) \end{aligned}$ | $\begin{gathered} 0.020 \\ (0.013) \end{gathered}$ | $\begin{aligned} & 0.032^{*} \\ & (0.018) \end{aligned}$ | $\begin{aligned} & 0.052^{* *} \\ & (0.073) \end{aligned}$ |
| Dummy: Comp.'s \#stores = 2 | $\begin{aligned} & 0.045^{*} \\ & (0.023) \end{aligned}$ | $\begin{aligned} & 0.052^{*} \\ & (0.029) \end{aligned}$ | $\begin{gathered} -0.017 \\ (0.031) \end{gathered}$ | $\begin{gathered} 0.041 \\ (0.029) \end{gathered}$ | $\begin{gathered} 0.076 \\ (0.046) \end{gathered}$ | $\begin{gathered} -0.007^{* *} \\ (0.093) \end{gathered}$ |
| Dummy: Comp.'s \#stores $\geq 3$ | $\begin{aligned} & 0.089^{*} \\ & (0.048) \end{aligned}$ | $\begin{aligned} & 0.101^{*} \\ & (0.059) \end{aligned}$ | $\begin{gathered} 0.011 \\ (0.084) \end{gathered}$ | $\begin{gathered} -0.041^{*} \\ (0.007) \end{gathered}$ | $\begin{gathered} -0.050^{* *} \\ (0.009) \end{gathered}$ | $\begin{gathered} -0.104^{* *} \\ (0.020) \end{gathered}$ |
| Pred. Prob. Y=1 at mean X | 0.024 | 0.027 | 0.014 | 0.045 | 0.054 | 0.085 |
| Time dummies | YES | YES | YES | YES | YES | YES |
| Control variables ${ }^{2}$ | YES | YES | YES | YES | YES | YES |
| County Fixed Effects | NO | YES | NO | NO | YES | NO |
| District Fixed Effects | NO | NO | YES | NO | NO | YES |
| Number of Observations ${ }^{3}$ | 2110 | 1715 | 535 | 2110 | 1855 | 640 |
| Number of Local Districts ${ }^{3}$ | 422 | 343 | 107 | 422 | 371 | 128 |
| log likelihood | -371.89 | -340.26 | -110.54 | -467.46 | -449.02 | -198.50 |
| Pseudo R-square | 0.229 | 0.252 | 0.624 | 0.159 | 0.161 | 0.441 |

Note 1: Estimated Marginal Effects are evaluated at the mean value of the rest of the explanatory variables.
Note 2: Every estimation includes as control variables log of population, log of GDP per capita, log of population density, proportion population 5-14, proportion population 15-29, average rent, and proportion of claimants of unemployment benefits. Note 3: Fixed effects estimations do not include districts for which the dependent variable does not have enough time variation.
(b) Forward looking behavior. Opening a store is a partly irreversible decision that involves a significant sunk costs. Therefore, it is reasonable to consider that firms are forward looking when they make this decision. Moreover, dynamic strategic effects may help explain the apparent lack of competitive effects when we look at these decisions from the point of view of a static model of
entry. Suppose that firms anticipate, with some uncertainty, the total number of hamburger stores that a local market can sustain in the long-run given the size and the socioeconomic characteristics of the market. For simplicity, suppose that this number of "available slots" does not depend on the ownership of the stores because the products sold by the two firms are very close substitutes. In this context, firms play a 'racing' game to fill as many 'slots' as possible with their own stores. Diseconomies of scale and scope may generate a negative effect of the own number of stores on the decision of opening new stores. However, in this model, during most of the period of expansion the number of slots of the competitor has zero effect on the decision of opening a new store. Only when the market is filled or close to being filled do the competitor's stores have a significant effect on entry decisions.
(c) Biased beliefs. As mentioned in the Introduction, competition in actual oligopoly industries is often characterized by strategic uncertainty. Firms face significant uncertainty about the strategies of their competitors. In the context of our application, it may be the case that MD's or/and BK's beliefs overestimate the negative effect of the competitor's stores on the competitor's entry decisions. For instance, if MD has one store in a local market, BK may believe that the probability that MD opens a second store is close to zero. These over-optimistic beliefs about the competitor's behavior may generate an apparent lack of response of BK's entry decisions to the number of MD's stores.

### 6.2 Model

Consider two retail chains competing in a local market. Each firm sells a differentiated product using its stores. Let $Y_{i t} \in\{0,1, \ldots, K\}$ be the number of stores of firm $i$ at period $t$. We abstract from store location within a local market and assume that every store of the same firm has the same demand. Let $q_{i t}$ be the quantity sold by all the stores of firm $i$. The demand for all the stores of firm $i$ is:

$$
q_{i t}= \begin{cases}W_{t}\left(1+b_{i t}-p_{i t}\right) & \text { if } \quad Y_{j t}=0  \tag{39}\\ W_{t}\left(\frac{Y_{i t}}{Y_{i t}+Y_{j t}}\right)\left(1+\left[b_{i t}+\delta_{i} b_{j t}\right]-\left[p_{i t}-p_{j t}\right]\right) & \text { if } \quad Y_{j t}>0\end{cases}
$$

where: $W_{t}$ is a measure of market size and it is exogenous; $b_{i t}$ is the 'quality' of product $i$ at period $t ;{ }^{16} p_{i t}$ is the 'price' of product $i$ at period $t$; and $\delta_{i} \geq-1$ is a parameter that captures the net effect of the quality of firm $j$ on the demand of firm $i$. This net effect is positive ( $\delta_{i}>0$ ) if the spillover effect from an increase in the quality of the competitor is larger than the competitive effect, and it is negative $\left(\delta_{i}<0\right)$ otherwise. When there is not any positive spillover effect we have that $\delta_{i}=-1$. In that case, if the two firms have the same qualities and prices, they share the market size $W_{t}$ in

[^9]proportion to their number of stores. A firm with better quality or/and lower price can get a larger proportion of the market. Production costs are linear in the quantity produced, i.e., $C_{i t}=c_{i} q_{i t}$, where $c_{i}$ is firm $i$ 's constant marginal cost. The variable profit of firm $i$ is $V P_{i t}=\left(p_{i t}-c_{i}\right) q_{i t}$. Given market size and qualities at period $t$, firms compete in prices ala Bertrand to maximize current variable profit.

If firm $i$ is a monopolist in the market (i.e., $Y_{j t}=0$ ), then the profit-maximizing price is $p_{i t}=c_{i}+1+b_{i t}$ and the variable profit is $V P_{i t}=W_{t}\left(\left[1+b_{i t}-c_{i}\right] / 2\right)^{2}$. If both firms are active in the market (i.e., $Y_{i t}>0$ and $Y_{j t}>0$ ), then the Bertrand equilibrium price is ${ }^{17} p_{i t}=c_{i}+1+$ $\left[b_{i t}+\delta_{i} b_{j t}\right]-(1 / 3)\left(\Delta c+\Delta B_{t}\right)$, where $\Delta B_{t} \equiv\left[b_{i t}+\delta_{i} b_{j t}\right]-\left[b_{j t}+\delta_{j} b_{i t}\right]$ and $\Delta c \equiv c_{i}-c_{j}$, and the equilibrium variable profit is $V P_{i t}=W_{t} Y_{i t}\left(Y_{i t}+Y_{j t}\right)^{-1}\left(1+\left[b_{i t}+\delta_{i} b_{j t}\right]-(1 / 3)\left(\Delta c+\Delta B_{t}\right)\right)^{2}$.

The visibility of a retail firm in a local market increases with its number of stores. We assume that a firm's quality increases with the number of stores that the firm has in the market. There are at least two ways in which the number of stores in the market affects the willingness to pay of the average consumer. First, an increase in the number of stores implies a reduction in consumer transportation cost to visit a store of the chain. Second, stores are like 'advertisements' in the sense that they increase the awareness of local consumers about the retail chain. We assume the following specification:

$$
\begin{equation*}
b_{i t}=b_{i}^{(0)}+b_{i}^{(1)} Y_{i t} \tag{40}
\end{equation*}
$$

where $b_{i}^{(0)} \geq 0$ is a parameter that represents the 'exogenous' quality of firm $i$ in every local market. ${ }^{18}$ And $b_{i}^{(1)} \geq 0$ is a parameter that measures the effect of the number of stores in the local market on the 'quality' of the firm in that market.

Firm $i$ is active in the market at period $t$ if $Y_{i t}$ is strictly positive. In order to distinguish decision and state variables, we use the variable $S_{i t}$ to represent the number of stores at period $t-1$, i.e., $S_{i t} \equiv Y_{i t-1}$. Every period, the two firms know the 'stocks' of stores in the market, $S_{i t}$ and $S_{j t}$, and choose simultaneously the new number of stores. Firm $i$ 's total profit function is:

$$
\begin{align*}
\Pi_{i t} & =V P_{i t}-1\left\{Y_{i t}>0 \text { and } S_{i t}=0\right\} \theta_{i}^{E C} \\
& -1\left\{Y_{i t}>0\right\}\left[\theta_{0 i}^{F C}-\theta_{1 i}^{F C} Y_{i t}-\theta_{2 i}^{F C} Y_{i t}^{2}\right]  \tag{41}\\
& -1\left\{Y_{i t}>S_{i t}\right\} \varepsilon_{i t}
\end{align*}
$$

where $1\{$.$\} is the indicator function, and \theta_{i}^{E C}, \theta_{0 i}^{F C}, \theta_{1 i}^{F C}$ and $\theta_{2 i}^{F C}$ are parameters in the cost function. $\theta_{0 i}^{E C}$ is an entry cost that is paid the first time that the firm opens a store in the local market.

[^10]$\theta_{0 i}^{F C}$ is a lump-sum cost associated with having any positive number of stores in the market. The function $\theta_{1 i}^{F C} Y_{i t}+\theta_{2 i}^{F C} Y_{i t}^{2}$ takes into account that operating costs increase with the number of stores in a linear or quadratic form. The variable $\varepsilon_{i t}$ is a private information shock in the cost of opening a new store, and it is i.i.d. normally distributed.

Note that our model implies the exclusion restriction that, given $Y_{j t}$, the profit of firm $i$ does not depend on $S_{j t}=Y_{j t-1}$. That is, a firm's current profit depends on his own and his opponents current number of stores in the market, but given these variables it does not depend on the number of stores of the competitors at period $t-1$. Of course a firm's beliefs about the probability distribution of $Y_{j t}$ depends on $S_{j t}$.

### 6.3 Estimation of the structural model

As described in section 5.1 above, we do not observe store closings in our sample. Furthermore, for almost all the observations with store openings the number of new stores is one. Therefore, we assume that $Y_{i m t} \in\left\{S_{i m t}, S_{i m t}+1\right\}$ or equivalently, $Y_{i m t}-S_{i m t} \in\{0,1\}$. Variable $Y_{i m t}-S_{i m t}$ is the binary indicator of the event "firm $i$ opens a new store in market $m$ at year $t$ ". The maximum value of $S_{i m t}$ in the sample is 13 , and we assume that the set of possible values of $S_{i m t}$ is $\{0,1, \ldots, 15\}$. Therefore, the state space $\mathcal{X}$ is $\{0,1, \ldots, 15\} \times\{0,1, \ldots, 15\}$ that has 256 grid points. . We assume that market characteristics are constant over time. The measure of market size $W_{m}$ is total population in the district. For some specifications, we allow the cost of investment to depend on market characteristics such as average rent, retail taxes, population density, or average income. Therefore, each market has its own vector of players' CCPs. The dimension of the vectors $\mathbf{P}_{i}$ in this model is equal to 108,032 , i.e., 422 markets times 256 states.

Tables 6 and 7 present estimates of the structural model under the assumption that firms are myopic, $\beta=0$, and under the assumption that firms are forward looking, $\beta=0.95$, respectively. We report two different sets of point estimates: estimates using a simple two-step Pseudo Maximum Likelihood method where the estimator of (equilibrium) players' beliefs in the first step is a nonparametric frequency estimator; and estimates using the Nested Pseudo Likelihood (NPL) method proposed in Aguirregabiria and Mira (2007). The NPL method imposes the equilibrium restrictions in the sample (i.e., the estimated beliefs should be equal to the estimated best response probabilities), while the two-step method only satisfies the equilibrium restrictions asymptotically. The NPL estimator has smaller asymptotic variance and finite sample bias than the two-step method. There are very substantial differences between two models, particularly in the estimates of the parameters that capture cannibalization and competition effects. While these effects have the 'wrong' sign in the myopic model, the signs are the expected ones in the forward looking model. All the parameter estimates in the forward looking model have the expected signs and have reasonable magnitudes. Therefore, it seems that forward looking behavior explains part of the puzzle in the reduced form
estimates.

| Table 6 <br> Myopic Game of Entry for McDonalds and Burger King Under the Assumption that Players' Beliefs are in Equilibrium Data: 422 markets, 2 firms, 5 years $=4,220$ observations |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |  |  |  |  |
|  | $\beta=0.00$ (not estimated) |  |  |  |  |  |  |  |
|  | Two Step Estimates Burger King McDonalds |  |  |  | NPL Estimates |  |  |  |
| Variable Profits: $\theta_{0}^{V P}$ | 4.904 | $(1.070)^{*}$ | 7.909 | $(2.289)^{*}$ | 4.864 | $(1.081)^{*}$ | 7.898 | $(2.287)^{*}$ |
| $\theta_{1}^{V P}$ cannibalization | 2.005 | $(0.869)^{*}$ | 3.510 | $(0.659) *$ | 2.035 | $(0.831)^{*}$ | 3.466 | $(0.647)^{*}$ |
| $\theta_{2}^{V P}$ competition | 0.014 | (0.046) | 0.032 | (0.051) | 0.016 | (0.044) | 0.037 | (0.053) |
| Fixed Costs: $\theta_{0}^{F C} \text { fixed }$ | 0.378 | $(0.212)^{*}$ | 0.806 | $(0.248) *$ | 0.374 | $(0.212)^{*}$ | 0.808 | $(0.247)^{*}$ |
| $\theta_{1}^{F C}$ linear | 3.099 | $(0.436)^{*}$ | 2.662 | $(0.405)^{*}$ | 3.103 | $(0.436)^{*}$ |  | $(0.405)^{*}$ |
| $\theta_{2}^{F C}$ quadratic | -0.054 | (0.064) |  | (0.041) | -0.052 | (0.063) | 0.087 | (0.041) |
| Pseudo R-square |  |  |  |  | 0.154 |  |  |  |
| Log-Likelihood | -895.5 |  |  |  | -895.4 |  |  |  |
| Distance $\left\\|\mathrm{P}^{K}-\mathrm{P}^{K-}\right\\|$ |  |  |  |  | 0.00 |  |  |  |
| \# NPL iterations | 1 |  |  |  | 5 |  |  |  |

## Table 7

## Dynamic Game of Entry for McDonalds and Burger King

 Under the Assumption that Players' Beliefs are in EquilibriumData: 422 markets, 2 firms, 5 years $=4,220$ observations

|  | $\beta=0.95$ (not estimated) |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Two Step Burger King |  | Estimates McDonalds |  | NPL Estimates |  |  |  |
|  |  |  | Burge | er King | McD | onalds |
| Variable Profits: $\theta_{0}^{V P}$ | 0.5849 | (0.1077)* |  |  | 0.8303 | (0.2968)* | 1.098 | $(0.2169)^{*}$ | 0.9737 | (0.3091)* |
| $\theta_{1}^{V P}$ cannibalization | -0.2096 | $(0.0552)^{*}$ | -0.0024 | (0.0392) | -0.0765 | (0.0725) | 0.2874 | $(0.0986) *$ |
| $\theta_{2}^{V P}$ competition | -0.0110 | $(0.0029)^{*}$ | 0.0008 | (0.0027) | -0.0129 | $(0.0065)^{*}$ | -0.0074 | (0.0073) |
| Fixed Costs: $\theta_{0}^{F C} \text { fixed }$ | 0.0784 | (0.0213)* | 0.0822 | $(0.0332)^{*}$ | 0.0788 | (0.0307)* | 0.0773 | (0.0261)* |
| $\theta_{1}^{F C}$ linear | 0.0790 | $(0.0420)^{*}$ | 0.1076 | $(0.0400)^{*}$ | 0.1509 | $(0.0282)^{*}$ | 0.1302 | $(0.0185)^{*}$ |
| $\theta_{2}^{F C}$ quadratic | -0.0078 | (0.0059) | -0.0034 | (0.0023) | -0.0054 | $(0.0026)^{*}$ | 0.0001 | (0.016) |
| Pseudo R-square | 0.323 |  |  |  | 0.146 |  |  |  |
| Log-Likelihood | -655.7 |  |  |  | -893.4 |  |  |  |
| Distance $\left\\|\mathrm{P}^{K}-\mathrm{P}^{K-}\right\\|$ | 4831.26 |  |  |  | 0.00 |  |  |  |
| \# NPL iterations | 1 |  |  |  | 31 |  |  |  |

Table 8 presents results of our test of equilibrium beliefs. We implement separate tests for MD and BK. We impose the restrictions that beliefs for $S_{j t}=0$ and $S_{j t}=3$ are unbiased.

Table 8
Estimated Bias in BK Beliefs
Difference Between $B_{M D}$ and $P_{M D}$

|  | Stores of BK |  |  |  |
| ---: | ---: | ---: | ---: | :--- |
|  | $\mathbf{0}$ |  | $\mathbf{1}$ |  |
| Stores of MD |  |  | $(0.06)$ |  |
| $\mathbf{1}$ | $\mathbf{- 0 . 1 7}$ | $\mathbf{( 0 . 0 4 )}$ | -0.10 | $(0.0)$ |
| $\mathbf{2}$ | -0.08 | $(0.07)$ | -0.06 | $(0.10)$ |

Estimated Bias in MD Beliefs
Difference Between $B_{B K}$ and $P_{B K}$

|  | Stores of MD |  |
| :---: | :---: | :---: |
|  | 0 | 1 |
| Stores of BK 1 | -0.03 (0.05) | 0.02 (0.04) |
| 2 | $0.03 \quad(0.10)$ | $0.04 \quad$ (0.12) |

## 7 Conclusion

This paper studies a class of dynamic games of incomplete information where players' beliefs about the other players' actions may not be in equilibrium. We present new results on identification, estimation, and inference of structural parameters and beliefs in this class of games when the researcher does not have data on elicited beliefs. Specifically, we derive sufficient conditions under which payoffs and beliefs are point identified. These conditions then lead naturally to a two-step estimator of payoffs and beliefs, which we show can be extended to provide a sequence of estimators with asymptotic variances and finite sample biases that decline monotonically. We also present a procedure for testing the null hypothesis that beliefs are in equilibrium. We illustrate our model and methods with an empirical application of a dynamic game of store location by McDonalds and Burger King. Our main interest in this application is to explain a puzzling empirical regularity, that the probability a firm opens a new store in a local market does not depend (or may even positively depend) on the number of stores its competitor currently has open in the location. In the context of our model we explore three alternative (but not mutually exclusive) explanations for these: cross-firm spillovers, forward looking behavior, and out of equilibrium (i.e., biased) beliefs. We find empirical evidence for the hypotheses of forward looking behavior and biased beliefs.

## APPENDIX

## [A.1] Integrated Value Functions and Continuation Values

The proofs of Propositions 1 and 2 apply the concepts of integrated value function and continuation value function as well as recursive formulas to calculate these functions. The integrated value function is defined as $\bar{V}_{i t}^{\mathbf{B}}\left(\mathbf{X}_{t}\right) \equiv \int V_{i t}^{\mathbf{B}}\left(\mathbf{X}_{t}, \varepsilon_{i t}\right) d G\left(\varepsilon_{i t}\right)$ (see Rust, 1994). Applying this definition to the Bellman equation, we obtained the integrated Bellman equation:

$$
\begin{align*}
\bar{V}_{i t}^{\mathbf{B}}\left(\mathbf{X}_{t}\right) & =\int \max _{Y_{i t} \in\{0,1\}}\left\{v_{i t}^{\mathbf{B}}\left(Y_{i t}, \mathbf{X}_{t}\right)+\varepsilon_{i t}\left(Y_{i t}\right)\right\} d G\left(\varepsilon_{i t}\right) \\
& =\int \max _{Y_{i t} \in\{0,1\}}\left\{\pi_{i t}^{\mathbf{B}}\left(Y_{i t}, \mathbf{X}_{t}\right)+\beta \sum_{\mathbf{X}^{\prime}} \bar{V}_{i t+1}^{\mathbf{B}}\left(\mathbf{X}^{\prime}\right) f_{i t}^{\mathbf{B}}\left(\mathbf{X}^{\prime} \mid Y_{i t}, \mathbf{X}_{t}\right)+\varepsilon_{i t}\left(Y_{i t}\right)\right\} d G\left(\varepsilon_{i t}\right) \tag{A.1.1}
\end{align*}
$$

For instance, if $\left\{\varepsilon_{i t}(0), \varepsilon_{i t}(1)\right\}$ have an independent double exponential distribution, the integrated Bellman equation becomes:

$$
\begin{align*}
\bar{V}_{i t}^{\mathbf{B}}\left(\mathbf{X}_{t}\right) & =\ln \left(\sum_{Y_{i t} \in\{0,1\}} \exp \left\{v_{i t}^{\mathbf{B}}\left(Y_{i t}, \mathbf{X}_{t}\right)\right\}\right)  \tag{A.1.2}\\
& =v_{i t}^{\mathbf{B}}\left(0, \mathbf{X}_{t}\right)-\ln \left(1-P_{i t}\left(\mathbf{X}_{t}\right)\right)
\end{align*}
$$

and $P_{i t}\left(\mathbf{X}_{t}\right)=\exp \left\{v_{i t}^{\mathbf{B}}\left(1, \mathbf{X}_{t}\right)-v_{i t}^{\mathbf{B}}\left(0, \mathbf{X}_{t}\right)\right\} /\left[1+\exp \left\{v_{i t}^{\mathbf{B}}\left(1, \mathbf{X}_{t}\right)-v_{i t}^{\mathbf{B}}\left(0, \mathbf{X}_{t}\right)\right\}\right]$. If we know payoffs and beliefs, we can use this formula to obtain the integrated value function by backwards induction, starting at the last period $T$ where $\bar{V}_{i T}^{\mathbf{B}}(\mathbf{X})=\pi_{i T}^{\mathbf{B}}(0, \mathbf{X})+\ln \left(1-P_{i T}(\mathbf{X})\right)$.

The continuation value function provides the expected and discounted value of future payoffs given future beliefs, current state, and current choices of both players. It is defined as:

$$
\begin{equation*}
d_{i t}^{\mathbf{B}}\left(Y_{i t}, Y_{j t}, \mathbf{X}_{t}\right) \equiv \beta \sum_{\mathbf{X}_{t+1}} \bar{V}_{i t+1}^{\mathbf{B}}\left(\mathbf{X}_{t+1}\right) f_{t}\left(\mathbf{X}_{t+1} \mid Y_{i t}, Y_{j t}, \mathbf{X}_{t}\right) \tag{A.1.3}
\end{equation*}
$$

Note that continuation values $d_{i t}^{\mathrm{B}}$ depend on beliefs at period $t+1$ and later, but not on beliefs at period $t$. By definition, the relationship between the conditional choice value function $v_{i t}^{\mathbf{B}}$ and the continuation value function $d_{i t}^{\mathbf{B}}$ is the following:

$$
\begin{align*}
v_{i t}^{\mathbf{B}}\left(Y_{i t}, \mathbf{X}_{t}\right) & =\left(1-B_{j t}\left(\mathbf{X}_{t}\right)\right)\left[\pi_{i t}\left(Y_{i t}, 0, \mathbf{X}_{t}\right)+d_{i t}^{\mathbf{B}}\left(Y_{i t}, 0, \mathbf{X}_{t}\right)\right]  \tag{A.1.4}\\
& +B_{j t}(X)\left[\pi_{i t}\left(Y_{i t}, 1, \mathbf{X}_{t}\right)+d_{i t}^{\mathbf{B}}\left(Y_{i t}, 1, \mathbf{X}_{t}\right)\right]
\end{align*}
$$

## [A.2] Proof of Proposition 1

For notational simplicity, we omit the time subindex $t$. Also, we omit the vector of state variables $\mathbf{X}$ as an argument in functions, unless it is needed. Define the function $q_{i}(\mathbf{X}) \equiv \Lambda^{-1}\left(P_{i}(\mathbf{X})\right)$. Given that the distribution function $\Lambda$ is invertible and it is known (up to scale) to the researcher, the function $q_{i}($.$) is identified everywhere in the support of \mathbf{X}$. The condition for player $i$ 's best response, implies that:

$$
\begin{align*}
q_{i} & =v_{i}^{\mathbf{B}}(1)-v_{i}^{\mathbf{B}}(0) \\
& =\left\{\left(1-B_{j}\right)\left[\pi_{i}(1,0)+d_{i}^{\mathbf{B}}(1,0)\right]+B_{j}\left[\pi_{i}(1,1)+d_{i}^{\mathbf{B}}(1,1)\right]\right\}  \tag{A.2.1}\\
& -\left\{\left(1-B_{j}\right)\left[\pi_{i}(0,0)+d_{i}^{\mathbf{B}}(0,0)\right]+B_{j}\left[\pi_{i}(0,1)+d_{i}^{\mathbf{B}}(0,1)\right]\right\} \\
& =\left(1-B_{j}\right)\left[\widetilde{\pi}_{i}(0)+\widetilde{d}_{i}^{\mathbf{B}}(0)\right]+B_{j}\left[\widetilde{\pi}_{i}(1)+\widetilde{d}_{i}^{\mathbf{B}}(1)\right]
\end{align*}
$$

where $\widetilde{\pi}_{i}\left(Y_{j t}, \mathbf{X}\right)$ is the function $\pi_{i}\left(1, Y_{j t}, \mathbf{X}\right)-\pi_{i}\left(0, Y_{j t}, \mathbf{X}\right)$, as defined in assumption ID-2, and $\widetilde{d}_{i}^{\mathbf{B}}\left(Y_{j t}, \mathbf{X}\right)$ is $d_{i}^{\mathbf{B}}\left(1, Y_{j}, \mathbf{X}\right)-d_{i}^{\mathbf{B}}\left(0, Y_{j}, \mathbf{X}\right)$.

Consider equation (A.2.1) evaluated at four different values of the vector $\mathbf{X}$, say $\mathbf{X}^{a}, \mathbf{X}^{b}, \mathbf{X}^{c}$, and $\mathbf{X}^{d}$. These four vectors have been constructed such that they have exactly the same value of the component $\left(S_{i}, W\right)$, but they can have different values for the element $S_{j}$, say $S_{j}^{a}, S_{j}^{b}, S_{j}^{c}$, and $S_{j}^{d}$. Note that the exclusion restriction in assumption $I D-2(A)$ implies that $\widetilde{\pi}_{i}\left(Y_{j}, \mathbf{X}\right)$ does not depend on variable $S_{j}$. Furthermore, by condition (i) in Proposition 1, the continuation value $\widetilde{d}_{i}^{\mathbf{B}}\left(Y_{j}, \mathbf{X}\right)$ does not depend on $S_{j}$ either. Therefore, we have that, evaluated at $\mathbf{X}^{a}, \mathbf{X}^{b}, \mathbf{X}^{c}$, and $\mathbf{X}^{d}$, function $\widetilde{\pi}_{i}(0, \mathbf{X})+\widetilde{d}_{i}^{\mathbf{B}}(0, \mathbf{X})$ takes exactly the same value. If we subtract equation (A.2.1) evaluated at $\mathbf{X}^{b}$ from the same equation evaluated at $\mathbf{X}^{a}$, we obtain the expression $q_{i}\left(\mathbf{X}^{a}\right)-q_{i}\left(\mathbf{X}^{b}\right)=\left[\tilde{\pi}_{i}\left(1, S_{i}, W\right)\right.$ $\left.-\tilde{\pi}_{i}\left(0, S_{i}, W\right)+\widetilde{d}_{i}^{\mathbf{B}}(1, W)-\widetilde{d}_{i}^{\mathbf{B}}(0, W)\right]\left[B_{j}\left(\mathbf{X}^{a}\right)-B_{j}\left(\mathbf{X}^{b}\right)\right]$. Similarly, if we subtract (??) evaluated at $\mathbf{X}^{d}$ from the same equation evaluated at $\mathbf{X}^{c}$, we get $q_{i}\left(\mathbf{X}^{c}\right)-q_{i}\left(\mathbf{X}^{d}\right)=\left[\tilde{\pi}_{i}\left(1, S_{i}, W\right)-\tilde{\pi}_{i}\left(0, S_{i}, W\right)\right.$ $\left.+\widetilde{d}_{i}^{\mathbf{B}}(1, W)-\widetilde{d}_{i}^{\mathbf{B}}(0, W)\right]\left[B_{j}\left(\mathbf{X}^{c}\right)-B_{j}\left(\mathbf{X}^{d}\right)\right]$. Finally, if $S_{j}^{a} \neq S_{j}^{b}$ or $S_{j}^{c} \neq S_{j}^{d}$, we can obtain the ratio between these two pairwise-difference equations, and this ratio implies that:

$$
\begin{equation*}
\frac{q_{i}\left(\mathbf{X}^{a}\right)-q_{i}\left(\mathbf{X}^{b}\right)}{q_{i}\left(\mathbf{X}^{c}\right)-q_{i}\left(\mathbf{X}^{d}\right)}=\frac{B_{j}\left(\mathbf{X}^{a}\right)-B_{j}\left(\mathbf{X}^{b}\right)}{B_{j}\left(\mathbf{X}^{c}\right)-B_{j}\left(\mathbf{X}^{d}\right)} \tag{A.2.2}
\end{equation*}
$$

This expression shows that under the conditions of Proposition 1 there is a function of beliefs that is identified, without having to impose the assumption of equilibrium beliefs. This result provides a nonparametric test for the null hypothesis of equilibrium beliefs. Define the function:

$$
\begin{equation*}
\delta_{i}\left(\mathbf{X}^{a}, \mathbf{X}^{b}, \mathbf{X}^{c}, \mathbf{X}^{d}\right) \equiv\left\{\frac{q_{i}\left(\mathbf{X}^{a}\right)-q_{i}\left(\mathbf{X}^{b}\right)}{q_{i}\left(\mathbf{X}^{c}\right)-q_{i}\left(\mathbf{X}^{d}\right)}\right\}-\left\{\frac{P_{j}\left(\mathbf{X}^{a}\right)-P_{j}\left(\mathbf{X}^{b}\right)}{P_{j}\left(\mathbf{X}^{c}\right)-P_{j}\left(\mathbf{X}^{d}\right)}\right\} \tag{A.2.3}
\end{equation*}
$$

It is clear that we can nonparametrically identify $\delta_{i}$. If beliefs are in equilibrium, $\delta_{i}$ should be equal to zero for any value of $\left(\mathbf{X}^{a}, \mathbf{X}^{b}, \mathbf{X}^{c}, \mathbf{X}^{d}\right)$. Let $\hat{\delta}_{i}$ be a root-M consistent and asymptotically normal nonparametric estimator of $\delta_{i}$. The hypothesis of equilibrium beliefs implies $\delta_{i}=0$, and we can test equilibrium beliefs using a simple LM test of the null hypothesis of $\delta_{i}=0$.

## [A.3] Proof of Proposition 2

Consider equation (A.2.1) that comes from player $i$ 's best response restriction. For the sake of notational simplicity, we omit the time subindex and the vector of state variables $\mathbf{X}$ as argument in all the functions.

$$
\begin{equation*}
q_{i}=\left(1-B_{j}\right)\left[\widetilde{\pi}_{i}(0)+\widetilde{d}_{i}^{\mathbf{B}}(0)\right]+B_{j}\left[\widetilde{\pi}_{i}(1)+\widetilde{d}_{i}^{\mathbf{B}}(1)\right] \tag{A.3.1}
\end{equation*}
$$

Substituting the restrictions imposed by assumption ID-3 into this equation, we have that:

$$
\begin{align*}
q_{i}^{L} & =\widetilde{\pi}_{i}(0)+\widetilde{d}_{i}^{L}(0)+P_{j}^{L}\left[\widetilde{\pi}_{i}(1)-\widetilde{\pi}_{i}(0)+\widetilde{d}_{i}^{L}(1)-\widetilde{d}_{i}^{L}(0)\right] \\
q_{i}^{H} & =\widetilde{\pi}_{i}(0)+\widetilde{d}_{i}^{H}(0)+P_{j}^{H}\left[\widetilde{\pi}_{i}(1)-\widetilde{\pi}_{i}(0)+\widetilde{d}_{i}^{H}(1)-\widetilde{d}_{i}^{H}(0)\right] \tag{A.3.2}
\end{align*}
$$

where $q_{i}^{L} \equiv q_{i}\left(S_{i}, S_{j}^{L}, W\right), q_{i}^{H} \equiv q_{i}\left(S_{i}, S_{j}^{H}, W\right), P_{j}^{L} \equiv P_{j}\left(S_{i}, S_{j}^{L}, W\right), P_{j}^{H} \equiv P_{j}\left(S_{i}, S_{j}^{H}, W\right)$, and similar definitions apply to $\widetilde{d}_{i}^{L}(0), \widetilde{d}_{i}^{L}(1), \widetilde{d}_{i}^{H}(0)$, and $\widetilde{d}_{i}^{H}(1)$. Subtracting the first equation to the second and solving for $\widetilde{\pi}_{i}(1)-\widetilde{\pi}_{i}(0)$ we can get:

$$
\begin{equation*}
\Delta_{i} \equiv \widetilde{\pi}_{i}(1)-\widetilde{\pi}_{i}(0)=\frac{\left[q_{i}^{H}-\left(1-P_{j}^{H}\right) \widetilde{d}_{i}^{H}(0)-P_{j}^{H} \widetilde{d}_{i}^{H}(1)\right]-\left[q_{i}^{L}-\left(1-P_{j}^{L}\right) \widetilde{d}_{i}^{L}(0)+P_{j}^{L} \widetilde{d}_{i}^{L}(1)\right]}{P_{j}^{H}-P_{j}^{L}} \tag{A.3.3}
\end{equation*}
$$

Given this identified function $\Delta_{i} \equiv \widetilde{\pi}_{i}(1)-\widetilde{\pi}_{i}(0)$, we can replace it into (A.3.2) and solve for $\widetilde{\pi}_{i}(0)$ to get:

$$
\begin{align*}
& \widetilde{\pi}_{i}(0)=q_{i}^{L}-\widetilde{d}_{i}^{L}(0)-P_{j}^{L}\left[\Delta_{i}+\widetilde{d}_{i}^{L}(1)-\widetilde{d}_{i}^{L}(0)\right]  \tag{A.3.4}\\
& \widetilde{\pi}_{i}(1)=\Delta_{i}+q_{i}^{L}-\widetilde{d}_{i}^{L}(0)-P_{j}^{L}\left[\Delta_{i}+\widetilde{d}_{i}^{L}(1)-\widetilde{d}_{i}^{L}(0)\right]
\end{align*}
$$

Finally, for states with $S_{j} \neq S_{j}^{L}$ and $S_{j} \neq S_{j}^{H}$ where beliefs can be biased, beliefs are identified by replacing the previous expressions for $\widetilde{\pi}_{i}(0)$ and $\widetilde{\pi}_{i}(1)$ into equation (A.3.1) and solving for $B_{j}$ :

$$
\begin{equation*}
B_{j}=\frac{q_{i}-\widetilde{\pi}_{i}(0)-\widetilde{d}_{i}^{\mathbf{B}}(0)}{\widetilde{\pi}_{i}(1)-\widetilde{\pi}_{i}(0)+\widetilde{d}_{i}^{\mathbf{B}}(1)-\widetilde{d}_{i}^{\mathbf{B}}(0)} \tag{A.3.5}
\end{equation*}
$$

The proof of identification is completed using a backwards induction argument. At the last period $T$, there is no future and the continuation values $\widetilde{d}_{i}^{\mathbf{B}}(0)$ and $\widetilde{d}_{i}^{\mathbf{B}}(1)$ are zero. Therefore, at the last period we have the following expressions for last-period payoffs in terms only of the known CCPs:

$$
\begin{align*}
\Delta_{i T} \equiv \widetilde{\pi}_{i T}(1)-\widetilde{\pi}_{i T}(0) & =\frac{q_{i T}^{H}-q_{i T}^{L}}{P_{j T}^{H}-P_{j T}^{L}} \\
\widetilde{\pi}_{i T}(0) & =q_{i T}^{L}-P_{j T}^{L} \Delta_{i T}  \tag{A.3.6}\\
\widetilde{\pi}_{i T}(1) & =\Delta_{i T}+q_{i T}^{L}-P_{j T}^{L} \Delta_{i T}
\end{align*}
$$

It is clear that last period payoffs are identified from last period CCPs. Similarly, zero continuation values imply that the expression for beliefs at the last period becomes:

$$
\begin{equation*}
B_{j T}=\frac{q_{i T}-\widetilde{\pi}_{i T}(0)}{\widetilde{\pi}_{i T}(1)-\widetilde{\pi}_{i T}(0)} \tag{A.3.7}
\end{equation*}
$$

Given payoffs and beliefs at period $T$, we can construct the continuation values at period $T-1$. First, we obtain the integrated value function:

$$
\begin{align*}
\bar{V}_{i T}^{\mathbf{B}} & =\ln \left(\sum_{Y_{i T} \in\{0,1\}} \exp \left\{\pi_{i T}^{\mathbf{B}}\left(Y_{i T}\right)\right\}\right)  \tag{A.3.8}\\
& =\ln \left(1+\exp \left\{\left(1-B_{j T}\right) \tilde{\pi}_{i T}(0)+B_{j T} \tilde{\pi}_{i T}(1)\right\}\right)
\end{align*}
$$

Given this value function, we obtain the continuation values at $T-1$ using the definition in equation (A.1.3). And given the continuation value functions, we apply the formulas in (A.3.4) and (A.3.5) to obtain payoffs and beliefs at $T-1$. By using backwards induction we identify beliefs and payoff functions at every period $t$.

## [A.4] Proof of Proposition 3

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[^1]:    ${ }^{1}$ See also Morris and Song (2002) for examples of models with strategic uncertainty and the related experimental evidence.
    ${ }^{2}$ For example, a firm would want its rival to believe that it is planning an expansion in a particular location to deter the rival from entering into the location, when in fact there is no such plan.

[^2]:    ${ }^{3}$ See Camerer (2003) and recent papers by Costa-Gomes and Weizsäcker (2008), and Palfrey and Wang (2009).
    ${ }^{4}$ Data on stated or elicited beliefs of firm managers is very rare and typically of low quality.

[^3]:    ${ }^{5}$ An exception is the recent paper by Goldfarb and Xiao (2011) that studies entry decisions in the US local telephone industry and finds significant heterogeneity in firms' beliefs about other firms' strategic behavior.
    ${ }^{6}$ The results in the paper can be generalized to models with more than two players or/and choice alternatives. A key result in our paper is the characterization of rational beliefs in section 3, that is used in the identification results and in the estimation methods in sections 4 and 5 . It is possible to extend that representation to multinomial choice models and games with more than two players.

[^4]:    ${ }^{7}$ See Bajari and Hong (2005), or Bajari et al (2010), among others.
    ${ }^{8}$ Based on results in Matzkin (1992 and 1994), Aguirregabiria (2010) provides conditions for the nonparametric identification of the distribution of the unobservables in single-agent dynamic structural models. Those conditions

[^5]:    can be applied to identify the distribution of the unobservables in our model.
    ${ }^{9}$ Note that $f_{t}^{S}\left(S^{\prime} \mid Y, \mathbf{X}\right)=\operatorname{Pr}\left(S_{m t+1}=S^{\prime} \mid Y_{m t}=Y, \mathbf{X}_{m t}=\mathbf{X}\right)$ and $f_{t}^{W}\left(\mathbf{W}^{\prime} \mid \mathbf{W}\right)=\operatorname{Pr}\left(\mathbf{W}_{m t+1}=\mathbf{W}^{\prime} \mid \mathbf{W}_{m t}=\mathbf{W}\right)$. We can estimate consistently these conditional distributions using, for intance, kernel methods.

[^6]:    ${ }^{10}$ See Aguirregabiria and Mira (2002), Pesendorfer and Schmidt-Dengler (2003), Bajari and Hong (2005), Bajari, Hong, and Ryan (2010), and Bajari et al. (2011), among others.
    ${ }^{11}$ It is relevant to point out that, in contrast to static decision problems, this normalization is not innocuous in dynamic decision models and it has implications for the predictions of some counterfactual experiments such as those that involve an hypothetical change in the transition probabilities of the state variables. See Aguirregabiria (2010) for an study of this issue in the context of single-agent dynamic decision models.

[^7]:    ${ }^{12}$ Appendix [A.1] provides more details on these two functions. See also Rust (1994).

[^8]:    ${ }^{13}$ We want to thank Otto Toivanen and Michael Waterson for generously sharing their data with us.
    ${ }^{14}$ The reason we exclude from our sample the districts in Greater London is that they do not satisfy the standard criteria of isolated geographic markets.
    ${ }^{15}$ As a definition of geographic market for the fast food retail industry, the district is perhaps a bit wide. However, an advantage of using district as definition of local market is that most of the markets in our sample are geographically isolated. Most districts contain a single urban area. And, in contrast to North America where many fast food restaurants are in transit locations, in UK these restaurants are mainly located in the centers of urban areas.

[^9]:    ${ }^{16}$ This 'quality' is just the willingness to pay for the product of the average consumer in the market.

[^10]:    ${ }^{17}$ The first order condition of profit maximization for firm $i$ is $p_{i}-c_{i}=B_{i}-\Delta p$, where $B_{i} \equiv 1+b_{i}+\delta_{i} b_{j}$, and $\Delta p \equiv p_{i}-p_{j}$. Similarly, the first order condition of profit maximization for firm $j$ is $p_{j}-c_{j}=B_{j}+\Delta p$. The difference between these first two conditions implies that $\Delta p=(1 / 3)(\Delta c+\Delta B)$ where $\Delta B \equiv B_{i}-B_{j}$ and $\Delta c \equiv c_{i}-c_{j}$. Solving this formula for $\Delta p$ into the first order condition of firm $i$, we get the following expression for the profit-maximizing price: $p_{i}=c_{i}+B_{i}-(1 / 3)(\Delta c+\Delta B)$.
    ${ }^{18}$ By 'exogenous' quality here we mean that it is the part of quality that does not depend on the number of stores in the market.

