On the Testability of Identication in Some Nonparametric Models with Endogeneity

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Testing Problems

- This paper consider three distinct hypothesis testing problem of identification.

- The first one concerns testing the necessary conditions for identification, also referred to the completeness condition in mean regression (indirectly).

- The second and third testing problems concern testing identification directly, in quantile regression.

Result

- Under some conditions and assumptions, there exists no nontrivial test for these hypothesis testing problems.

Introduction

- $\{V_i\}_{i=1}^{n}$ is an i.i.d. sequence of random variables with distribution $P \in \mathbf{P}$
- Each of these hypothesis testing problems may be written as

$$H_0 : P \in \mathbf{P}_0 \quad versus \quad H_1 : P \in \mathbf{P}_1$$

$$\mathbf{P} = \mathbf{P}_0 \bigcup \mathbf{P}_1$$

Main idea of this paper

- Under some conditions and assumptions different from the literatures, this paper concludes that the following result still holds.
- Any sequence of tests $\{\phi_i\}_{i=1}^{n}$ that controls size at level $\alpha \in (0,1)$ in the sense that

$$\limsup_{n\to\infty} \sup_{P\in\mathbf{P}_0} E_{P^n}[\phi_n] < \alpha \qquad (1)$$

- also satisfies

$$\limsup_{n\to\infty} \sup_{P\in\mathbf{P}_1} E_{P^n}[\phi_n] < \alpha \qquad (2)$$

Case 1

- Let $V_i = (X_i, Z_i)$ and **P** be a set of probability measure on $\mathbb{R}^{d_x} \times \mathbb{R}^{d_z}$
- $Y_i = g(W_i) + \varepsilon_i$ and $E[\varepsilon_i \mid Z_i] = 0$
- For Z_i^x a (possible empty) subvector of Z_i and $W_i = (X_i, Z_i^x) \in \mathbb{R}^{d_w}$, define,

$$\mathbf{P}_1 = \mathbf{P} \setminus \mathbf{P}_0 = \{P \in \mathbf{P} : E_p[\theta(W_i) \mid Z_i] = 0 \ for \ \theta \in \Theta(P) \Rightarrow \theta = 0 \ P - a.s.\} \quad (3)$$

- Here, $\Theta(P)$ is understood to be the subset of the set of all functions from $\mathbb{R}^{d_w} \rightarrow \mathbb{R}$, and

$$\theta(W_i) = \tilde{g}(W_i) - g(W_i)$$

Case 2

- Let $V_i = (Y_i, X_i, Z_i)$ and **P** be a set of probability measure on $\mathbb{R} \times \mathbb{R}^{d_x} \times \mathbb{R}^{d_z}$
- For Z_i^x a (possible empty) subvector of Z_i and $W_i = (X_i, Z_i^x) \in \mathbb{R}^{d_w}$. Consider an outcome of interest Y_i and endogenous variable X_i, and an instrumental variable Z_i, and there is some $\theta \in \Theta(P)$ for which
- $Y_i = \theta(W_i) + \varepsilon_i$ and $P\{\varepsilon_i < 0 \mid Z_i\} = \tau$ w.p.1 under P (4)

  and
- $\mathbf{P}_0 = \mathbf{P} \setminus \mathbf{P}_1 = \{P \in \mathbf{P} : \exists! \theta \in \Theta(P)$ s.t.（4）holds under $P\}$ (5)

Case 3

- And for case 3, we have

$$Y_i = \theta(W_i, \varepsilon_i) \text{ and } P\{\theta(W_i, \varepsilon_i) - \theta(W_i, \tau) < 0 \mid Z_i\} = \tau \text{ w.p.1 under P} \quad (6)$$

- $\mathbf{P}_0 = \mathbf{P} \setminus \mathbf{P}_1 = \{P \in \mathbf{P} : \exists! \theta \in \Theta(P) \text{ s.t.(6) holds under } P\}$ (7)

A useful lemma

- Let M denote the space of Borel probability measures on a metric space. Suppose $\mathbf{P}$ is a subset of M and $\mathbf{P} = \mathbf{P}_0 \bigcup \mathbf{P}_1$. If for each $P \in \mathbf{P}_1$ there exist a sequence $\{P_k\}_{k=1}^{\infty}$ in $\mathbf{P}_0$ with $H(P, P_k) = o(1)$ then every sequence of test functions $\{\phi_i\}_{i=1}^{n}$ satisfies:

$$\limsup_{n\to\infty} \sup_{P\in\mathbf{P}_1} E_{P^n}[\phi_n] \le \limsup_{n\to\infty} \sup_{P\in\mathbf{P}_0} E_{P^n}[\phi_n]$$

- H is the Hellinger distance

key point: $\mathbf{P}_0$ being dense in $\mathbf{P}_1$

A useful lemma (continuous)

- A modification of theorem in Romano(2004)
- Hellinger distance as opposed to Total Variation distance.
- Large-sample result as opposed to a finite-sample result.
- The power of the test is bounded by the asymptotic size

Assumptions

- Let $M_{X,Z}$ be the set of all the probability measures on $\mathbb{R}^{d_x} \times \mathbb{R}^{d_z}$, and, for ν a Borel measure on $\mathbb{R}^{d_x} \times \mathbb{R}^{d_z}$, define

$$M_{X,Z}(\nu) \equiv \{P \in M_{X,Z} : P \ll \nu\} \qquad (8)$$

- A1: ν is a positive σ-finite Borel measures on $\mathbb{R}^{d_x} \times \mathbb{R}^{d_z}$
- A2: $\nu = \nu_X \times \nu_Z$, where ν_X and ν_Z are Borel measures on $\mathbb{R}^{d_x}$ and $\mathbb{R}^{d_z}$
- A3: The measure ν_X is atomless (on $\mathbb{R}^{d_x}$)

Theorem 1

- Suppose ν satisfies assumption 1,2 and 3. Define $M_{X,Z}(\nu)$ as in (8) and let $\mathbf{P}= M_{X,Z}(\nu)$. Further define $\mathbf{P}_0$ and $\mathbf{P}_1$ as in (3) with $\Theta(P)=L^{\infty}(P)$. Then the sequence of test function $\{\phi_i\}_{i=1}^{n}$ satisfies

$$\lim_{n\to\infty}\sup \sup_{P\in\mathbf{P}_1} E_{P^n}[\phi_n] \le \lim_{n\to\infty}\sup \sup_{P\in\mathbf{P}_0} E_{P^n}[\phi_n]$$

- If $L^{\infty}(P)\subseteq\Theta(P)$ or $\Theta(P)=L^{q}(P)$ this theorem still holds

Assumptions

- Let $M_{Y,X,Z}$ be the set of all the probability measures on $\mathbb{R}\times\mathbb{R}^{d_x}\times\mathbb{R}^{d_z}$, and, for ν a Borel measure on $\mathbb{R}\times\mathbb{R}^{d_x}\times\mathbb{R}^{d_z}$, define

$$M_{Y,X,Z}(\nu)\equiv\{P\in M_{Y,X,Z}:P\ll\nu\} \qquad (9)$$

- A4: ν is a positive σ-finite Borel measures on $\mathbb{R}\times\mathbb{R}^{d_x}\times\mathbb{R}^{d_z}$
- A5: $\nu=\nu_Y\times\nu_X\times\nu_Z$, where ν_Y , ν_X and ν_Z are Borel measures on $\mathbb{R}$, $\mathbb{R}^{d_x}$ and $\mathbb{R}^{d_z}$

Theorem 2

- Suppose ν satisfies assumptions 3,4 and 5. Then define $M_{Y,X,Z}(\nu)$ as in (9) and let $\mathbf{P} = M_{Y,X,Z}(\nu)$ such that for each $P \in \mathbf{P}$ there is some $\theta \in \Theta(P) = L^{\infty}(P)$ for which (4) holds. Further define $\mathbf{P}_0$ and $\mathbf{P}_1$ in (5). Then the sequence of test functions $\{\phi_i\}_{i=1}^{n}$ satisfies

$$\lim_{n\to\infty} \sup \sup_{P\in\mathbf{P}_1} E_{P^n}[\phi_n] \leq \lim_{n\to\infty} \sup \sup_{P\in\mathbf{P}_0} E_{P^n}[\phi_n]$$

Theorem 3

- T denotes the set of all functions $\theta : \mathbb{R}^{d_w} \times [0,1] \to \mathbb{R}$ and define

$$T(P) \equiv \{\theta \in T : \theta(W_i, \cdot) \text{ is strictly increasing and } \sup_{0 \le \tau \le 1} \|\theta(\cdot, \tau)\|_{L^\infty(P)} < \infty\}$$

- Suppose ν satisfies assumptions 3,4 and 5. Then define $\mathbf{P} = M_{Y,X,Z}(\nu)$ such that for each $P \in \mathbf{P}$ there is some $\theta \in \Theta(P) = T(P)$ for which (6) holds. Further define $\mathbf{P}_0$ and $\mathbf{P}_1$ as in (7). Then the sequence of test functions $\{\phi_i\}_{i=1}^n$ satisfies

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}_1} E_{P^n}[\phi_n] \le \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} E_{P^n}[\phi_n]$$

Conclusion

- Three distinct hypothesis testing about identification.
- Assumptions 1,2,3,4 and 5.
- Functional spaces is $L^{\infty}(P)$ or $L^{q}(P)$ (in case 3 $\sup_{0\leq\tau\leq1}\|\theta(\cdot,\tau)\|_{L^{\infty}(P)}<\infty$ or $\sup_{0\leq\tau\leq1}\|\theta(\cdot,\tau)\|_{L^{q}(P)}<\infty$)
- Case 1: utilizes completeness condition.
- Case 2 & Case 3: utilize definitions directly.
- No nontrivial tests for these three cases.

Remarks

- Conditions are satisfied under commonly used assumptions.
- Not rule out the existence of reasonable tests under more restrictive assumptions.
- Help shape the development of nontrivial tests of the hypotheses this paper considers.