Identification and Estimation of Dynamic Games when Players' Beliefs are not in Equilibrium

Victor Aguirregabiria and Arvind Magesan

Presented by Weilong Zhang

Hanqing Institute, Renmin University of China

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Cases when player's belief is not in the equilibrium

- Competition in oligopoly industries: Firm managers have incentives to misrepresent their own strategies and face significant uncertainty about the strategies of their competitors.
- Policy change in a strategic environment: Firms need to take time to learn about strategies of competitors after the policy change.
- In laboratory experiments, there exists significant heterogeneity in agents' elicited beliefs, and that this heterogeneity is often one of the most important factors in explaining heterogeneity in observed behavior.

Main Problem we want to deal with

- Relax the assumption of equilibrium beliefs. When players beliefs are not in equilibrium they are different from the actual distribution of players actions.
- This paper concentrates on the identification of players payoff functions and beliefs and does not want to make any arbitrary assumption on beliefs.
- However, without other restrictions, beliefs cannot be identified and estimated by simply using a nonparametric estimator of the distribution of players' actions. (Order condition for identification is not satisfied, the number of restrictions is less than the number of parameters).

Main Steps

- First, a standard exclusion restriction can provide testable nonparametric restrictions of the null hypothesis of equilibrium beliefs.
- Second, new results on the nonparametric point-identification of payoff functions and beliefs. (no strategic uncertainty at two extreme points should be imposed).
- Third, a simple two-step estimation method of structural parameters and beliefs is proposed.
- Fourth, an empirical application of a dynamic game of store location by retail chains is illustrated. (Omitted because of time limitation)

- Two player,i,j. Finite period, T. $Y_{it} \in 0, 1$ represents the choice of player i in period t.
- Optimal expected utility $E_t(\sum_{s=0}^{T} \beta_i^s \prod_{i,t+s})$.
- One period payoff function $\prod_{i,t} = \pi_{it}(Y_{jt}, X_t) - \varepsilon_{it} \quad if \quad Y_{it} = 1, \prod_{i,t} = 0 \quad if \quad Y_{it} = 0.$
- Y_{jt} represents the current action of the other player. X_t is a vector of state variables which are common knowledge for both players.

- Common knowledge X_{it} has three parts: $X_t \in (W_t, S_{it}, S_{jt})$. W_t is a vector of state variables that evolve exogenously according to a Markov process with transition probability function. ("Market Size")
- S_{it}, S_{jt} are endogenous state variables. They evolve over time according to a transition probability function $f_{St}(S_{t+1}|Y_{it}, Y_{jt}, X_t)$. ("the number of consecutive years of player i in market").
- Y_{jt} represents the current action of the other player. X_t is a vector of state variables which are common knowledge for both players.

An example

- Deterministic transition rule: $S_{it+1} = Y_{it}(S_{it} + Y_{it})$.
- Current payoff function
  $\prod_{it} = \pi(Y_{jt}, S_{it})W_t - \varepsilon_{it}$ *if* $Y_{it} = 1$, *and* $\prod_{it} = 0$ *if* $Y_{it} = 0$.
- we got $\pi(Y_{jt}, s_{it}) = ((1 - Y_{jt})\theta_i^M + Y_{jt}\theta_i^D) - \theta_{i0}^{FC} - \theta_{i1}^{FC} exp(-S_{it}) - 1(S_{it} = 0)\theta_{i1}^{EC}$

Markov Perfect Equilibrium (MPE)

ASSUMPTION 1 (Payoff relevant state variables): Players' strategy functions depend only on payoff relevant state variables: $\mathbf{X}_t$ and ε_{it}.

ASSUMPTION 2 (Rational beliefs on own future behavior): Players are forward looking, maximize expected intertemporal payoffs, and have rational expectations on their own behavior in the future.

ASSUMPTION 'EQUIL': (Rational or equilibrium beliefs on other players' actions): Strategy functions are common knowledge, and players' have rational expectations on the current and future behavior of other players. That is, players beliefs about other players' behavior are consistent with the actual behavior of other players.

- Given the strategy function σ_{it}, we have the Conditional Choice Probability function $P_{it}(X_t) = \int 1\{\sigma_{it}(X_t, \varepsilon_{it}) = 1\} d\Lambda_i(\varepsilon_{it})$.
- When assumption 'Equil' not holds, the belief of choice probability $B_{jt}(X_t) = \int 1\{b_{jt}(X_t, \varepsilon_{it}) = 1\} d\Lambda_i(\varepsilon_{it})$ will be not equal to the actual choice probability P_{jt}.

Given the expected one-period payoff function
$\pi_{it}^{B}(X_t) = (1 - B_{jt}(X_t))\pi_{it}(0, X_t) + B_{jt}(X_t)\pi_{it}(1, X_t)$

Using backwards induction in the following Bellman equation:

$$V_{it}^{\mathbf{B}}(\mathbf{X}_t, \varepsilon_{it}) = \max_{Y_{it}\in\{0,1\}} Y_{it}(\pi_{it}^{\mathbf{B}}(\mathbf{X}_t) - \varepsilon_{it}) + \beta_i \int V_{it+1}^{\mathbf{B}}(\mathbf{X}_{t+1}, \varepsilon_{it+1})\, d\Lambda_i(\varepsilon_{it+1})\, df_{it}^{\mathbf{B}}(\mathbf{X}_{t+1}|Y_{it}, \mathbf{X}_t)$$

We denote $v_{it}^{B}(X_t)$ a threshold value function because it represents the threshold value that makes player i indifferent between the choice of alternatives 0 and 1.

$$v_{it}^{\mathbf{B}}(\mathbf{X}_t) \equiv \pi_{it}^{\mathbf{B}}(\mathbf{X}_t) + \beta_i \sum_{\mathbf{X}_{t+1}} \left[f_{it}^{\mathbf{B}}(\mathbf{X}_{t+1}|1, \mathbf{X}_t) - f_{it}^{\mathbf{B}}(\mathbf{X}_{t+1}|0, \mathbf{X}_t)\right]\ \bar{V}_{it+1}^{\mathbf{B}}(\mathbf{X}_{t+1})$$

- Given the threshold, the optimal response function is $Y_{it} = 1 \quad iff \quad \{\varepsilon_{it} \leq v_{it}^{B}(X_t)\}$.
- Under assumption 1 and 2 the actual behavior of player i, satisfying $P_{it}(X_t) = \Lambda_i(v_{it}^{B}(X_t))$.
- When player' belief are in equilibrium, we have that $B_{jt}(X_t) = P_{jt}(X_t)$.

Basic Assumption

- We concentrate on the identification of the payoff functions π_{it} and belief function B_{it} and assume that $\{f_{St}, f_{Wt}, \Lambda_i, \beta_i : i = 1, 2\}$ are known.
- The transition probability base on belief is the combination between belief function and actual transition function $f_{it}^B(S_{t+1}|Y_{it}, X_t) = (1 - B_{jt}(X_t))f_{it}(S_{t+1}|Y_{it}, 0, X_t) + B_{jt}(X_t)f_{it}(S_{t+1}|Y_{it}, 1, X_t)$

ASSUMPTION 4 (Exclusion Restriction): The one-period payoff function of player i depends on the actions of both players, Y_{it} and Y_{jt}, the common state variables $\mathbf{W}_t$, and the own stock variable, S_{it}, but it does not depend on the stock variable of the other player, S_{jt}.

$$\pi_{it}(Y_{jt}, \mathbf{X}_t) = \pi_{it}(Y_{jt}, \mathbf{W}_t, S_{it})$$

- Under the assumption, the null hypothesis of equilibrium belief is testable.
- Best response condition:
  $P_i(X) = \Lambda_i((1 - B_{jt}(X_t))\pi_{it}(0, X_t) + B_{jt}(X_t)\pi_{it}(1, X_t))$
- define
  $q_i(X) = \Lambda_i^{-1}(P_i(X)) = \pi_i(0, X) + [\pi_i(1, X) - \pi_i(0, X)]B_j(X)$
- given four values of vector X, say X^a, X^b, X^c, X^d, with same (S_i, W) but different S_j, then they will have same payoff π_i. Then $\frac{q_i(X^a)-q_i(X^b)}{q_i(X^c)-q_i(X^d)} = \frac{B_j(X^a)-B_j(X^b)}{B_j(X^c)-B_j(X^d)}$

A nonparametric test for the null hypothesis of equilibrium beliefs.

$$\delta(X^a, X^b, X^c, X^d) = \frac{q_i(X^a) - q_i(X^b)}{q_i(X^c) - q_i(X^d)} - \frac{P_j(X^a) - P_j(X^b)}{P_j(X^c) - P_j(X^d)}$$

Additional Assumption We Need

ASSUMPTION 5a (Monotonic beliefs and large support): (i) The beliefs function $B^0_{jt}(\mathbf{X}_t)$ *is strictly monotonically decreasing (or increasing) in the stock variable* S_{jt}, i.e., $\partial B^0_{jt}(\mathbf{X}_t)/\partial S_{jt} < 0$. *(ii) Conditional on* $(\mathbf{W}_t, S_{it})$ *the probability distribution of* S_{jt} *has unbounded support over the whole real line.*

ASSUMPTION 5b (No strategic uncertainty at two 'extreme' points): There are two values in the support of the distribution of S_{jt}, *say* S_j^{low} *and* S_j^{high}, *such that for every value of* (S_i, W) *we have that beliefs are in equilibrium, i.e.,* $B^0_{jt}(S_j^{low}, S_i, W) = P^0_{jt}(S_j^{low}, S_i, W)$, *and* $B^0_{jt}(S_j^{high}, S_i, W) = P^0_{jt}(S_j^{high}, S_i, W)$.

PROPOSITION 2: Under Assumptions 1-4 and either 5a or 5b the payoff functions $\{\pi^0_{it}$ *for any* $i,t\}$ *and the beliefs functions* $\{B^0_{it}$ *for any* $i,t\}$ *are nonparametrically identified.*

Proof

Given we have known the belief in two extreme points, we have that for any value of (Sj, W):

$$\begin{aligned} q_i^0(S_j^{low}, S_i, \mathbf{W}) &= \pi_i^0(0, S_i, \mathbf{W}) + \left[\pi_i^0(1, S_i, \mathbf{W}) - \pi_i^0(0, S_i, \mathbf{W})\right] \; P_j^0(S_j^{low}, S_i, \mathbf{W}) \\ q_i^0(S_j^{high}, S_i, \mathbf{W}) &= \pi_i^0(0, S_i, \mathbf{W}) + \left[\pi_i^0(1, S_i, \mathbf{W}) - \pi_i^0(0, S_i, \mathbf{W})\right] \; P_j^0(S_j^{high}, S_i, \mathbf{W}) \end{aligned} \tag{14}$$

Combining these two equations, we obtain $[\pi_i^0(1, S_i, \mathbf{W}) - \pi_i^0(0, S_i, \mathbf{W})] = [q_i^0(S_j^{high}, S_i, \mathbf{W}) - q_i^0(S_j^{low}, S_i, \mathbf{W})] \,/\, [P_j^0(S_j^{high}, S_i, \mathbf{W}) - P_j^0(S_j^{low}, S_i, \mathbf{W})]$. And substituting this expression into the equations in (14), we identify the payoff function as:

$$\begin{aligned} \pi_i^0(0, S_i, \mathbf{W}) &= q_i^0(\mathbf{X}^{low}) - P_j^0(\mathbf{X}^{low}) \left[\frac{q_i^0(\mathbf{X}^{high}) - q_i^0(\mathbf{X}^{low})}{P_j^0(\mathbf{X}^{high}) - P_j^0(\mathbf{X}^{low})}\right] \\ \pi_i^0(1, S_i, \mathbf{W}) &= q_i^0(\mathbf{X}^{low}) + \left[1 - P_j^0(\mathbf{X}^{low})\right] \left[\frac{q_i^0(\mathbf{X}^{high}) - q_i^0(\mathbf{X}^{low})}{P_j^0(\mathbf{X}^{high}) - P_j^0(\mathbf{X}^{low})}\right] \end{aligned} \tag{15}$$

where $\mathbf{X}^{low} \equiv (S_j^{low}, S_i, \mathbf{W})$ and $\mathbf{X}^{high} \equiv (S_j^{high}, S_i, \mathbf{W})$. Again, given the identification of the payoff function, we can obtain beliefs as $B_j^0(\mathbf{X}) = [q_i^0(\mathbf{X}) - \pi_i^0(0, \mathbf{X})] \,/\, [\pi_i^0(1, \mathbf{X}) - \pi_i^0(0, \mathbf{X})]$.

Proof continuing

Given beliefs and payoffs at period T, we can use backwards induction to prove the identification of the payoff function and beliefs at any period $t < T$. Suppose that the functions π^0_{it+s} and B^0_{jt+s} are known for every $s \geq 1$. Given this information, we want to identify functions π^0_{it} and B^0_{jt}. Player i's best response implies that $P^0_{it}(\mathbf{X}_t) = \Lambda_i(v^{\mathbf{B}^0}_{it}(\mathbf{X}_t))$, where $v^{\mathbf{B}^0}_{it}(\mathbf{X}_t) = (1 - B^0_{jt}(\mathbf{X}_t))\ \pi^0_{it}(0, \mathbf{X}_t) + B^0_{jt}(\mathbf{X}_t)\ \pi^0_{it}(1, \mathbf{X}_t) + \ \beta_i \sum_{\mathbf{X}_{t+1}} [f^{\mathbf{B}^0}_{it}(\mathbf{X}_{t+1}|1, \mathbf{X}_t) - \ f^{\mathbf{B}^0}_{it}(\mathbf{X}_{t+1}|0, \mathbf{X}_t)]\ \bar{V}^{\mathbf{B}^0}_{it+1}(\mathbf{X}_{t+1})$, and $\bar{V}^{\mathbf{B}^0}_{it+1}$ is the integrated values function $\int V^{\mathbf{B}^0}_{it+1}(\mathbf{X}_{t+1}, \varepsilon_{it+1}) d\Lambda_i(\varepsilon_{it+1})$. It is straightforward to show that given $\{P^0_{it+s}, \pi^0_{it+s}, B^0_{jt+s} : s = 1, 2, ..., T-t\}$ the integrated value function $\bar{V}^{\mathbf{B}^0}_{it+1}$ is known. Then, define the function:

$$q^0_{it}(\mathbf{X}) \equiv \Lambda_i^{-1}(P^0_{it}(\mathbf{X})) - \beta_i \sum_{\mathbf{X}_{t+1}} [f^{\mathbf{B}^0}_{it}(\mathbf{X}_{t+1}|1, \mathbf{X}_t) - f^{\mathbf{B}^0}_{it}(\mathbf{X}_{t+1}|0, \mathbf{X}_t)]\ \bar{V}^{\mathbf{B}^0}_{it+1}(\mathbf{X}_{t+1}) \tag{16}$$

This function is identified everywhere in the support of $\mathbf{X}$. Furthermore, by the best response condition and the definition of q^0_{it}, we have that:

$$q^0_{it}(\mathbf{X}) = \pi^0_{it}(0, \mathbf{X}) + \left[\pi^0_{it}(1, \mathbf{X}) - \pi^0_{it}(0, \mathbf{X})\right]\ B^0_{jt}(\mathbf{X}) \tag{17}$$

This equation has the same structure as the best response condition at last period T. Therefore, we can use exactly the same arguments as above to show the identification of functions π^0_{it} and B^0_{jt}.

Estimation with nonparametric payoff function

Step 1: Nonparametric estimation of CCPs, P_{it}^{u}, for every player, time period, and state $\mathbf{X}$, and (if needed) of the transition probabilities f_{St} and f_{Wt}.

Step 2: Estimation of preferences and beliefs. At the last period T, we construct $\hat{q}_{iT}^{0}(\mathbf{X}) = \Lambda_i^{-1}(\hat{P}_{iT}^{0}(\mathbf{X}))$ for any value of $\mathbf{X}$ in the support $\mathcal{X}$. The estimated payoff function at point $(S_i, \mathbf{W})$ is:

$$\begin{aligned}
\hat{\pi}_{iT}^{0}(0, S_i, \mathbf{W}) &= \hat{q}_{iT}^{0}(\mathbf{X}^{low}) - \hat{P}_{jT}^{0}(\mathbf{X}^{high}) \left[\frac{\hat{q}_{iT}^{0}(\mathbf{X}^{high}) - \hat{q}_{iT}^{0}(\mathbf{X}^{low})}{\hat{P}_{jT}^{0}(\mathbf{X}^{high}) - \hat{P}_{jT}^{0}(\mathbf{X}^{low})} \right] \\
\hat{\pi}_{iT}^{0}(1, S_i, \mathbf{W}) &= \hat{q}_{iT}^{0}(\mathbf{X}^{low}) + \left[1 - \hat{P}_{jT}^{0}(\mathbf{X}^{high}) \right] \left[\frac{\hat{q}_{iT}^{0}(\mathbf{X}^{high}) - \hat{q}_{iT}^{0}(\mathbf{X}^{low})}{\hat{P}_{jT}^{0}(\mathbf{X}^{high}) - \hat{P}_{jT}^{0}(\mathbf{X}^{low})} \right]
\end{aligned} \tag{24}$$

where $\mathbf{X}^{low} \equiv (S_j^{low}, S_i, \mathbf{W})$ and $\mathbf{X}^{high} \equiv (S_j^{high}, S_i, \mathbf{W})$. The estimated beliefs function is:

$$\hat{B}_{jT}^{0}(\mathbf{X}) = \frac{\hat{q}_{iT}^{0}(\mathbf{X}) - \hat{\pi}_{iT}^{0}(0, \mathbf{X})}{\hat{\pi}_{iT}^{0}(1, \mathbf{X}) - \hat{\pi}_{iT}^{0}(0, \mathbf{X})} \tag{25}$$

Estimation with parametric payoff function

To estimate θ_i^0 we propose a simple three steps method. The first two-steps are the same as for the nonparametric model.

Step 3: Given the estimates from step 2, we can apply a pseudo maximum likelihood method in the spirit of Aguirregabiria and Mira (2002, 2007) to estimate the structural parameters θ^0. Define the pseudo likelihood function:

$$Q(\theta, \mathbf{B}, \mathbf{P}) \equiv \sum_{m=1}^{M} \sum_{t=1}^{T} \sum_{i=1}^{2} Y_{imt} \ \log \Lambda \left(\tilde{z}_{imt}^{\mathbf{B},\mathbf{P}} \ \theta_i + \tilde{e}_{imt}^{\mathbf{B},\mathbf{P}} \right) + (1 - Y_{imt}) \ \log \left(1 - \Lambda \left(\tilde{z}_{imt}^{\mathbf{B},\mathbf{P}} \ \theta_i + \tilde{e}_{imt}^{\mathbf{B},\mathbf{P}} \right) \right)$$

$\tilde{z}_{imt}^{\mathbf{B},\mathbf{P}}$ is the sum of expected and discounted stream of $\{z_{it'}(Y_{jt'}, \mathbf{X}_{t'}) : t' = t, t+1, ..., T\}$ given that player i behaves according to the choice probabilities $P_{it'}(.)$ in $\mathbf{P}$, and player j behaves according to the probabilities $B_{jt'}(.)$ in $\mathbf{B}$. Similarly, $\tilde{e}_{imt}^{\mathbf{B},\mathbf{P}}$ is the sum of expected and discounted stream of $\{e(P_{it'}(\mathbf{X}_{t'})) : t' = t, t+1, ..., T\}$, and for the logit model $e(P_{it'}(\mathbf{X}_{t'})) = \gamma - \ln P_{it'}(\mathbf{X}_{t'})$ where γ is Euler's constant. From steps 1 and 2, we have consistent estimates of CCPs, $\hat{\mathbf{P}}^0$, and beliefs,

Iteration to Converge

We can apply step 3 and the updating of CCPs and beliefs recursively to obtain a sequence of estimators $\{\hat{\boldsymbol{\theta}}^{(K)}, \hat{\mathbf{B}}^{(K)}, \hat{\mathbf{P}}^{(K)} : K \geq 1\}$. At each iteration K of this iterative procedure we perform tasks (i) to (v):

(i) Update of preference parameters: $\hat{\boldsymbol{\theta}}^{(K)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \hat{\mathbf{B}}^{(K-1)}, \hat{\mathbf{P}}^{(K-1)})$.

(ii) Update of CCP functions: $\hat{P}_{it}^{(1)}(\mathbf{X}) = \Lambda\left(\tilde{z}_{it}^{\hat{\mathbf{B}}^{(K-1)},\hat{\mathbf{P}}^{(K-1)}}(\mathbf{X})\ \hat{\boldsymbol{\theta}}_i^{(K)} + \tilde{e}_{it}^{\hat{\mathbf{B}}^{(K-1)},\hat{\mathbf{P}}^{(K-1)}}\right)$.

(iii) Update of value functions: $\hat{V}_{iT}^{(K)}(\mathbf{X}) = -\ln\left(1 - \hat{P}_{iT}^{(K)}(\mathbf{X})\right)$ and for $t < T$, $\hat{V}_{it}^{(K)}(\mathbf{X}) = -\ln\left(1 - \hat{P}_{it}^{(K)}(\mathbf{X})\right) + \beta_i \sum_{\mathbf{X}_{t+1}} f_{it}(\mathbf{X}_{t+1}|0,\mathbf{X})\ \hat{V}_{it+1}^{(K)}(\mathbf{X})$.

(iv) Update of q functions: $\hat{q}_{iT}^{(K)}(\mathbf{X}) = \Lambda^{-1}(\hat{P}_{it}^{(K)}(\mathbf{X}))\ -\beta_i \sum_{\mathbf{X}_{t+1}}[f_{it}(\mathbf{X}_{t+1}|1,\mathbf{X}) - f_{it}(\mathbf{X}_{t+1}|0,\mathbf{X})]\ \hat{V}_{it+1}^{(K)}(\mathbf{X}_{t+1})$.

(v) Update of beliefs functions:

$$\hat{B}_{jt}^{(K)}(\mathbf{X}) = \hat{P}_{jt}^{(K)}(\mathbf{X}^{low}) + \left[\hat{P}_{jt}^{(K)}(\mathbf{X}^{high}) - \hat{P}_{jt}^{(K)}(\mathbf{X}^{low})\right]\ \left[\frac{\hat{q}_{it}^{(K)}(\mathbf{X}) - \hat{q}_{it}^{(K)}(\mathbf{X}^{low})}{\hat{q}_{it}^{(K)}(\mathbf{X}^{high}) - \hat{q}_{it}^{(K)}(\mathbf{X}^{low})}\right]$$