# 1 Hotz-Miller approach: avoid numeric dynamic programming

- Rust, Pakes approach to estimating dynamic discrete-choice model very computer intensive. Requires using numeric dynamic programming to compute the value function(s) for every parameter vector $\theta$.

- Alternative method of estimation, which avoids explicit DP. Present main ideas and motivation using a simplified version of (Hotz and Miller 1993), (Hotz, Miller, Sanders, and Smith 1994).

- Related to recent emphasis in literature of non/semi-parametric identification of structural models: given data, what structural parameters can be identified without making excessive parametric assumptions?

- For simplicity, think about Harold Zurcher model.

- What do we observe in data from DDC framework? For agent $i$, time $t$, observe:

  - $\{\tilde{x}_{it}, d_{it}\}$: observed state variables $\tilde{x}_{it}$ and discrete decision (control) variable $d_{it}$. For simplicity, assume $d_{it}$ is binary, $\in \{0, 1\}$

    Let $i = 1, \ldots, N$ index the buses, $t = 1, \ldots, T$ index the time periods.

  - For Harold Zurcher model: $\tilde{x}_{it}$ is mileage on bus $i$ in period $t$, and $d_{it}$ is whether or not engine of bus $i$ was replaced in period $t$.

  - Given renewal assumptions (that engine, once repaired, is good as new), define transformed state variable $x_{it}$: mileage since last engine change.

  - Unobserved state variables: $\epsilon_{it}$, $i.i.d.$ over $i$ and $t$. Assume that distribution is known (Type 1 Extreme Value in Rust model)

- In the following, let quantities with hats $\hat{}$'s denote objects obtained just from data.

  Objects with tildes $\tilde{}$'s denote "predicted" quantities, obtained from both data and calculated from model given parameter values $\theta$.

- From this data alone, we can estimate (or "identify"):

- Transition probabilities of observed state and control variables: $G(x'|x,d)$[1], estimated by conditional empirical distribution

$$\hat{G}(x'|x,d) \equiv \begin{cases} \sum_{i=1}^{N}\sum_{t=1}^{T-1} \frac{1}{\sum_i \sum_t \mathbf{1}(x_{it}=x,d_{it}=0)} \cdot \mathbf{1}\left(x_{i,t+1} \leq x', x_{it}=x, d_{it}=0\right), & \text{if } d=0 \\ \sum_{i=1}^{N}\sum_{t=1}^{T-1} \frac{1}{\sum_i \sum_t \mathbf{1}(d_{it}=1)} \cdot \mathbf{1}\left(x_{i,t+1} \leq x', d_{it}=1\right), & \text{if } d=1. \end{cases}$$

- Choice probabilities, conditional on state variable: $\text{Prob}\left(d=1|x\right)$[2], estimated by

$$\hat{P}(d=1|x) \equiv \sum_{i=1}^{N}\sum_{t=1}^{T-1} \frac{1}{\sum_i \sum_t \mathbf{1}\left(x_{it}=x\right)} \cdot \mathbf{1}\left(d_{it}=1, x_{it}=x\right).$$

Since $\text{Prob}\left(d=0|x\right) = 1 - \text{Prob}\left(d=1|x\right)$, we have $\hat{P}(d=0|x) = 1 - \hat{P}(d=1|x)$.

- With estimates of $\hat{G}(\cdot|\cdot)$ and $\hat{p}(\cdot|\cdot)$, as well as a parameter vector $\theta$, you can "estimate" the choice-specific value functions by constructing the sum

$$\begin{aligned}\tilde{V}(x,d=1;\theta) =& u(x,d=1;\theta) + \beta E_{x'|x,d=1} E_{d'|x'} E_{\epsilon'|d',x'}\left[u(x',d';\theta) + \epsilon'\right. \\ & \left.+ \beta E_{x''|x',d'} E_{d''|x''} E_{\epsilon'|d'',x''}\left[u(x'',d'';\theta) + \epsilon'' + \beta \cdots\right]\right]\end{aligned}$$

$$\begin{aligned}\tilde{V}(x,d=0;\theta) =& u(x,d=0;\theta) + \beta E_{x'|x,d=1} E_{d'|x'} E_{\epsilon'|d',x'}\left[u(x',d';\theta) + \epsilon'\right. \\ & \left.+ \beta E_{x''|x',d'} E_{d''|x''} E_{\epsilon'|d'',x''}\left[u(x'',d'';\theta) + \epsilon'' + \beta \cdots\right]\right].\end{aligned}$$

Here $u(x,d;\theta)$ denotes the per-period utility of taking choice $d$ at state $x$, *without* the additive logit error. Note that the observation of $d'|x'$ is crucial to being able to forward-simulate the choice-specific value functions. Otherwise, $d'|x'$ is multinomial with probabilities given by Eq. (1) below, and is impossible to calculate without knowledge of the choice-specific value functions.

Also, the expectation $E_{\epsilon'|d'}$ denotes the expectation of the $\epsilon$ conditional on choice $d$ being taken. For the logit case, there is a closed form:

$$E[\epsilon|d,x] = \gamma - \log(Pr(d|x))$$

[1]By stationarity, note we do not index the $G$ function explicitly with time $t$.

[2]By stationarity, note we do not index this probability explicitly with time $t$.

where $\gamma$ is Euler's constant $(0.577...)$ and $Pr(d|x)$ is the choice probability of action $d$ at state $x$.

Both of the other expectations in the above expressions are observed directly from the data.

- Both choice-specific value functions can be simulated by (for $d = 1, 2$):

$$\tilde{V}(x, d; \theta) \approx = \frac{1}{S} \sum_s \Big[ u(x, d; \theta) + \gamma - \log(\hat{P}(d|x)) + \beta \Big[ u(x'^s, d'^s; \theta) + \gamma - \log(\hat{P}(d'^s|x'^s)) + \beta \Big[ u(x''^s, d''^s; \theta) + \gamma - \log(\hat{P}(d''^s|x''^s)) + \beta \cdots \Big]\Big]\Big]$$

where

  - $x'^s \sim \hat{G}(\cdot|x, d)$
  - $d'^s \sim \hat{p}(\cdot|x'^s)$, $x''^s \sim \hat{G}(\cdot|x'^s, d'^s)$
  - &etc.

In short, you simulate $\tilde{V}(x, d; \theta)$ by drawing $S$ "sequences" of $(d_t, x_t)$ with a initial value of $(d, x)$, and computing the present-discounted utility correspond to each sequence. Then the simulation estimate of $\tilde{V}(x, d; \theta)$ is obtained as the sample average.

- In practice, "truncate" the infinite sum at some large $T$.

- Given an estimate of $\tilde{V}(\cdot, d; \theta)$, you can get the predicted choice probabilities:

$$\tilde{p}(d = 1|x; \theta) \equiv \frac{exp\left(\tilde{V}(x, d = 1; \theta)\right)}{\exp\left(\tilde{V}(x, d = 1; \theta)\right) + \exp\left(\tilde{V}(x, d = 0; \theta)\right)} \tag{1}$$

and analogously for $\tilde{p}(d = 0|x; \theta)$. Note that the predicted choice probabilities are different from $\hat{p}(d|x)$, which are the actual choice probabilities computed from the actual data.

- One way to estimate $\theta$ is to minimize the distance between the predicted conditional choice probabilities, and the actual conditional choice probabilities:

$$\hat{\theta} = \text{argmin}_\theta ||\hat{\mathbf{p}}(d = 1|x) - \tilde{\mathbf{p}}\left(d = 1|x; \theta\right)||$$

where $\mathbf{p}$ denotes a vector of probabilities, at various values of $x$.

- Another way to estimate $\theta$ is very similar to the Berry/BLP method. Given the logit assumption, we can equate the actual conditional choice probabilities $\hat{p}(d|x)$ to the model's predicted choice probabilities $\tilde{p}(d|x;\theta)$ to obtain that

$$\hat{\delta}_x \equiv \log \hat{p}(d=1|x) - \log \hat{p}(d=1|x) = [V(x,d=1) - V(x,d=0)] .$$

An alternative estimator could proceed by doing

$$\bar{\theta} = \text{argmin}_\theta ||\hat{\delta}_x - \left[\hat{V}(x,d=1;\theta) - \hat{V}(x,d=0;\theta)\right]||.$$

## 2 Identification of DDC Models

- The argument here follows (Bajari and Hong 2005). Go back to Eq. (1): for each $x$

$$\begin{aligned}\tilde{p}(d=1|x) &= \frac{exp\,(V(1;x))}{\exp\,(V(0;x)) + \exp\,(V(1;x))} \\ &= \frac{exp\,(V(1;x) - V(0;x))}{1 + \exp\,(V(1;x) - V(0;x))}\end{aligned} \tag{2}$$

(Since we will not rely on parametric identification, remove parameter vector $\theta$ from notation.)

For every $x$, we observe choice probability $\tilde{p}(d=1|x)$, so we can solve for $V(1;x) - V(0;x)$: difference in choice-specific value functions evaluated at $x$.

(Note: above is easily generalized to more than two choices, and also generalizable to non-logit error terms.)

- We still need to identify $V(0:x)$, for every $x$. From Bellman equation, note that the choice-specific value function for $d=0$ is defined as:

$$\begin{aligned}V(0;x) &= u(x,0) + \beta E_{x',\epsilon'|x,d=0}[\max_d\,(V(0;x') + \epsilon'_0, V(1;x') + \epsilon'_1)] \\ &= u(x,0) + \beta E_{x'|x,0}[\log \sum_{d'=0,1} \exp V(d';x')] \\ &= u(x,0) + \beta E_{x'|x,0}[\log\,(1 + \exp(V(1;x) - V(0;x)) + V(0;x)]\end{aligned} \tag{3}$$

If we normalize $u(x,0)=0$ for all $x$, then given knowledge of $V(1;x) - V(0;x)$ for all $x$, we can iterate over the last line of Eq. (3) to obtain $V(0;x)$ for all $x$.

- Then $u(x,d)$ for all $(x,d)$ can be calculated as

$$u(x,d) = V(d;x) - \beta E_{x'|x,d}[\log \sum_{d'=0,1} \exp V(d';x')] \tag{4}$$

- Note that in order to calculate the expectations in Eqs. (3) and Eqs. (4), we use estimate of transition probabilities.

## References


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