Chapter 1

The Lorenz Model

1.1 Introduction

Lorenz was interested in the predictability of the solutions to hydrodynamics equations. He was a meteorologist studying weather forecasting—and the question of the fundamental limitations to this endeavor. The model he introduced [1] can be thought of as a gross simplification of one feature of the atmosphere, namely the fluid motion driven by thermal buoyancy known as convection, although his original paper seems to use the model simply as a set of equations "whose solutions afford the simplest example of a deterministic nonperiodic flow of which the writer is aware".

The model describes the convection motion of a fluid in a small, idealized "Rayleigh-Bénard" cell. The idealization is that the boundary conditions of the fluid at the upper and lower plates are taken to "stress free" rather than the realistic "no-slip", the lateral boundary conditions are taken to be "periodic" rather than corresponding to a realistic side walls, and the motion is assumed to be two dimensional rather than three. These three modifications greatly simplify the mathematical analysis.

The full description of the motion of the fluid is replaced by dynamical equations for a few of the simplest modes of the system, so that the temperature T is approximated by

$$T(x,z,t) \simeq -rz + 9\pi^3\sqrt{3}Y(t)\cos(\pi z)\cos\left(\frac{\pi}{\sqrt{2}}x\right) + \frac{27\pi^3}{4}Z(t)\sin(2\pi z) \tag{1.1}$$

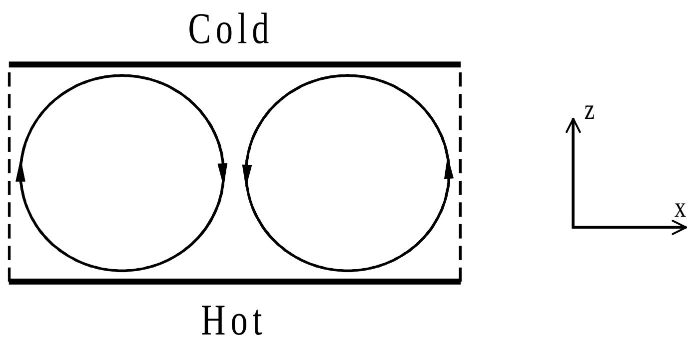

Figure 1.1: The physical system modelled by Lorenz

and the fluid velocity components u (x direction) and w (z direction) are given conveniently in terms of the stream function ψ

$$u = -\frac{\partial \psi}{\partial z}, \qquad w = \frac{\partial \psi}{\partial x}$$

with the stream function

$$\psi(x, z, t) = 2\sqrt{6}X(t)\cos(\pi z)\sin\left(\frac{\pi}{\sqrt{2}}x\right) \quad . \tag{1.2}$$

For the technically sophisticated it should be remarked that the variables have been "de-dimensionalized" with respect to a convenient length (the depth of the cell), time (the heat diffusion time across the depth of the cell) and temperature. The first term in the temperature equation is the linear profile that gives the conduction of heat in the absence of fluid motion, and the parameter r is the temperature difference across the cell (in the scaled units). The mode structure, which comes from solving the linearized fluid and heat equations, is sinusoidal in both horizontal and vertical directions. The horizontal wave length has been chosen as the one that optimizes the onset of convection, and the width of the system is fixed at one wavelength. The rather complicated looking numerical prefactors are simply a convenient choice of normalization of the modes that simplifies the dynamical equations.

The scheme now is to substitute these expressions into the coupled fluid and heat equations. Since these equations are non-linear, there will be terms coupling the different modes, and also terms generating higher harmonics than are represented in equations (1.1),(1.2). The latter terms are ignored—the major approximation in the scheme. This reduces the complicated *partial differential equations* describing the fluid motion and heat flow to three *ordinary differential equations*

$$\begin{aligned} \dot{X} &= -\sigma(X - Y) \\ \dot{Y} &= rX - Y - XZ \\ \dot{Z} &= b(XY - Z) \end{aligned} \tag{1.3}$$

where the "dot" denotes the time derivative d/dt. The parameter σ depends on the properties of the fluid (in fact the ratio of the viscous to thermal diffusivities): Lorenz took the value to be 10 in his paper; for water the value is typically between 1 and 4, for an ideal gas the value is about 0.7, and for oils the value is $10 - 1000$ or even higher. The number $b = 8/3$: this would be different for a different choice of horizontal wavelength or roll diameter. Again the temperature difference r appears and is the important control parameter: for $r < 1$ the solution at long times is asymptotic to $X = Y = Z = 0$, i.e. no convection. For $r > 1$ more interesting solutions occur!

The important features of these equations are:

- They are autonomous—time does not explicitly appear on the right hand side;
- They involve only first order time derivatives so that (with the autonomy) the evolution depends only on the instantaneous value of (X, Y, Z);
- They are non-linear, here through the quadratic terms XZ and XY in the second and third equations;
- They are dissipative—crudely the "diagonal" terms such as $\dot{X} = -\sigma X$ correspond to decaying motion, but more systematically we will see that "volumes in phase space" shrink in the dynamics;
- The solutions are bounded.

Lorenz simulated these equations and found chaos. (Actually he references his colleague Saltzman for telling him about "aperiodic solutions" to similar systems of equations, although a paper [2] by Saltzman published one year earlier has no mention of such dynamics. More importantly Lorenz realized the importance of the aperiodic motion, and developed diagnostic tools to make sense of it.)

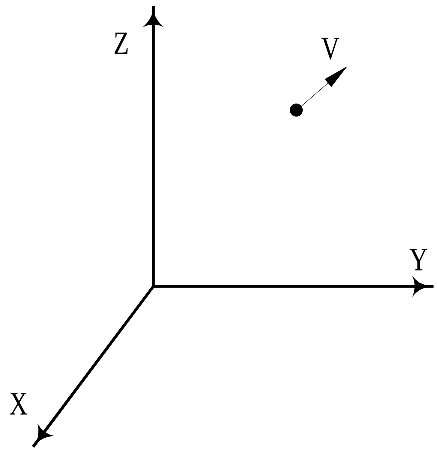

Figure 1.2: Phase Space for the Lorenz Model

We will now look at some examples of the dynamics. Since the equations for the dynamics of (X, Y, Z) are first order and autonomous we will call this set of variables the "phase space". The dynamics at each point in phase space, specified by the "velocity in phase space" vector $\vec{V} = (\dot{X}, \dot{Y}, \dot{Z})$, is unique. The evolution in time then traces out a path in the three dimensional phase space. An immediate result is that *phase space trajectories cannot cross*.

1.2 Demonstrations

Please see the accompanying programs

1.3 Final Words

The "sensitive dependence on initial conditions" found by Lorenz is now known affectionately as Lorenz's "butterfly effect". In fact in a later paper [3] Lorenz remarked:

> One meteorologist remarked that if the theory were correct, one flap of the sea gull's wings would be enough to alter the course of the weather

> forever.

By the time of Lorenz's talk at the December 1972 meeting of the American Association for the Advancement of Science in Washington, D.C. the sea gull had evolved into the more poetic butterfly - the title of his talk was [4]:

> Predictability: Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?

Lorenz's work was largely ignored for ten years, but can now be seen as a prescient beginning to the study of chaos. Using this model we have identified some characteristics of chaos, that we will want to quantify further:

- apparent ”randomness” in the time variation;
- broad band components to the power spectrum;
- sensitive dependence on initial conditions.

In addition we have briefly looked at:

- interesting structures in phase space known as strange attractors;
- the Poincaré section to show the structure most simply;
- the use of 1d maps.

It is now known that the Lorenz equations are not an accurate description of the original (idealized) convection system for temperature differences, expressed by the dimensionless measure r, large enough to yield chaos—the equations may be derived systematically as an expansion in $r-1$, but 27 is not a small number! For example Curry [5] has shown that if the mode truncation is not done, but instead sufficient modes are retained to give numerical convergence, the chaos disappears. In this work the two dimensionality of the flow (in the $x-z$ plane) was maintained. On the other hand McLauglin and Martin [6] showed that chaos is obtained for a three dimensional version—but of course the mode equations are then not simply the Lorenz equations.

An alternative approach has been to construct experimental systems for which the Lorenz equations *are* a good description. Since the Lorenz equations break down due to the excitation of higher spatial harmonics, one scheme has been to investigate convection in a circular glass tube held vertically and heated over the lower half and cooled over the upper half. This is the thermosyphon. Indeed behavior qualitatively similar to the predictions of the Lorenz model are obtained, with chaotic reversal of the circulation around the tube. However since the boundary condition over the lower half is constant heat input, rather than constant temperature, it is found that a fourth mode (the mean temperature) must be added to make the predictions quantitative [7][8][9][10][11].

Another experimental system described by the Lorenz equations is the Rititake dynamo—a homopolar generator with the output fed back through inductors and resistors to the coil generating the magnetic field [11]. The disc is driven by a constant torque. The coupled circuit and rotation equations can be reduced to the Lorenz form, and experiments [12] indeed show an apparently chaotic reversal of the coil current and the magnetic field. Analogies with the earth's magnetic field, which shows irregular reversals on a time scale of millions of years, are certainly intriguing, although recent numerical simulations [13] suggest that a three mode truncation will not be a good approximation for the turbulent dynamics of the earth's interior.

1.4 Appendix: Numerical considerations

We need to solve ordinary differential equations (ODEs) of the form

$$\frac{dy}{dt} = f(y, t). \tag{1.4}$$

Coupled ODEs take this form if we think of y as a vector. The solution is based on the simple idea of dividing time up into small steps of size h: $t_n = nh$, $y_n = y(t_n)$ and then using a Taylor expansion about the present time:

$$y_{n+1} = y_n + h\, f(y_n, t_n) + \cdots \tag{1.5}$$

where the $\cdots$ represent higher terms in the expansion. Subtleties arise when trying to improve the *stability* and improve or control the *accuracy*.

A popular choice is the "4th order Runge Kutta" scheme where successive estimates of the increment are made, and combined in a way that effectively includes higher order terms in the Taylor expansion. We evaluate

$$\begin{aligned} k_1 &= h\,f(y_n, t_n) \\ k_2 &= h\,f(y_n + \tfrac{1}{2}k_1, t_n + \tfrac{1}{2}h) \\ k_3 &= h\,f(y_n + \tfrac{1}{2}k_2, t_n + \tfrac{1}{2}h) \\ k_4 &= h\,f(y_n + k_3, t_n + h) \end{aligned} \tag{1.6}$$

and combine them to give

$$y_{n+1} = y_n + \tfrac{1}{6}k_1 + \tfrac{1}{3}k_2 + \tfrac{1}{3}k_3 + \tfrac{1}{6}k_4 + O(h^5). \tag{1.7}$$

The size of the $O(h^5)$ error depends on the derivatives of f, i.e. the smoothness of the function.

Now we can choose a step size h, and iterate to get the solution. There are numerous tricks to improve the procedure and our knowledge of how well we are doing. An obvious improvement is to do at each time step an auxiliary calculation of two half steps, and to compare the two results to get an estimate of the error. We can then also combine the two results to get an estimate equivalent to a fifth order Runge Kutta! This is the routine used in the demonstrations. For more details see the book "Numerical Recipes" [14] chapter 15. A further enhancement is to dynamically vary the step size to keep the error estimate below some chosen limit: this is a good idea generally, but would further distort the loose connection between computer time and iteration time that is used in displaying the dynamic plots, so has not been implemented in the demonstrations. Completely different numerical schemes [14] may also have advantages.

In evolving ODEs displaying chaos the question of "numerical accuracy" is more subtle—the sensitive dependence on initial conditions implies that a small numerical error will be amplified over time to give an $O(1)$ error after times of order $\log(1/\text{error})$, usually well within the times of interest. Thus we cannot, with our usual care, expect to get the "correct" answer from a given initial condition, only one that is locally a good approximation and statistically representative in some sense. The question of how well numerical solutions "track" an accurate solution is still an active area of work [15] and is discussed further in chapter 26. This takes us back to Lorenz's original question, and is a good place to close this introduction!

December 24, 1999

Bibliography

[1] E. Lorenz, J. Atmos. Sciences, **20**, 130 (1963)

[2] B. Salzman, J. Atmos. Sci. **19**, 239 (1962)

[3] E. Lorenz, Transactions of the New York Academy of Sciences, **??,** 409 (1963)

[4] As quoted in "Chaos and Nonlinear Dynamics" by R.C.Hilborn (Oxford University Press, 1994). This information was kindly sent to me by Corrie Modell.

[5] J.H. Curry, Commun. Math. Phys. **60**, 193 (1978)

[6] J.B. McLaughlin and P.C. Martin, Phys. Rev. **A12**, 186 (1975)

[7] H.F. Creveling, J.F. dePaz, J.Y. Baladi, and R.J. Schoenhal, J. Fluid Mech. **67**, 65 (1975)

[8] J.A. Yorke and E.D. Yorke, J. Stat. Phys. **21**, 263 (1979)

[9] M. Gorman, P.J. Widmann, and K.A. Robbins, Physica **19D**, 255 (1986); **36D**, 157 (1989)

[10] J.E. Hart, Int. J. Heat and Mass Transfer **27**, 759 (1984)

[11] "Perspectives in Nonlinear Dynamics" by E.A. Jackson (Cambridge University Press, 1991), vol. 2, ch. 7.

[12] K.A. Robbins, Math. Proc. Camb. Phil. Soc. **82**, 309 (1977)

[13] G.A. Glatzmaier and P.H. Roberts, Nature **377**, 203 (1995)

[14] W.H. Press, B.P. Flannery, S.A. Teukolsky, and W.T. Vetterling, "Numerical Recipes", (Cambridge University Press, 1986)

[15] T. Sauer, C. Grebogi, and J.A. Yorke, Phys. Rev. Lett. **79**, 59 (1997)