Do Tutte Polynomials Satisfy The Kontsevich Conjecture?

Jessica Su (supervised by Dr. Matilde Marcolli)

Abstract - Long

0.0.1 What is the Kontsevich conjecture?

From any graph, it is possible to obtain a polynomial called the graph polynomial. Kontsevich conjectured that the number of zeros of the graph polynomials over a finite field was polynomially dependent on the order of the field. This conjecture was disproven. However, it is still interesting because it is true for graphs with up to twelve edges.

0.0.2 What is a Tutte polynomial?

Since Kontsevich's conjecture was moderately successful for graph polynomials, we decided to see if it worked for the related Tutte polynomials. The Tutte polynomial of a graph is defined as follows:

Definition. Let $G(V, E)$ be a graph with vertex set V and edge set E. Assign an edge weight v_e to each $e \in E$. Then the multivariate Tutte polynomial of G is

$$T(q, \mathbf{v}) = \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} v_e$$

where the sum is over the subgraphs of G that span all vertices and where $k(A)$ is the number of connected components in each subgraph A. q is an indeterminate.

Example. The Tutte polynomial of C_3 is $q^3 + q^2(v_1 + v_2 + v_3) + q(v_1v_2 + v_1v_3 + v_2v_3) + v_1v_2v_3$.

0.0.3 Why do we care?

Graph polynomials and Tutte polynomials are both important in physics. Graph polynomials are useful in quantum field theory. Tutte polynomials are useful in statistical physics. Understanding how they are related would illuminate a link between statistical physics and quantum field theory.

One way to compare these polynomials is to compare the number of zeros they have over finite fields. One way to do that is to see whether Tutte polynomials satisfy the Kontsevich conjecture. In this talk, we find cases for which the Kontsevich conjecture is true, false, and partially false.

0.0.4 The Monte Carlo method

The naive way to count the number of roots of a Tutte polynomial is to plug in all possible tuples and count how many yield 0. (This only works for fields of prime order.) This is a terrible idea because there are $p^{|E|}$ of them (p the order of the field, $|E|$ the number of edges).

Instead, we used the Monte Carlo method. The idea is if you evaluate the polynomial at tuples chosen randomly from $F^{|E|}$, the proportion of zeros in the sample can be made arbitrarily close to the proportion of actual zeros. Typically, 50000 trials will yield a count within 1 percent of the correct value - though you have to be careful, because if there are large numbers of roots this can translate to large margins of error.

Using this method, we found some interesting results. For certain classes of graphs, the numbers of roots were the same for all values of $q \neq 0, 1$. Furthermore, plenty of graphs did seem to satisfy the Kontsevich conjecture, but we couldn't be sure because (a) the numbers of roots were approximate (b) we couldn't try infinitely many points. To remedy this uncertainty, we tried actually proving things.

0.0.5 Kontsevich's conjecture succeeds for $q = 1$

Proposition. Kontsevich's conjecture is always successful when $q = 1$ (for fields of prime order).

Proof. You'll just have to wait until the talk to find out.

0.0.6 Kontsevich's conjecture mostly fails for $q \neq 1$, but sometimes the failures are trivial

One could conceive that the number of roots might be polynomially dependent on the field size except at points where $q = 0$. If this happens, Kontsevich's conjecture must fail, as shown in the following proposition:

Proposition. If a set of points satisfies a polynomial at infinitely many values but does not satisfy that polynomial at certain other values, the entire set of points cannot satisfy a polynomial (even if it's a different polynomial).

Example. No tree can satisfy the Kontsevich conjecture for $q \neq 1$.

It stands to reason that if the Kontsevich conjecture doesn't even work for trees, it probably won't work for many other graphs either. But this was sort of a trivial failure of the conjecture, and I was later redirected towards studying the "modified" Kontsevich conjecture (which said failures caused by q being 0 didn't count). The modified Kontsevich conjecture works for trees. It also works for cycles, but a proof of this is beyond the scope of this talk.

0.0.7 The modified Kontsevich conjecture probably fails for K_4

Definition. For a multivariate polynomial f, $Z[f]$ is the number of roots of the polynomial. An expression of the form $Z[f]$ is called a *Z-expression.*

Lemma 1. There are infinitely many primes of the form $4k + 3$ (or $4k - 1$).

Proof. Number theory!

Lemma 2. For all primes $4k + 3$, the polynomial $x^2 + 2x + 2$ has no solution.

Proof. More number theory!

Lemma 3. Z_{x^2+2x+2} is not a polynomial.

Proof. This follows from Lemmas 1 and 2. Details in the talk.

Now we are ready to show the theorem.

Theorem. Kontsevich's conjecture fails for K_4 when $q = 2$.

Proof. John Stembridge wrote code to reduce Z-expressions to simpler Z-expressions. We used this code to compute the Kontsevich polynomial for K_4.

For $q = 2$, the Kontsevich polynomial was $p^6 Z[2] - p^5 Z[2] + p^5 Z[4x_2x_4x_5 + 8x_2x_3x_4x_5 + 4x_2x_3x_4 + 4x_2x_3x_5 + 4x_3x_4x_5 + 4x_3x_4 + 4x_2x_5 + 2x_3x_4^2 + 2x_3^2x_4 + x_3^2x_4^2 + 2x_2x_3^2x_4 + x_2x_3^2x_4^2 + 2x_2x_3x_4^2 + 2x_3x_4^2x_5 + 2x_3^2x_4x_5 + x_3^2x_4^2x_5 + 2x_2^2x_4x_5x_3 + 2x_2x_4^2x_5x_3 + x_3^2x_4^2x_5x_2 + 2x_3^2x_4x_5x_2 + 2x_2^2x_5 + 2x_2^2x_4x_5 + 2x_2^2x_5x_3 + 2x_2x_5^2 + x_2^2x_5^2 + x_2^2x_4x_5^2 + 2x_2x_4x_5^2 + x_2^2x_3x_5^2 + 2x_2x_3x_5^2 + x_2^2x_4x_5^2x_3 + 2x_2x_4x_5^2x_3] - p^5 Z[2, 4x_2x_4x_5 + 8x_2x_3x_4x_5 + 4x_2x_3x_4 + 4x_2x_3x_5 + 4x_3x_4x_5 + 4x_3x_4 + 4x_2x_5 + 2x_3x_4^2 + 2x_3^2x_4 + x_3^2x_4^2 + 2x_2x_3^2x_4 + x_2x_3^2x_4^2 + 2x_2x_3x_4^2 + 2x_3x_4^2x_5 + 2x_3^2x_4x_5 + x_3^2x_4^2x_5 + 2x_2^2x_4x_5x_3 + 2x_2x_4^2x_5x_3 + x_3^2x_4^2x_5x_2 + 2x_3^2x_4x_5x_2 + 2x_2^2x_5 + 2x_2^2x_4x_5 + 2x_2^2x_5x_3 + 2x_2x_5^2 + x_2^2x_5^2 + x_2^2x_4x_5^2 + 2x_2x_4x_5^2 + x_2^2x_3x_5^2 + 2x_2x_3x_5^2 + x_2^2x_4x_5^2x_3 + 2x_2x_4x_5^2x_3] + p^5 + p^4 Z[2] - p^4 + 3p^3 Z[2] + 2p^3 Z[2 + 2x_4 + x_4^2] + 2p^3 Z[2 + 2x_5 + x_5^2] - 2p^3 Z[2, 2 + 2x_4 + x_4^2] - 2p^3 Z[2, 2 + 2x_5 + x_5^2] - 3p^3 - 13p^2 Z[2] + 13p^2 + pZ[2] - p + Z[2] - 1$.

The $Z[2]$ terms are only 1 at points where $2 = 0$, which we are ignoring. So ignoring the $Z[2]$ terms we get something of the form

$p^5 Z$[really long polynomial] + terms polynomial in p + $2p^3 Z[2 + 2x_4 + x_4^2] + 2p^3 Z[2 + 2x_5 + x_5^2]$.

The $Z[2 + 2x + x^2]$ terms stack and produce something that isn't a polynomial. So unless Z[really long polynomial] has a number of roots that cancel those out, which I doubt, the Z-expression for K_4 is not a polynomial. Further work will focus on numerically verifying that things do not in fact cancel out.